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ON LORENTZIAN PARA-KENMOTSU MANIFOLDS WITHIN THE FRAMEWORK OF ZAMKOVOY CANONICAL PARACONTACT CONNECTION

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Abstract

In the present study, we investigated the study of the Zamkovoy canonical para contact connection with respect to the Lorentzian para-Kenmotsu manifold. The purposed scheme deals with Zamkovoy canonical paracontact connection ∇^z on LP-Kenmotsu manifold and obtain relationship between ∇^z and Levi-Civita connection ∇. It is proven that the notion of φ- symmetric and concircular φ- symmetric with respect to ∇^z and ∇ are equivalent and some other prepositions and corollaries are stated. It is shown that the manifold is Ricci symmetric and hence manifold is Einstein with respect to locally φ-Ricci symmetric manifold concerning connection∇^z. Eventually we gave an example of 3-dimensional LP-Kenmotsu manifold accepting connection∇^z for verifying our results.

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1. Introduction

In recent years, several authors initiated the study of paracontact geometry due to its unexpectedrelation with contact geometry that is most activated now days. As a result, Kaneyuki, S. and Williams, F. L. [21] proposed the paracontact manifold as a suitable alternative to the well-known contact metricmanifold Zamkovoy, S. [24] carried out a thorough analysis of paracontact metric manifolds and theirsubclasses, emphasizing on the similarities and contrasts in the context of the contact case. Next, Kenmotsu, K. [18] studied a contact Riemannian manifold which satisfies a special type of condition, characterized different geometric properties of the manifolds of class (3), the obtained structure called a Kenmotsu structure. In 1972, again Kenmotsu, K. gave the notion of Kenmotsu manifolds [18]. Recently Dileo G. [13] and Pastore, A. M. [12] studied almost Kenmotsu manifolds satisfying η-parallelism and locally symmetry respectively.

In a similar way Matsumoto, K. [16] introduced the notion of paracontact and in particular, Lorentzian para-Sasakian manifolds. Later these manifolds have been widely studied by many geometers such as Matsumoto and Mihai [17] and Rosca

[15], Mihai, Shaikh and De [14], Venkatesha and Bagewadi [30], Venktesha, Pradeep Kumar and Bagewadi [31, 32] and obtained several interesting results for same kind of manifolds.

In the framework of Sasakian geometry, Biswas and Baishya [4, 5] introduced and analysed a unique connection, known as the general connection and described as follows:

$$\nabla^G_X Y = \nabla_X Y + k_1[(\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi] + k_2\eta(X)\varphi Y, \tag{1.1}$$

for all $X, \Upsilon \in \chi(M)$ and the pair (λ, μ) being real constants. The beauty of such connection ∇^G lies in the fact that it has the flavour of:

(i) Quarter symmetric metric connection [7, 8] for $(k_1, K_2) = (0, -1)$;

(ii) Schouten-Van Kampen connection [9] for $(k_1, K_2) = (1, 0)$;

(iii) Tanaka Webster connection [10] for $(k_1, K_2) = (1, -1)$; and

(iv) Zamkovoy connection [24] for$(k_1, K_2) = (1, 1)$.

In the framework of Kenmotsu geometry, A. M. Blaga [7] examined various canonical linear connections (Levi civita, Schouten-van Kampen, Golab, and Zamkovoy connections) with a specific view φ- conjuction also in [9]. We are now exploring the Zamkovoy canonical paracontact connection on Lorentzian para-Kenmotsu manifolds as a continuation of this work. This connection on a para-Kenmotsu manifold was adapted and studied by rigorously by S. Zamkovoy [24]. In paracontact geometry, this connection replaces the generalised Tanaka Webster connection [20]. Throughout the paper we denote Zamkovoy canonical connection as ∇^Z .

On the other hand, T. Takahashi [25] introduced the notion of local φ-symmetry of a sasakian manifolds. Since then, a number of authors have investigated this idea in relation to various structures [1, 2, 8, 10, 11, 26, 28].

Definition 1.1. A Lorentzian para-Kenmotsu manifold M is said to be locally φ-symmetric if its curvature tensor satisfies the condition [29]

$$\varphi^2((\nabla_w R)(X, \Upsilon)U) = 0, \tag{1.2}$$

for all X, Υ, U, W orthogonal to ξ.

Recently, De, U. C. and Sarkar, A. [27] introduce the notion of locally φ-Ricci symmetry on Sasakian manifolds. Further, this notion was studied by S. Ghosh and U. C. De [20] in the context of (k, μ)- contact metric manifolds with interesting results.

Definition 1.2. A Lorentzian para-Kenmotsu manifold M is said to be locally φ-Ricci symmetric if the Ricci operator satisfy the condition [29]

$$\varphi^2((\nabla_w Q)\Upsilon) = 0, \tag{1.3}$$

for all X, Υ on M such that $S(X, \Upsilon) = g(QX, \Upsilon)$. The concircular curvature tensor C on a LP-Kenmotsu manifold is defined as:

$$C(X, \Upsilon)U = R(X, \Upsilon)U - \frac{r}{n(n-1)}[S(\Upsilon, U)X - S(X, U)\Upsilon]. \tag{1.4}$$

The current study is concerned with a Lorentzian para-Kenmotsu manifold accepting the Zamkovoy canonical paracontact connection ∇^Z.

2. Preliminaries

Let M be a n-dimensional Lorentzian metric manifold, with a structure tensor (φ, ξ, η, g), where φ is a (1,1) tensor field, ξ is a Vector field, η is a 1 form on M and g is a Lorentzian metric, it is well known that the following condition holds:

$$\varphi^2(X) = X + \eta(X)\xi, \quad \eta(\xi) = -1, \tag{2.1}$$

$$\varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad rank(\varphi) = n - 1, \tag{2.2}$$

$$g(\varphi X, \varphi \Upsilon) = g(X, \Upsilon) + \eta(X)\eta(\Upsilon), \tag{2.3}$$

for all X, Υ on M, then it is said to be Lorentzian almost para-contact manifold. If ξ is a killing vector field, the para contact structure is called K-para contact.

From (2.3) it is obivious that

$$g(X, \varphi \Upsilon) = g(\varphi X, \Upsilon) \quad and \quad g(X, \xi) = \eta(X), \tag{2.4}$$

for all X, Υ on M.

Definition 2.1. A Lorentzian almost paracontact manifold M will be Lorentzian para-Kenmotsu manifold if

$$g(X, \varphi \Upsilon) = g(\varphi X, \Upsilon) \quad and \quad g(X, \xi) = \eta(X), \tag{2.5}$$

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.6}$$

$$(\nabla_X \eta)\Upsilon = -g(X, \Upsilon) - \eta(X)\eta(\Upsilon), \tag{2.7}$$

for all vector fields $X, \Upsilon \in M$ and ∇ denotes covariant differentiation with respect to the Lorentzian metric g.

Agreement 2.1. In Lorentzian para-Kenmotsu manifold M, the following relations hold [29]:

$$R(X, \Upsilon)\xi = \eta(\Upsilon)X - \eta(X)\Upsilon, \tag{2.8}$$

$$g(R(X, \Upsilon)Z, \xi) = \eta(R(X, \Upsilon)Z) = g(\Upsilon, Z)\eta(X) - g(X, Z)\eta(\Upsilon), \tag{2.9}$$

$$R(\xi, X)\Upsilon = g(X, \Upsilon)\xi - \eta(\Upsilon)X, \tag{2.10}$$

$$S(X, \xi) = (n - 1)\eta(X), \tag{2.11}$$

$$S(\varphi X, \Upsilon) = S(X, \Upsilon) + (n - 1)\eta(X)\eta(\Upsilon) \tag{2.12}$$

$$S(\xi, \xi) = -(n - 1), \tag{2.13}$$

$$Q\xi = (n - 1)\xi, \tag{2.14}$$

for all vector fields $X, \Upsilon, Z \in M. R(X, \Upsilon)Z$ and S are the Riemannian curvature tensor and Ricci tensor respectively.

Definition 2.2. A Lorentzian para-Kenmotsu manifold will be an η-Einstein manifold if its Ricci tensor S takes the following form

$$S(X, \Upsilon) = ag(X, \Upsilon) + b\eta(X)\eta(\Upsilon), \tag{2.15}$$

a, b are scalar function on M.

3. Lorentzian para-Kenmotsu manifold admitting the Zamkovoy canonical paracontact connection

We consider a n-dimensional Lorentzian para-Kenmotsu manifold admitting a connection ∇^Z using Levi Civita connection ∇. Now using (1.1) and case (iv), we have Zamkovoy canonical paracontact connection as:

$$\nabla^Z_X \Upsilon = \nabla_X \Upsilon + (\nabla_X \eta)(\Upsilon)\xi - \eta(\Upsilon)\nabla_X \xi + \eta(X)\varphi\Upsilon. \tag{3.1}$$

By using (2.6) and (2.7) in (3.1),we obtain

$$\nabla^Z_X \Upsilon = \nabla_X \Upsilon - g(X, \Upsilon)\xi - \eta(\Upsilon)X + \eta(X)\varphi\Upsilon, \tag{3.2}$$

for all vector fields $X, \Upsilon \in \chi(M)$.

The curvature tensor R_{∇^Z} , with respect to the connection ∇^Z is defined as

$$R_{\nabla^Z}(X, \Upsilon)U = \nabla^Z_X \nabla^Z_\Upsilon U - \nabla^Z_\Upsilon \nabla^Z_X U - \nabla^Z_{[X,\Upsilon]} U. \tag{3.3}$$

Similarly we obtain in Lorentzian para-Kenmotsu manifold

$$R_{\nabla^Z}(X, \Upsilon)U = R(X, \Upsilon)U - g(\Upsilon, U)X + g(X, U)\Upsilon, \tag{3.4}$$

where $R(X, \Upsilon)U = \nabla_X \nabla_\Upsilon - \nabla_\Upsilon \nabla_X U - \nabla_{[X,\Upsilon]} U$, is the curvature tensor of M with respect to the connection ∇ and (3.4) is the curvature tensor of Lorentzian para-Kenmotsu manifold with respect to the connection ∇^Z.

$$S_{\nabla^Z}(\Upsilon, U) = S(\Upsilon, U) + (n-1)g(\Upsilon, U). \tag{3.5}$$

Now contracting (3.4), we obtain Ricci tensor S_{∇^Z} of a Lorentzian para-Kenmotsu manifold with respect to the Zamkovoy canonical connection ∇^Z, this gives

$$Q_{\nabla^Z}\Upsilon = Q\Upsilon + (n-1)\Upsilon. \tag{3.6}$$

Then contracting (3.4) over Υ and U we obtain

$$r_{\nabla^Z} = r + n(n-1), \tag{3.7}$$

where S and r denote the Ricci tensor ans scalar curvature of Levi civita connection ∇ respectively .

Agreement 3.1. For a Lorentzian para-Kenmotsu manifold M with respect to the Zamkovoy canonical paracontact connection ∇^Z :

(i) R_{∇^Z} is skew-symmetric in first two slots i.e. $R'_{\nabla^Z}(X, \Upsilon, U, V) = -R'_{\nabla^Z}(\Upsilon, X, U, V)$.
(ii) R_{∇^Z} is skew-symmetric in last two slots i.e. $R'_{\nabla^Z}(X, \Upsilon, U, V) = -R'_{\nabla^Z}(\Upsilon, X, V, U)$.
(iii) R_{∇^Z} is symmetric in pair of slots i.e. $R'_{\nabla^Z}(U, V, X, \Upsilon) = R'_{\nabla^Z}(X, \Upsilon, U, V)$.
(iv) $R_{\nabla^Z}(X, \xi)U = R_{\nabla^Z}(\xi, \Upsilon)U = R_{\nabla^Z}(X, \Upsilon)\xi$.
(v) $S_{\nabla^Z}(\Upsilon, \xi) = 2(n-1)\eta(\Upsilon)$.
(vi) The Ricci tensor S_{∇^Z} is symmetric.

Preposition 3.1. A Lorentzian para-Kenmotsu manifold accepting connection ∇^Z is Ricci flat, iff it is an Einstein manifold of the form $S(\Upsilon, U) = -(n-1)g(\Upsilon, U)$.

Proof. In a Lorentzian para-Kenmotsu manifold Ricci tensor is defined by (3.5), next suppose that a Lorentzian para-Kenmotsu manifold is Ricci flat i.e. $S_{\nabla^Z} = 0$, then from (3.5) we get

$$S(\Upsilon, U) = -(n-1)g(\Upsilon, U). \tag{3.8}$$

Conversely, if the manifold is Einstein manifold of the form $S(\Upsilon, U) = -(n-1)g(\Upsilon, U)$, then from (3.5) it follows that $S_{\nabla^Z} = 0$. □

Preposition 3.2. In a Lorentzian para-Kenmotsu, if the curvature tensor of the Zamkovoy canonical paracontact connection vanishes, then the sectional curvature of the plane determined by two vectors $X, \Upsilon \in \xi^\perp$ is positive (i.e. 1).

Proof. Let $\xi^\perp$ denote the $(n-1)$-dimensional distribution orthogonal to ξ in a LP-Kenmotsu manifold with respect to Zamkovoy canonical paracontact connection, whose curvature tensor vanishes. Then for any $X \in \xi^\perp$, $g(X, \xi) = 0$ or $\eta(X) = 0$.

Now we shall determine the sectional curvature $'R$ of the plane determine by the vector $X, \Upsilon \in \xi^\perp$. Taking inner product on both sides of (3.4) with X and then for $U = \Upsilon$, we have

$$R_{\nabla^Z}(X, \Upsilon, U, X) = R(X, \Upsilon, \Upsilon, X) - g(\Upsilon, \Upsilon)g(X, X) + g(X, \Upsilon)g(X, \Upsilon). \tag{3.9}$$

Putting $R_{\nabla^Z} = 0$, in (3.9) we have

$$'R(X, \Upsilon) = \frac{R(X, \Upsilon, \Upsilon, X)}{g(X, \Upsilon)^2 - g(X, X)g(\Upsilon, \Upsilon)} = 1. \tag{3.10}$$

This proves the required result. □

4. Locally φ -symmetry and locally concircular φ -symmetry admitting connection ∇^Z and ∇

Definition 4.1. A Lorentzian para-Kenmotsu manifold is said to be locally φ-symmetry concerning Zamkovoy canonical paracontact connection ∇^Z, if its curvature tensor R_{∇^Z} satisfies the condition as [9]:

$$\varphi^2((\nabla^Z_W R_{\nabla^Z})(X, \Upsilon)U) = 0, \tag{4.1}$$

for any vector fields X, Υ, U, W orthogonal to ξ.

Let us take positively; Lorentzian para-Kenmotsu manifold M is locally φ- symmetric for the Zamkovoy canonical paracontact connection ∇_Z . Then in view of (3.2), (4.1) have the form

$$\begin{aligned}(\nabla^Z_W R_{\nabla^Z})(X, \Upsilon)U &= (\nabla_W R_{\nabla^Z})(X, \Upsilon)U - g(W, R_{\nabla^Z}(X, \Upsilon)U)\xi \\ &+\eta(R_{\nabla^Z}(X, \Upsilon)U)W + \eta(W)\varphi R_{\nabla^Z}(X, \Upsilon)U.\end{aligned} \tag{4.2}$$

Then (4.2) becomes as a result of $\eta(R_{\nabla^Z})(X,\Upsilon)U)=0$.

$$(\nabla^Z_W R_{\nabla^Z})(X,\Upsilon)U = (\nabla_W R_{\nabla^Z})(X,\Upsilon)U - g(W, R_{\nabla^Z}(X,\Upsilon)U)\xi \\ +\eta(W)\varphi R_{\nabla^Z}(X,\Upsilon)U. \tag{4.3}$$

Now covariant differentiation of (3.4) over W, we get

$$(\nabla_W R_{\nabla^Z})(X,\Upsilon)U = (\nabla_W R)(X,\Upsilon)U. \tag{4.4}$$

Using (4.4) in (4.2), then we have

$$(\nabla^Z_W R_{\nabla^Z})(X,\Upsilon)U = (\nabla_W R)(X,\Upsilon)U - [R'(X,\Upsilon,U,W) - g(\Upsilon,U)g(X,W) \\ +g(X,U)g(\Upsilon,W)]\xi + \eta(W)[\varphi R(X,\Upsilon)U - g(\Upsilon,U)\varphi X + g(X,U)\varphi\Upsilon]. \tag{4.5}$$

Defining φ^2 on both side of above equation, then using (2.1) and (2.2) we obtain

$$\varphi^2(\nabla^Z_W R_{\nabla^Z})(X,\Upsilon)U = \varphi^2(\nabla_W R)(X,\Upsilon)U \\ +\eta(W)[\varphi R(X,\Upsilon)U - g(\Upsilon,U)\varphi X + g(X,U)\varphi\Upsilon]. \tag{4.6}$$

If we consider φ^2 orthogonal to ξ gives expression as

$$\varphi^2(\nabla^Z_W R_{\nabla^Z})(X,\Upsilon)U = \varphi^2(\nabla_W R)(X,\Upsilon)U. \tag{4.7}$$

Then we have concluded following as:

Theorem 4.1. *A Lorentzian para-Kenmotsu manifold is locally φ- symmetric with regard to the Zamkovoy canonical paracontact connection ∇^Z, iff it is so with respect to the Levi civita-Connection ∇.*

Corollary 4.2. *A Lorentzian para-Kenmotsu manifold is locally φ-symmetric iff its scalar curvature is constant (positive i.e.1) [29] and in view of (3.10) it is so for Zamkovoy canonical paracontact connection.*

Also in view of (4.1) it is obvious that:

Corollary 4.3. *A Lorentzian para-Kenmotsu manifold satisfies the condition $R(X,\Upsilon).S = 0$ with respect to ∇ , then the manifold is locally φ-symmetric and so with respect to Zamkovoy canonical paracontact connection.*

Definition 4.2. In a n-dimensional Lorentzian para-Kenmotsu manifold, the concircular curvature tensor C_{∇^Z} in fever of Zamkovoy canonical paracontact connection is defined as

$$C_{\nabla^Z}(X,\Upsilon)U = R_{\nabla^Z}(X,\Upsilon)U - \frac{r_{\nabla^Z}}{n(n-1)}[g(X,U)\Upsilon - g(\Upsilon,U)X], \tag{4.8}$$

where R_{∇^Z} and r_{∇^Z} are the Riemannian curvature tensor and scalar curvature with respect to the connection ∇^Z respectively.

Taking in account (3.4) and (3.7), (4.8) gives

$$C_{\nabla^Z}(X,\Upsilon)U = C(X,\Upsilon)U, \tag{4.9}$$

where $C(X,\Upsilon)U$ is concircular curvature tensor defined by (1.4) . Then from (1.4) it is clear that concircular curvature tensor with respect to ∇^Z and ∇ is same.

Definition 4.3. A Lorentzian para-Kenmotsu manifold accepting Zamkovoy canonical paracontact connection ∇^Z , is said to be locally concircular φ- symmetric if its concircular curvature tensor C_{∇^Z} satisfies the condition as [9]:

$$\varphi^2((\nabla^Z_W C_{\nabla^Z}(X,\Upsilon)U) = 0, \tag{4.10}$$

for any vector fields X, Υ, U, W orthogonnal to ξ.

Let us suppose that a Lorentzian para-Kenmotsu manifold M is locally concircular φ- symmetric concerning connection ∇^Z. Then using (3.2), (4.10) simplifies as

$$\begin{aligned}(\nabla^Z_W C_{\nabla^Z})(X,\Upsilon)U) &= (\nabla_W C_{\nabla^Z})(X,\Upsilon)U - g(W, C_{\nabla^Z}(X,\Upsilon)U)\xi \\ &+\eta(C_{\nabla^Z}(X,\Upsilon)U)W + \eta(W)\varphi C_{\nabla^Z}(X,\Upsilon)U.\end{aligned} \tag{4.11}$$

Now taking covariant differentiation of (4.9) over W, yields

$$(\nabla_W C_{\nabla^Z})(X,\Upsilon)U = (\nabla_W C)(X,\Upsilon)U. \tag{4.12}$$

Making use of (4.9) and (4.12) in (4.11), we obtain

$$\begin{aligned}(\nabla^Z_W C_{\nabla^Z})(X,\Upsilon)U) &= (\nabla_W C)(X,\Upsilon)U - g(W, C(X,\Upsilon)U)\xi \\ &+\eta(C(X,\Upsilon)U)W + \eta(W)(\varphi C)(X,\Upsilon)U.\end{aligned} \tag{4.13}$$

Using (1.4), in the above expression, we have

$$\begin{aligned}(\nabla^Z_W C_{\nabla^Z})(X,\Upsilon)U) &= (\nabla_W C)(X,\Upsilon)U - R'(X,\Upsilon,U,W)\xi + \eta(W)\varphi R(X,\varphi)U \\ &+\frac{r}{n(n-1)}[g(\varphi,U)(g(X,W)\xi - \eta(W)\varphi X) - g(X,U)(g(\Upsilon,W))]\xi - \eta(W)\varphi\Upsilon] \\ &+\left[1 - \frac{r}{n(n-1)}\right][g(\Upsilon,U)\eta(X)W - g(X,U)\eta(\Upsilon)W].\end{aligned} \tag{4.14}$$

Applying φ on both sides of above equation, then using (2.1) and (2.2) in (4.4) we have

$$\begin{aligned}\varphi^2(\nabla^Z_W C_{\nabla^Z})(X,\Upsilon)U &= \varphi^2(\nabla_W C)(X,\Upsilon)U + \eta(W)\varphi R(X,\Upsilon)U \\ &+\frac{r}{n(n-1)}[g(X,U)\eta(W)\varphi\Upsilon - g(\Upsilon,U)\eta(W)\varphi X] \\ &+\left[1 - \frac{r}{n(n-1)}\right][g(\Upsilon,U)\eta(X) - g(X,U)\eta(\Upsilon)][W + \eta(W)\xi].\end{aligned} \tag{4.15}$$

If we consider X, Υ, U, W orthogonal to ξ , above equation reduces to

$$\varphi^2(\nabla^Z_W C_{\nabla^Z})(X, \Upsilon)U = \varphi^2(\nabla_W C)(X, \Upsilon)U. \tag{4.16}$$

Then we can concluded that:

Theorem 4.4. *A Lorentzian para-Kenmotsu manifold accepting Zamkovoy canonical paracontact connection ∇^Z is locally concircular φ- symmetric, iff it is so for the Levi-Civita connection ∇.*

Preposition 4.1. Let us take an n-dimensional locally concircular φ- symmetric Lorentzian para-Kenmotsu manifold M admitting Zamkovoy canonical paracontact connection ∇^Z, and if the scalar curvature r with respect to Levi-Civita connection ∇ is constant, then M is locally φ- symmetric.

Proof. From (1.4), we have

$$(\nabla_W C)(X, \Upsilon)U = (\nabla_W R_\nabla)(X, \Upsilon)U - \frac{\nabla_W r}{n(n-1)}[g(\Upsilon, U)X - g(X, U)\Upsilon]. \tag{4.17}$$

Take (4.17) in (4.16), we obtain

$$\begin{aligned}\varphi^2((\nabla_W C)(X, \Upsilon)U) &= \varphi^2((\nabla_W R_\nabla)(X, \Upsilon)U)\\ &-\frac{\nabla_W r}{n(n-1)}[g(\Upsilon, U)\varphi^2 X - g(X, U)\varphi^2\Upsilon].\end{aligned} \tag{4.18}$$

Considering (2.1) in (4.18), and for X, Υ, U, W orthogonal to ξ , we get

$$\begin{aligned}\varphi^2((\nabla_W C)(X, \Upsilon)U) &= \varphi^2((\nabla_W R_\nabla)(X, \Upsilon)U)\\ &-\frac{\nabla_W r}{n(n-1)}[g(\Upsilon, U)X - g(X, U)\Upsilon].\end{aligned} \tag{4.19}$$

If r is constant, then $(\nabla_W r) = 0$, therefore we have

$$\varphi^2((\nabla_W C)(X, \Upsilon)U) = \varphi^2((\nabla_W R_\nabla)(X, \Upsilon)U). \tag{4.20}$$

It completes the proof. □

5. Locally φ- Ricci symmetric Lorentzian para-Kenmotsu manifold concerning the connections ∇^Z and ∇

Definition 5.1. A Lorentzian para-Kenmotsu manifold will be locally φ-Ricci symmetric for Zamkovoy canonical paracontact connection ∇^Z, if its Ricci operator Q_{∇^Z} satisfies the given condition as [9]:

$$\varphi^2((\nabla^Z_W Q_{\nabla^Z})X) = 0. \tag{5.1}$$

We assume that the LP-Kenmotsu manifold is locally φ- Ricci symmetric with respect to Zamkovoy canonical paracontact connection ∇^Z. Then in view of (2.1) it is obivious that (5.1) takes the form

$$(\nabla^Z_W Q_{\nabla^Z})X + \eta((\nabla^Z_W Q_{\nabla^Z})X)\xi = 0. \tag{5.2}$$

Now from (3.5), we have

$$(\nabla^Z_W Q_{\nabla^Z})X = \nabla^Z_W Q_{\nabla^Z} X - Q_{\nabla^Z}(\nabla^Z_W X). \tag{5.3}$$

Using (3.6) in (5.3), we get

$$(\nabla^Z_W Q_{\nabla^Z})X = (\nabla^Z_W Q)X). \tag{5.4}$$

Taking account of (5.4), (5.3) reduces to

$$(\nabla^Z_W Q_{\nabla^Z})X + \eta((\nabla^Z_W Q)X)\xi = 0. \tag{5.5}$$

From (3.2) it follows that

$$\begin{aligned}(\nabla^Z_W Q)(X) &= \nabla^Z_W QX - Q(\nabla^Z_W X)\\ &= (\nabla_W Q)X - S(W,X)\xi + (n-1)(\eta(X)W + [g(W,X)\xi)\\ &\quad +\eta(X)QW + \eta(W)(\varphi QX - Q\varphi X),\end{aligned} \tag{5.6}$$

and

$$\eta((\nabla^Z_W Q)X) = \eta((\nabla_W Q)X) - S(W,X)\xi + (n-1)g(W,X)\xi. \tag{5.7}$$

Using (5.6) and (5.7) we obtain from (5.5) that

$$(\nabla_W Q)X + (n-1)\eta(X)W + \eta(W)(\varphi QX - Q\varphi X) + \eta(X)QW - \eta((\nabla_W Q)X)\xi = 0. \tag{5.8}$$

In (5.8), take inner product over U, with in mind that X, W, U orthogonal to ξ, we get

$$(\nabla_W S)(X,U) = 0. \tag{5.9}$$

This implies that the manifold is Ricci symmetric in view of Levi-civita connection, hence we have a result:

Theorem 5.1. *A locally φ-Ricci symmetric Lorentzian para-Kenmotsu manifold with respect to Zamkovoy canonical paracontact connection ∇^Z, is Ricci symmetric manifold.*

In view of [30], we can state a corollary as:

Corollary 5.2. *Any φ-Ricci symmetric Lorentzian para-Kenmotsu manifold with respect to Levi-Civita connection is an Einstein manifold and this is also apparent when considering the Zamkovoy canonical paracontact connection ∇^Z .*

Corollary 5.3. *Any Lorentzian para-Kenmotsu manifold whose curvature tensor is covariant constant for the Zamkovoy canonical paracontact connection ∇^Z and recurrent for Levi-Civita connection ∇, is an η-Einstein manifold [9].*

6. Example

We consider the 3-dimensional manifold

$$M^3 = \{(x, y, z) \in R^3, z > 0\},$$

where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = \frac{\delta}{\delta x}, \quad e_2 = \frac{\delta}{\delta y}, \quad e_3 = x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y} + \frac{\delta}{\delta z},$$

are linearly independent at each point of M^3. The one form defined as $\eta(X) = g(X, \xi) = g(X, e_3)$. Let $g,\ \varphi$ be the Riemannian metric and (1, 1) tensor field defined as

$$g(e_1, e_1) = 1,\ g(e_2, e_2) = 1,\ g(e_3, e_3) = -1,\ g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0,$$

$$\varphi(e_1) = -e_2,\ \varphi(e_2) = -e_1,\ \varphi(e_3) = 0.$$

Let ∇ be the Levi-Civita Connection with respect to the Lorentzian metric g, then

$$[e_1, e_2] = 0,\ [e_2, e_1] = 0,\ [e_1, e_3] = -e_1,\ [e_3, e_1] = e_1, [e_2, e_3] = -e_2,\ [e_3, e_2] = e_2.$$

Now by using Koszul's formula, we have

$$\begin{gathered} \nabla_{e_1} e_1 = -e_3,\ \ \nabla_{e_1} e_2 = 0,\ \ \nabla_{e_1} e_3 = -e_1, \\ \nabla_{e_2} e_1 = 0,\ \ \nabla_{e_2} e_2 = -e_3,\ \ \nabla_{e_2} e_3 = -e_2, \\ \nabla_{e_3} e_1 = 0,\ \ \nabla_{e_3} e_2 = 0,\ \ \nabla_{e_3} e_3 = 0. \end{gathered}$$

Also we have

$$\begin{gathered} R(e_1, e_2)e_1 = -e_2,\ \ R(e_1, e_3)e_1 = -e_3,\ \ R(e_2, e_3)e_1 = 0, \\ R(e_1, e_2)e_2 = e_1,\ \ R(e_1, e_3)e_2 = 0,\ \ R(e_2, e_3)e_2 = -e_3, \\ R(e_1, e_2)e_3 = 0,\ \ R(e_1, e_3)e_3 = -e_1,\ \ R(e_2, e_3)e_3 = -e_2. \end{gathered}$$

It is obvious that the manifold is Lorentzian para-Kenmotsu manifold [6].

Using above equation we have scalar curvature

$$\begin{gathered} \nabla^Z_{e_1} e_1 = -e_3,\ \ \nabla^Z_{e_1} e_2 = 0,\ \ \nabla^Z_{e_1} e_3 = 0, \\ \nabla^Z_{e_2} e_1 = 0,\ \ \nabla^Z_{e_2} e_2 = 0,\ \ \nabla^Z_{e_2} e_3 = 0, \\ \nabla^Z_{e_3} e_1 = 0,\ \ \nabla^Z_{e_3} e_2 = 0,\ \ \nabla^Z_{e_3} e_3 = 0. \end{gathered}$$

Also from the above result we can compute the components of curvature tensor as

$$\begin{gathered} R_{\nabla^Z}(e_1, e_2)e_1 = 0,\ \ R_{\nabla^Z}(e_1, e_3)e_1 = 0,\ \ R_{\nabla^Z}(e_2, e_3)e_1 = 0, \\ R_{\nabla^Z}(e_1, e_2)e_2 = 0,\ \ R_{\nabla^Z}(e_1, e_3)e_2 = 0,\ \ R_{\nabla^Z}(e_2, e_3)e_2 = 0, \\ R_{\nabla^Z}(e_1, e_2)e_3 = 0,\ \ R_{\nabla^Z}(e_1, e_3)e_3 = 0,\ \ R_{\nabla^Z}(e_2, e_3)e_3 = 0. \end{gathered}$$

Using this we have $r_{\nabla^Z} = 6$.

7. Conclusion

In this study, we examine the Zamkovoy canonical para contact connection with respect to the Lorentzian para-Kenmotsu manifold. We obtained relationship between ∇^Z and Levi-Civita connection ∇ and deduce some geometric results. Also we explore our results by an example of 3-dimensional LP-Kenmotsu manifold accepting connection ∇^Z.

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