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THE SINH-COSH METHOD FOR TRAVELING WAVE SOLUTIONS OF SOME PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

This paper investigates the application of the Sinh-Cosh method to derive traveling wave solutions for three nonlinear partial differential equations: the Schäfer-Wayne Short Pulse Equation (SPE), the (2+1)-dimensional Kadomtsev-Petviashvili (KP) equation, and the Modified Equal Width (MEW) equation. The method effectively generates a variety of solutions, including multisoliton, singular soliton, and periodic waveforms, characterized by free parameters. By reducing these partial differential equations to nonlinear ordinary differential equations and solving them using hyperbolic sine and cosine functions, we demonstrate the reliability and efficiency of the method. The resulting solutions are visualized graphically to showcase the soliton behaviors. The study highlights the method's potential in solving complex nonlinear evolution equations relevant to various fields of science and engineering.

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1. Introduction

Nonlinear evolution equations (NLEEs) have emerged as invaluable tools for describing the natural phenomena of science and engineering. The traveling wave solutions of NLEEs play a crucial role in modeling and understanding various physical phenomena across diverse scientific disciplines, including plasma physics, the propagation of shallow water waves, fluid mechanics, electricity, chemical kinematics, biology, optical fibers, among others. The traveling wave solutions contribute significantly to our understanding of intricate physical phenomena and provide insights into their internal mechanisms. Numerous methods have been derived in the literature to find traveling wave solutions, such as Hirota's bilinear transformation method [1], the tanh-function method [2], the (G'/G)-expansion method [3], the F-expansion method [4], the sine-cosine method [5], the sinh-cosh method [6], and so on.

The Sch*ä*fer-Wayne Short Pulse Equation (SPE) was introduced by Sch*ä*fer and Wayne as a model equation that characterizes the propagation of ultra-short light pulses in silica optical fibers. It is an alternative to the nonlinear Schrödinger equation, offering an approximation for very short optical pulses in nonlinear media [7]. V. Kuetche, T. Bouetou, and T. Kofane utilized Hirota's Method and the Hodnett–Moloney Approach to derive a loop soliton solution for the Schäfer-Wayne Short Pulse Equation (SWSPE) [8]. Additionally, A. Sakovich and S. Sakovich employed SGE method to generate non-singular solitary wave solutions for the SPE [9]. In [10] O. Gaxiola applied the LSM to obtain bright and dark optical solitons of the Schäfer-Wayne short-pulse equation.

The Kadomtsev-Petviashvili (KP) equation represents generalization of a Korteweg-de Vries (KdV) equation. It describes the evolution of two-dimensional, small-amplitude, long waves in weakly dispersive and weakly nonlinear media. In [11], the authors employed the Exp($-\phi(\xi)$) -expansion method to obtain various traveling wave solutions of the KP equation. W. Ma used the Hirota bilinear form and generated a class of lump solutions of the KP equation [12]. In [13], the authors derived quasiperiodic solutions of the KP equation using the multi-dimensional Baker–Akhiezer function generated by the Broer–Kaup hierarchy. M. Kumar et al. obtained some new exact solutions of the KP equation by employing the similarity transformations method [14]. Klein and Roidot obtained the singular solutions of the dispersionless KP equation numerically for different classes of initial data [15].

The modified equal width (MEW) equation was initially proposed by Morrison et al. as a model in partial differential equations for simulating the transmission of one-dimensional waves in a nonlinear dispersion process [16]. B. Saka solved the MEW equation numerically using quintic B-spline collocation algorithms for various boundary and initial conditions [17]. S. Karakoç and T. Geyikli [18] proposed a lumped Galerkin method based on the cubic B-spline finite element method to obtain a numerical solution of the MEW equation and also studied solitary wave motion and the interaction of two solitary waves. In [19], the authors derived dispersive traveling wave solutions of the MEW equation by employing two mathematical techniques, namely, the Extended Simple Equation method and the Exp($-\phi(\xi)$) expansion method.

The outline of this paper is as follows: Introduction and literature survey presented in Section 1. In section 2, the sinh-cosh method introduced. Solutions to the said equations proposed in Section 3. Graphical presentation and their discussion provided in Section 4. Finally, the conclusion section offers a brief summary of the results.

2. Analysis of the method

Consider a nonlinear partial differential equation

$$P(u, u_t, u_x, u_x x, u_x t, \dots) = 0, \tag{2.1}$$

where $u(x, t)$ is traveling wave solution of (2.1). The basic steps of the sinh-cosh method are listed as follows:

1. To find a traveling wave solution we introduce a new wave variable $\xi = x - ct$ where c is speed of traveling wave, so that

$$u(x, t) = u(\xi) = u(x - ct) \tag{2.2}$$

then we get,

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \quad \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \xi^2}, \quad \frac{\partial^2}{\partial t} = \frac{\partial}{\partial \xi^2}, \dots \quad \text{and so on.}$$

2. By using above relations one can reduce the PDE (2.1) into nonlinear ordinary differential equation (ODE)

$$Q\left(u,\ u_\xi,\ u_{\xi\xi},\ \dots\right) = 0. \tag{2.3}$$

3. Integrate the ODE (2.3) as long as all the terms contain derivatives, by setting the constant of integration is zero.
4. Solution of ODE can be express as

$$u\,(x,t) = A\cosh^p(\mu\xi), \tag{2.4}$$

or in the form

$$u\,(x,t) = A\sinh^p(\mu\xi). \tag{2.5}$$

5. Derivative of (2.4) become ,

$$\begin{aligned}
u\,(\xi) &= A\cosh^p(\mu\xi)\,,\\
u^n\,(\xi) &= A^n\cosh^{np}(\mu\xi)\,,\\
(u^n)_\xi &= A^n np\mu\cosh^{np-1}(\mu\xi)\,sinh\,(\mu\xi)\,,\\
(u^n)_{\xi\xi} &= A^n n^2p^2\mu^2\cosh^{np}(\mu\xi) + A^n np(np-1)\mu^2\cosh^{np-2}(\mu\xi)\,,
\end{aligned} \tag{2.6}$$

and the derivative of (2.5) become

$$\begin{aligned}
u\,(\xi) &= A\sinh^p(\mu\xi)\,,\\
u^n\,(\xi) &= A^n\sinh^{np}(\mu\xi)\,,\\
(u^n)_\xi &= A^n np\mu\sinh^{np-2}(\mu\xi)\cosh(\mu\xi)\,,\\
(u^n)_{\xi\xi} &= A^n np(np-1)\mu^2\sinh^{np-2}(\mu\xi) + A^n n^2p^2\mu^2\sinh^{np}(\mu\xi)\,,
\end{aligned} \tag{2.7}$$

other derivatives can be calculated in similar manner.

6. Substitute(2.6) or (2.7) into ODE (2.3), and balancing the terms of cosine hyperbolic when (2.4) is used, or balance the terms of sine hyperbolic when (2.5) is used and and solve the resulting system of algebraic equations. Next we collect the coefficient of like powers of $\sinh^k(\mu\xi)$ or $\cosh^k(\mu\xi)$ and set to zero to get a system of algebraic equations in A, p, and μ.

3. Applications

3.1. The Schӓfer-Wayne Short Pulse Equation (SPE) In this subsection, sinh-cosh method in used to find exact solutions of SPE,

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}. \tag{3.1}$$

The traveling wave transformation $u(x,t) = u(\xi)$, where $\xi = x - ct$ transform the equation (3.1) into following ordinary differential equation.

$$\left(2c + u^2\right) u_{\xi\xi} + 2u\left(1 + u_{\xi}^2\right) = 0. \tag{3.2}$$

Let solution of (3.1) is of the form

$$\begin{aligned} u &= A\cosh^p(\mu\xi), \\ \text{so,} & \\ u_{\xi} &= Ap\mu\cosh^{p-1}(\mu\xi)\sinh(\mu\xi), \\ u_{\xi\xi} &= Ap^2\mu^2\cosh^p(\mu\xi) - Ap(p-1)\mu^2\cosh^{p-2}(\mu\xi). \end{aligned} \tag{3.3}$$

Substitute (3.3) in (3.1) we get,

$$A^3p^2\mu^2\cosh^{3p}(\mu\xi) - A^3p(p-1)\mu^2\cosh^{3p-2}(\mu\xi) + 2cAp^2\mu^2\cosh^p(\mu\xi) - 2cAp(p-1)\mu^2\cosh^{p-2}(\mu\xi) + 2A\cosh^p(\mu\xi) + 2A^3p^2\mu^2\cosh^{3p}(\mu\xi) - 2A^3p^2\mu^2\cosh^{3p-2}(\mu\xi) = 0. \tag{3.4}$$

Equating the exponents and coefficients of cosh, we get following algebraic equations

$$\begin{aligned} p(p-1) &\neq 0, \\ 3p &= p-2, \\ 2cAp^2\mu^2 + 2A &= 0, \\ 3A^3p^2\mu^2 - 2cAp(p-1)\mu^2 &= 0. \end{aligned} \tag{3.5}$$

Solving above system we get,

$$\begin{aligned} p &= -1, \\ A &= \pm 2\sqrt{\frac{c}{3}}, \end{aligned} \tag{3.6}$$

$$\mu = \pm\frac{i}{\sqrt{c}}. \tag{3.7}$$

Exact solution equation (3.1) is

$$u(x,t) = \pm 2\sqrt{\frac{c}{3}}\cosh^{-1}\left(\pm\frac{i}{c}(x-ct)\right) \tag{3.8}$$

Now substitute (2.7) in (3.1), we get

$$\begin{aligned} u &= A\sinh^p(\mu\xi), \\ u_{\xi} &= Ap\mu\sinh^{p-1}(\mu\xi)\cosh(\mu\xi), \\ u_{\xi\xi} &= Ap^2\mu^2\sinh^p(\mu\xi) + Ap(p-1)\mu^2\sinh^{p-2}(\mu\xi) \end{aligned}$$

$$A^3p62\mu^2\sinh^{3p}(\mu\xi)+A^3p(p-1)\mu^2\sinh^{3p-2}(\mu\xi)+2cAp^2\mu^2\sinh^{p}(\mu\xi)+ 2cAp(p-1)\mu^2\sinh^{p-2}2Asinh^{p}(\mu\xi)+2A^3p^2\mu^2\sinh^{3p-2}(\mu\xi) +2A^3p^2\mu^2\sinh^{3p}(\mu\xi)=0. \quad (3.9)$$

Again by same process we get following system of equation,

$$\begin{aligned} p(p-1) &\neq 0,\\ 3p &= p-2,\\ 2cAp^2\mu^2+2A &= 0,\\ 3A^3p^2\mu^2+2cAp\,(p-1)\,\mu^2 &= 0. \end{aligned} \quad (3.10)$$

Solving this system yield,

$$\begin{aligned} p &= -1,\\ A &= \pm 2i\sqrt{\frac{c}{3}},\\ \mu &= \pm\frac{i}{\sqrt{c}}. \end{aligned} \quad (3.11)$$

Result (3.12) gives the solution of (3.1),

$$u\,(x,t)=\pm 2i\sqrt{\frac{c}{3}}\sinh^{-1}\left(\pm\frac{i}{c}\,(x-ct)\right). \quad (3.12)$$

3.2. (2+1) Dimensional Kadomtev-Petviashvili (KP) Equation Consider (2+1) dimensional KP equation,

$$u_{xt}-6u_x^2-6uu_{xx}+u_{xxxx}+3\delta^2u_{yy}=0. \quad (3.13)$$

Substitute traveling wave transformation $u\,(x,y,t)=u\,(x+y-ct)$ *and* $\delta^2=1$ in (3.13) yield,

$$[-cu'-6uu'+u''']'+3u''=0. \quad (3.14)$$

Integrating twice (3.14) with constant of intgration is zero we get,

$$-cu-3u^2+u''+3u=0. \quad (3.15)$$

Substituting (2.4) in (3.15) gives

$$-cA\cosh^{p}(\mu\xi)-3A^2\cosh^{2p}(\mu\xi)+A\mu^2p^2\cosh^{p}(\mu\xi)-Ap(p-1)\mu^2\cosh^{p-2} +3A\cosh^{p}(\mu\xi)=0. \quad (3.16)$$

Which give following system of equations,

$$\begin{aligned} p(p-1) &\neq 0\\ 2p &= p-2,\\ -cA+A\mu^2p^2+3A &= 0\\ -3A^2-Ap(p-1)\mu^2 &= 0 \end{aligned} \quad (3.17)$$

By solving above system we get,

$$p = -2 \quad \mu = \pm\sqrt{\frac{A}{2}}i \quad c = 3 - 2A. \tag{3.18}$$

Solution of (3.13) is

$$u(x, y, t) = A\cosh^{-2}\left(\pm\frac{\sqrt{A}}{2}i\,(x + y - (3 - 2A)\,t)\right) \tag{3.19}$$

Similarly, euqation(2.5) give following periodic solution

$$u(x, y, t) = A\sinh^{-2}\left(\pm\frac{\sqrt{A}}{2}\,(x + y + (3 + 2A)\,t)\right) \tag{3.20}$$

3.3. The Modified Equal Width Equation In this section we employed sinh-cosh method to find exact solution of MEW equation,

$$u_t + 3u^2u_x - \alpha u_{xxt} = 0. \tag{3.21}$$

The traveling wave transformation (2.2) yield

$$-cu' + 3u^2u' - \mu cu'' = 0. \tag{3.22}$$

Integrating (3.22) and using (2.6) we get following system of algebraic equations,

$$\begin{aligned} p(p-1) &\neq 0, \\ 3p &= p - 2 \\ -cA - \alpha cAp(p-1)\mu^2 &= 0, \\ A^3 + \alpha cAp(p-1)\mu^2 &= 0. \end{aligned} \tag{3.23}$$

By solving above system we get,

$$\begin{aligned} p &= -1, \\ \mu &= \pm\frac{i}{\sqrt{\alpha}}, \\ c &= \frac{A^2}{2}. \end{aligned} \tag{3.24}$$

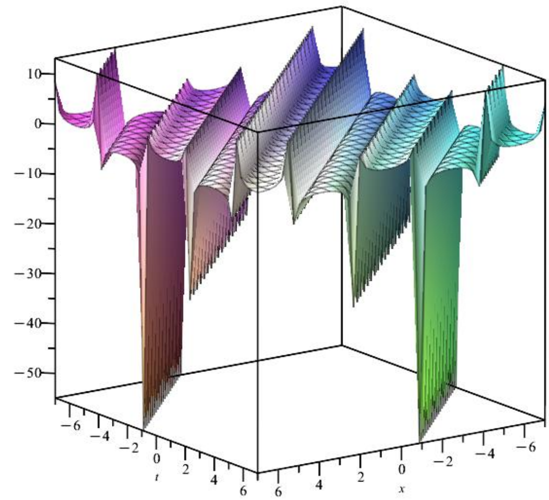
Exact solution of (3.21) is,

$$u(x, t) = A\cosh^{-1}\left(\pm\frac{i}{\sqrt{\alpha}}\left(x - \frac{A^2}{2}\right)t\right). \tag{3.25}$$

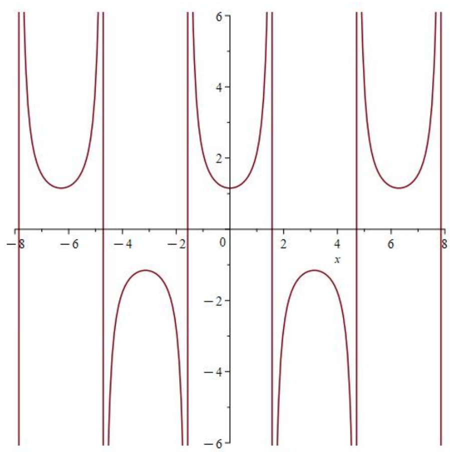
Exact solution of (3.21) By using (2.5) is,

$$u(x, t) = A\sinh^{-1}\left(\pm\frac{1}{\sqrt{\alpha}}\left(x + \frac{A^2}{2}t\right)\right). \tag{3.26}$$

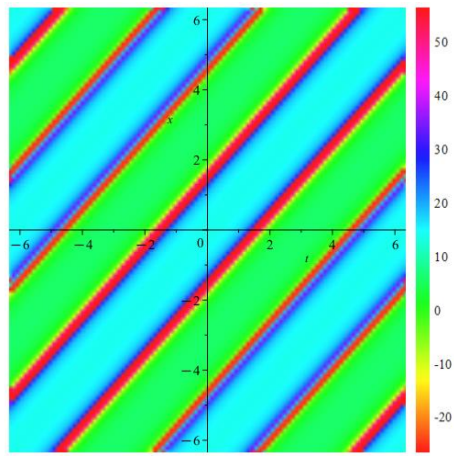
4. Graphical Presentation and Dicussion



(a) 3D Plot

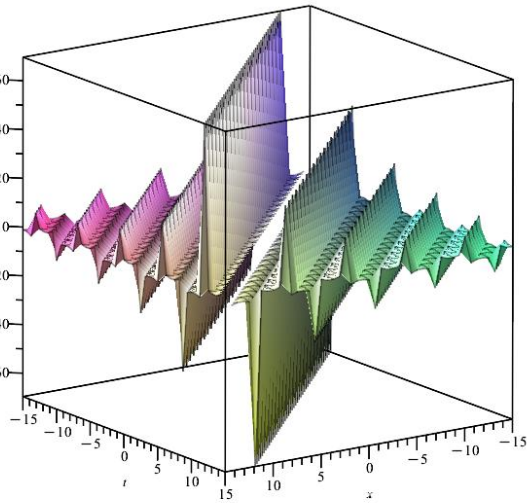


(b) 2D Plot

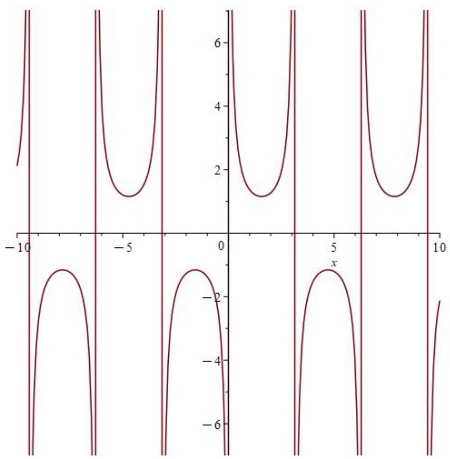


(c) Density Plot

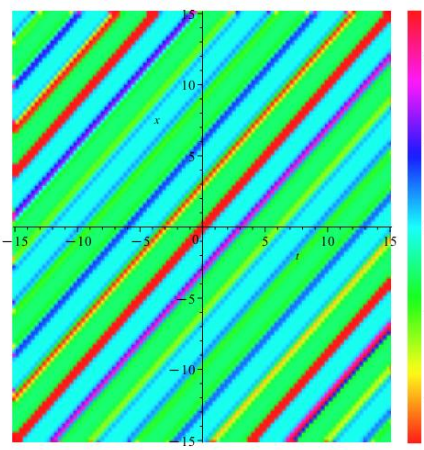
FIGURE 1. Graphical Presentation of (3.8)



(a) 3D Plot

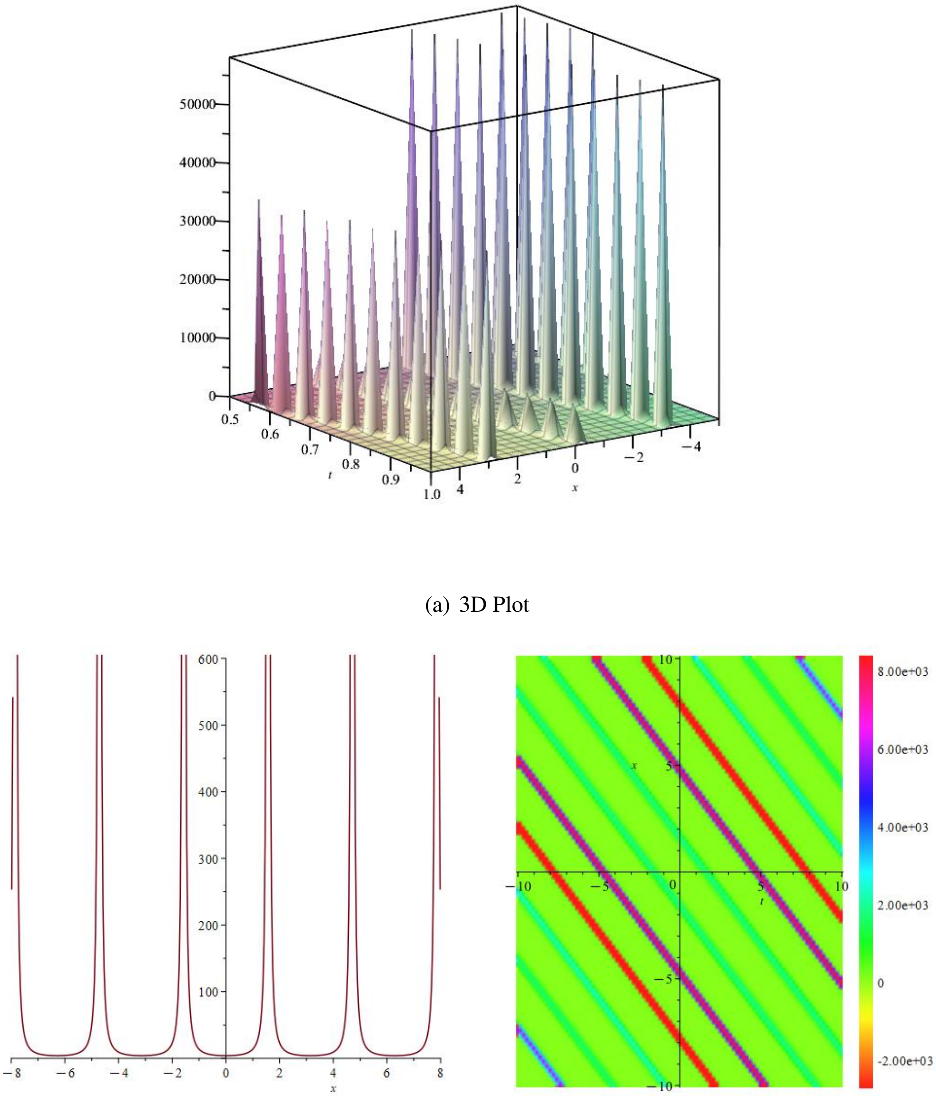


(b) 2D Plot



(c) Density Plot

FIGURE 2. Graphical Presentation of (3.12)

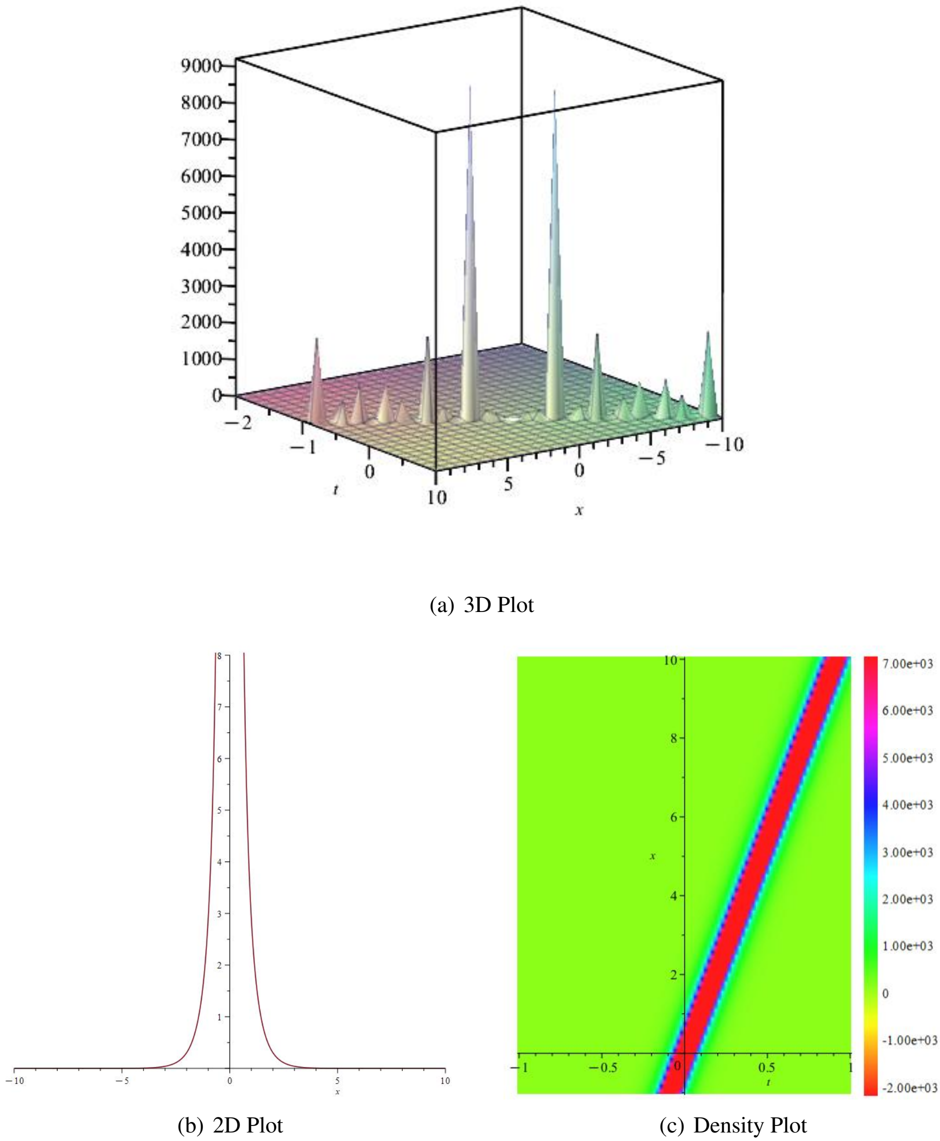


(a) 3D Plot

(b) 2D Plot

(c) Density Plot

FIGURE 3. Graphical Presentation of (3.19)

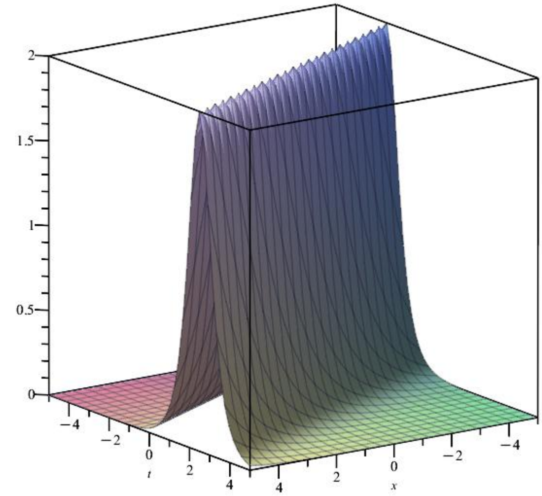


(a) 3D Plot

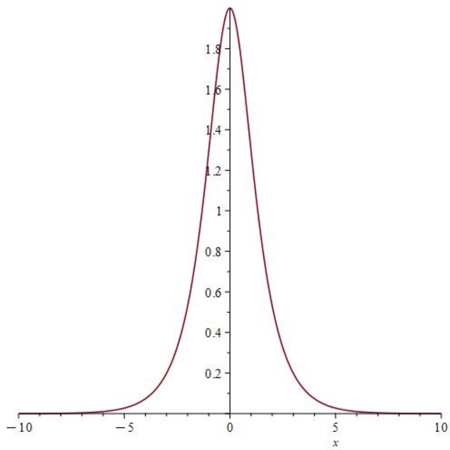
(b) 2D Plot

(c) Density Plot

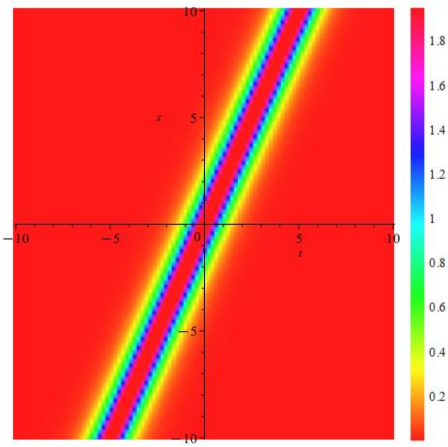
FIGURE 4. Graphical Presentation of (3.20)



(a) 3D Plot

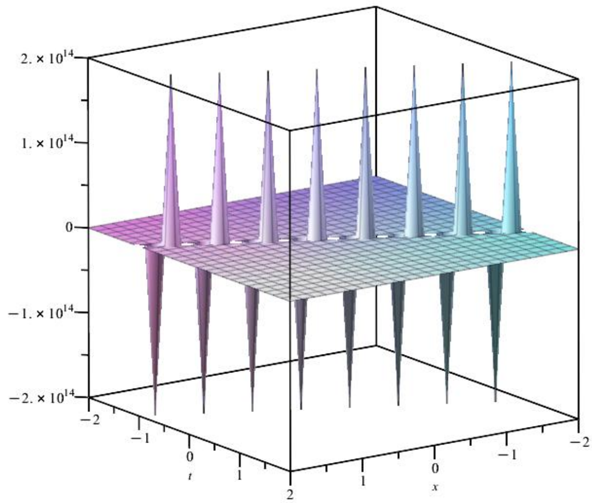


(b) 2D Plot

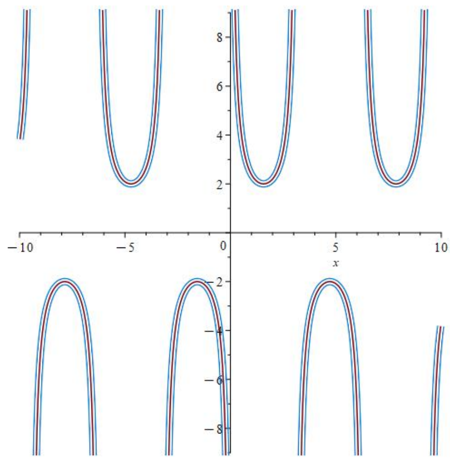


(c) Density Plot

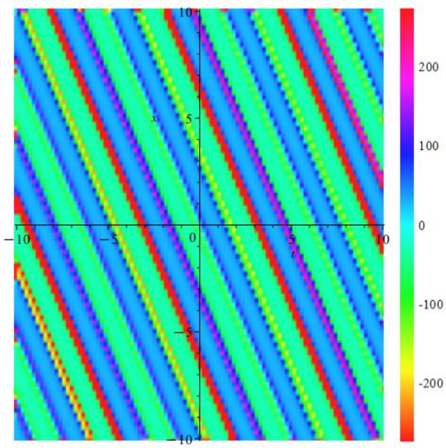
FIGURE 5. Graphical Presentation of (3.25)



(a) 3D Plot

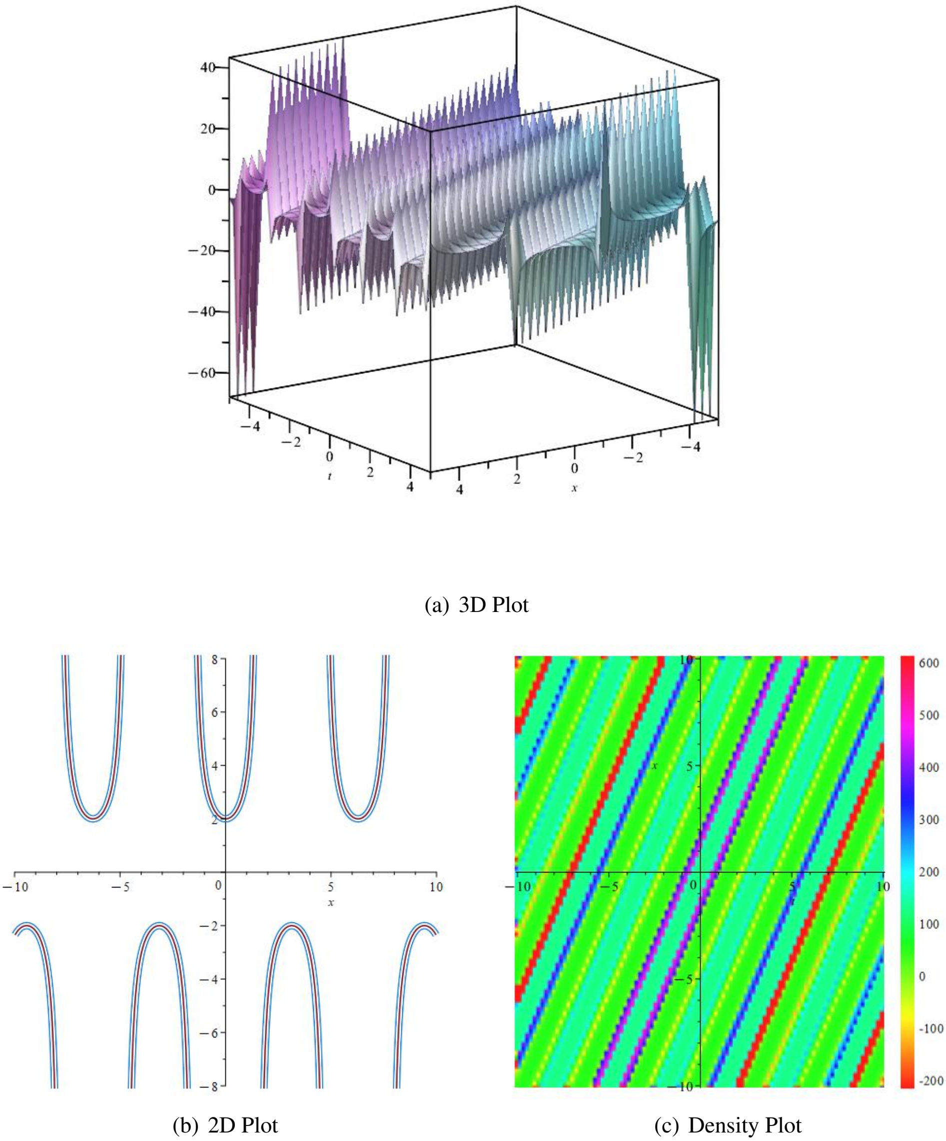


(b) 2D Plot



(c) Density Plot

FIGURE 6. Graphical Presentation of (3.26)

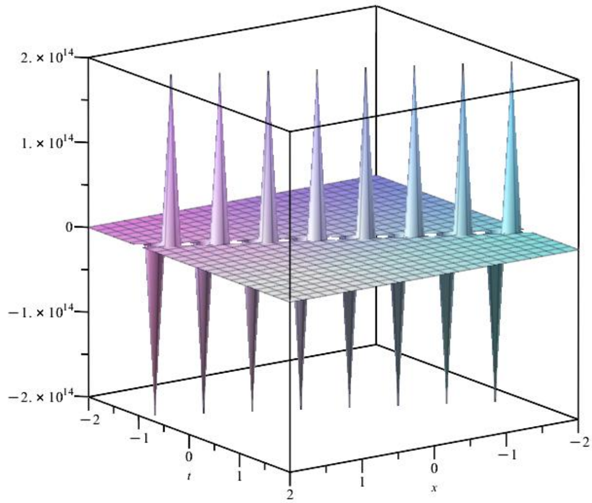


(a) 3D Plot

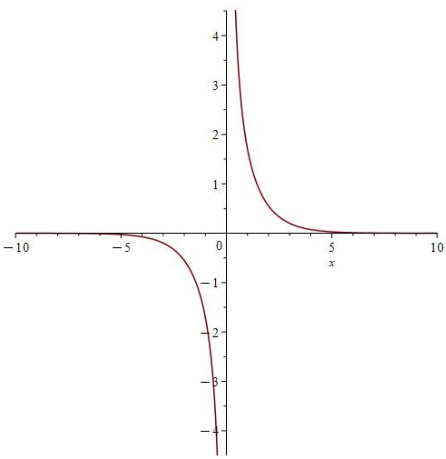
(b) 2D Plot

(c) Density Plot

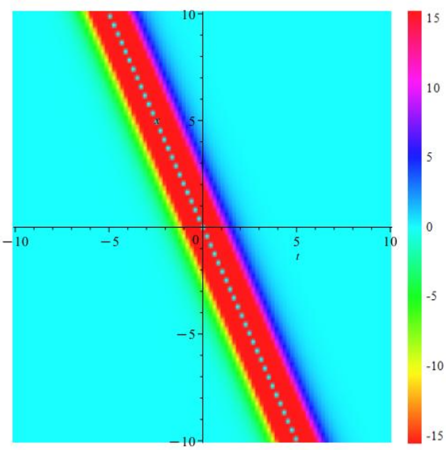
FIGURE 7. Graphical Presentation of (3.25)



(a) 3D Plot



(b) 2D Plot



(c) Density Plot

Figure 8. Graphical Presentation of (3.26)

In Figure 1, (a) represent graph of (3.8), solitary wave solution of the Sch*ä*fer-Wayne short pulse equation for $c = 1$ and $-7 \leq x, t \leq 7$. (b) gives 2D represention of (3.8) for $t = 0$ and $-8 \leq 8$, and (c) represent the density plot of (3.8). Equation

(3.12) gives singular solution for (3.1) which is represented in figure 2 for $c = 1$ and $-15 \leq x, t \leq 15$. Figure 3 represent multisoliton solution of KP equation given by (3.19) for $A = 4, y = 0$ and $-5 \leq x \leq 5, 0 \leq t \leq 1$. Figure 4 represnt traveling wave solution of KP equation given by (3.20) for $A = 4, y = 0$ and $-10 \leq x \leq 10, 0 \leq t \leq 10$. For the modified equal width equation we consider to cases $\alpha = -1$ and $\alpha = 1$ for these two consideration different set of solutions. For $\alpha = -1$ equation (3.25) gives bright soliton solution, where $A = 2, -5 \leq x, t \leq 5$ and (3.26) periodic solution for (3.21). Equation (3.25) represent solitary wave solution for $\alpha = 1, A = 2, -5 \leq x, t \leq 5$ where as (3.26) gives periodic solution for (3.21).

5. Conclusion

The Sinh-Cosh method has proven to be an efficient tool for finding exact traveling wave solutions of nonlinear evolution equations. Through its application to the Schäfer-Wayne Short Pulse Equation, the (2+1)-dimensional Kadomtsev-Petviashvili equation, and the Modified Equal Width equation, we obtained a range of soliton and periodic solutions. These results underline the method's versatility and effectiveness in dealing with nonlinear equations, offering significant insight into their soliton behaviors. Graphical presentations of the solutions further validate the approach. The findings suggest that the Sinh-Cosh method can be widely applied to solve other nonlinear partial differential equations in physics, engineering, and related fields.

Author contributions:
Conceptualisation: S. S. Handibag, R. M. Wayal ; *Software:* S. S. Handibag, R. M. Wayal ; *Writing-Original Draft:* S. S. Handibag, R. M. Wayal

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] R. Hirota, *Exact envelope soliton solutions of a nonlinear wave equation*, J. Math. Phys. **14** (1973) 805-810.

[2] M. Malfliet, *Solitary wave solutions of nonlinear wave equations*, A. J. Phys.,**60** (1992) 650-654.

[3] M. Akbar, N. Ali, E. Zayed,*Abundant exact traveling wave solutions of the generalized Bretherton equation via* (G'/G)*-expansion method*, Commun. Theor. Phys., **57** (2012) 173-178.

[4] Y. Zhou, M. Wang, Y. Wang,*Periodic wave solutions to coupled KdV equations with variable coefficients*, Phys. Lett. A., **308** (2003) 31-36.

[5] A. Wazwaz, *A Sine-Cosine Method for Handling Nonlinear Wave Equations*, Math. Comp. mode., **40** (2004) 499-508.

[6] A. Taqi, M. Shalla, B. Jomaa, *Traveling wave solution for some partial differential equations*, AIP Conference Proceedings., **2096,020015** (2019).

[7] T. Schäfer, C. E. Wayne, *Propagation of ultra-short optical pulses in nonlinear media*,Physica D: Nonlinear phenomena, **196**, No. 1-2(2004)90-105.

[8] V. Kuetche, T. Bouetou, and T. Kofane, *On Two-Loop Soliton Solution of the Schäfer–Wayne Short-Pulse Equation Using Hirota's Method and Hodnett–Moloney Approach*, J. of the phy. Soci. Of Japan, **76**, No.2 (2007) 024004.

[9] A. Sakovich and S. Sakovich, *Solitary wave solutions of the short pulse equation*, J. Phys. A:Math. Gen. **39** (2006) 361-367.

[10] O. Gaxiola,*Bright and dark optical solitons of the Schäfer–Wayne short-pulse equation by Laplace substitution method*, Optik-Int. J. for light and electron optics, **200** (2020)163414.

[11] K. Khan and M. Akbar,*Exact traveling wave solutions of Kadomtsev–Petviashvili equation*, J. Egypt. Math. Soc., **23**, No.2(2015) 278–281.

[12] W-X Ma,*Lump solutions to the Kadomtsev–Petviashvili equation*, Phy. Lett. A, **379**, No. 36 (2015) 1975-1978.

[13] P. Zhao, E. Fan, L. Luo, *Quasiperiodic solutions of the Kadomtsev-Petviashvili equation via the multidimensional Baker-Akhiezer function generated by the Broer-Kaup hierarchy*, J. Math. Anal. Appl., **435**, No. 1 (2016) 38-60.

[14] M. Kumar, A. Tiwari, R. Kumar, *Some more solutions of Kadomtsev–Petviashvili equation*, Comp. Math. Appl.,**74**, No.10 (2017) 2599-2607.

[15] C. Klein, K. Roidot, *Numerical study of shock formation in the dispersion less Kadomtsev-Petviashvili equation and dispersive regularizations*, Physica D, **265** (2013) 1-25.

[16] P.J. Morrison, J.D. Meiss, J.R. Carey, *Scattering of RLW solitary waves*, Physica D, **11**, No.3 (1984) 324–336.

[17] B. Saka, *Algorithms for numerical solution of the modified equal width wave equation using collocation method*, Math. Comp. Modelling, **45** (2007) 1096–1117.

[18] S Karakoç and T. Geyikli,*Numerical solution of the modified equal width wave equation*, Int. J. diff. eq., **2012**, article id 587208.

[19] D. Lu, A. Seadawy, A. Ali, *Dispersive traveling wave solutions of the Equal-Width and Modified Equal-Width equations via mathematical methods and its applications*, Results in Physics, **9** (2018) 313-320.

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