GANITA, Vol.74(2), 2024, 155-167

RESULTS ON PICARD'S SOLUTION DEPENDENCE ON THE INITIAL CONDITIONS OF FRACTIONAL ORDER SYSTEMS

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Abstract

Fractional differential equations find widespread application in engineering domains, including control engineering, electronic system development, electronic circuit design, and speech modeling. Due to the analytical intractability of many such equations, numerical methods have been developed to obtain solutions. This study explores the sensitivity of solutions obtained through Picard's method for a fractional order system represented as ${}^{C}_{x_0}D^{\alpha}_{x}y(x) = f(x, y)$, subject to the initial condition $y(x_0) = y_0$, in which derivative has been taken in caputo sense. The research investigates how slight variations in the initial condition and the function $f(x, y)$affect the solutions. This analysis provides valuable insights into the stability and robustness of solutions for fractional differential equations, enhancing their practical applicability in diverse engineering applications.

2010 *Mathematics subject classification:* primary 26A33; secondary 34A08.
Keywords and phrases: Caputo Fractional Order Differential Equations, Picard's Iterative Method, Initial Value Problem, Existence and Uniqueness Theorem.

1. Introduction

Fractional order differential equations have gained significant importance recently, owing to their versatile applications across various scientific and engineering domains. These equations find practical use in control theory, signal processing, electric circuits, and modeling viscoelastic materials. Additionally, they offer valuable insights in population modeling, particularly in situations like epidemics or wars, where traditional integer-order models fall short [1–11] . In various real-life scenarios, integer order differential equations fall short in accurately representing the complexities of the problems. In such cases, fractional order models emerge as valuable alternatives, offering improved results. However, similar to their integer counterparts, many fractional order differential equations lack analytic solutions, necessitating the use of numerical methods for finding solutions. Given the challenges and importance of fractional order differential equations, our focus has been on developing numerical techniques to tackle these complexities effectively. As a result, in recent times, a diverse range of

We sincerely appreciate the assistance and input provided by the Research and Development cell Sant Baba Bhag Singh University, Jalandhar, India

numerical methods has been employed to obtain numerical solutions for fractional order differential equations [12]. Some of these methods are iterative methods [13–15], the Adomian decomposition method [16], the Variatonal iteration method [17] and Homotomy perturbation method [18]. In 2017 Rainey et al. [15] developed Picard's iterative method for Caputo fractional differential equation

Polynomial interpolation has remained a ceaseless topic of research since decades due to its various numerical applications such as forming the basis for algorithms in numerical derivation and numerical quadrature.

$$\begin{cases} {}_{x_0}^{C}D_x^{\alpha} y(x) = f(x, y) \\ y(x_0) = y_0, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$ and ${}_{x_0}^{C}D_x^{\alpha} y(x)$ represents the Caputo Fractional Derivative of order α of the of the function $y(x)$.

In this paper, we explore the sensitivity of the solution to the initial value problem presented above concerning slight variations in the right-hand side function, denoted as $f(x, y)$, and the initial condition $y(x_0) = y_0$. Our analysis demonstrates that even small changes in these components can lead to corresponding slight modifications in the solution, subject to specific suitable conditions. To substantiate our findings, we illustrate the relationship between the solution and the initial condition, as well as the function $f(x, y)$, through graphical representations.

The structure of our research paper is as follows:section 2 provides essential preliminary definitions, laying the groundwork for our subsequent analyses. In Section 3, we present our main results, consisting of two theorems that elucidate the dependency of the solution for equation 1.1 on the initial condition and the function $f(x, y)$. Finally, in Section 4, we furnish examples that serve as supporting evidence for our main findings.

2. Preliminaries

Before presenting our main result, it is imperative to introduce some fundamental definitions that are essential for its proof. Fractional derivatives and fractional integrations have been extensively studied, and various definitions can be found in the literature [19–21]. In our current research, we focus on two specific definitions: the Riemann-Liouville and Caputo fractional derivatives and integrals.

The Riemann-Liouville fractional derivative is one of the earliest and most commonly used definitions, while the Caputo fractional derivative provides a convenient approach when initial conditions are involved. Both definitions play crucial roles in understanding and solving fractional order differential equations. In the subsequent sections, we will present these definitions in detail and demonstrate their significance in the context of our main result.

Definition 2.1. [22] The Gamma function is the generalization of factorial $n!$ and it allows n to take any real or even complex values. The Gamma function is defined by the formula

$$\Gamma(x) = \int_0^\infty e^{-t} t^{(x-1)} dt, x \in \mathbb{R}^+.$$

Definition 2.2. [22] The Beta function is defined by the definite integral

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, x, y \in \mathbb{R}^+.$$

Definition 2.3. [23] Let $f(x)$ be a piecewise continuous function on $(0, \infty)$, integrable on any finite sub-interval of $[0, \infty]$ and α be a non-negative real number. Then for $t > 0$, Riemann-Liouville fractional integral of $f(x)$ of order α can be defined by the formula

$${}_{x_0}^{RL}D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} f(t)\, dt, \alpha > 0.$$

In our present work, we used the following symbols ${}_{x_0}^{RL}I_x^{\alpha} f(x)$ for Riemann-Liouville integration of $f(x)$, that is ${}_{x_0}^{RL}I_x^{\alpha} f(x) = {}_{x_0}^{RL}D_x^{-\alpha} f(x)$.
A special case of Riemann-Liouville fractional integral is when $x_0 = 0$. Then the Riemann-Liouville operator becomes ${}_{0}^{RL}D_x^{-\alpha} y(x)$ and the formula becomes

$${}_{0}^{RL}D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} f(t)\, dt, \alpha > 0.$$

From the above equation, we can easily see that

$${}_{x_0}^{RL}I_x^{\alpha} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} x^{(\mu+\alpha)}, \alpha > 0, \mu > -1, x > 0.$$

Definition 2.4. [24] Let $f(t)$ be continuous on $[0, \infty]$ and $\mu, \nu > 0$. Then for all t,

$$D^{-\mu}\left[D^{-\nu} f(t)\right] = D^{-(\mu+\nu)} f(t) = D^{-\nu}\left[D^{-\mu} f(t)\right].$$

Above equation represents the law of exponents of Fractional Integrals.

Definition 2.5. **(Riemann-Liouville Fractional Derivatives)**[25] The fractional derivative of order, ν can be defined by using the fractional integration as

$${}_{x_0}^{RL}D_x^{\alpha} f(x) = D^n \left[{}_{x_0}^{RL}D_x^{-(n-\alpha)} f(x)\right], \quad \alpha \in R^+, n-1 < \alpha < n.$$

Definition 2.6. [25] The Caputo derivative ${}_{a}^{C}D_x^{\alpha} f(x)$, of order $\nu > 0$ for the real-valued function $f(x)$ is defined as

$${}_{x_0}^{C}D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \cdot \int_{x_0}^{x} f^n(t) \frac{1}{(x-t)^{\alpha+1-n}} dt, \quad n-1 < \alpha < n.$$

Definition 2.7. [26] The Mittag-Leffler function is the generalization of the exponential function, e^x and it plays a very important role in fractional calculus. The Mittag-Leffler function with one parameter and two parameters can be defined in terms of a power series as

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \alpha > 0,$$

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \alpha > 0, \beta > 0.$$

On the bases of the above definition, the following relations hold

$$E_{\alpha,1}(x) = E_\alpha(x)\, \alpha > 0,$$

$$E_{1,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

Theorem 2.8. [15] For initial value problem (1.1) let the function $f(x,y)$ is continuous and satisfy Lipschitz condition with respect to y on the closed rectangle $R = \{(x,y)|x \in [x_0 - a, x_0 + a], y \in [y_0 - b, y_0 + b]\}$ and let $|f(x,y)| < M$, where $M > 0$.Then (1.1) has a unique solution in the interval $[x_0, x_0 + h]$, where $h = \min\{a, \frac{b}{M}\Gamma(\alpha+1)^{\frac{1}{\alpha}}\}$.

3. Results

In this section, we have established the dependence of the solution for the initial value problem (1.1) on both the initial condition and the right-hand side function $f(x,y)$. Our results have been rigorously demonstrated through the examination of slight variations in both the initial condition and the right-hand side function, taking into account certain suitable conditions. The proofs provided solidify our findings and support the assertion of the solution's sensitivity to these changes.
Consider the initial value problem (1.1), then the integral form of (1.1) is written as

$$\begin{aligned} y(x) &= y_0 + {}_{x_0}I_x^\alpha f(t, y(t)) \\ &= y_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, y(t))\, dt. \end{aligned} \tag{3.1}$$

Theorem 3.1. Suppose $f(x,y)$ is a continuous function and satisfies Lipschitz condition w.r.t. variable y in a domain D of $xy-$ plane and $(x_0, y_0) \in D$ be a fixed point. Suppose that there exist $h > 0$ and $\delta > 0$ such that the initial value problem (1.1) has a unique solution $\phi(x, Y_0)$ on $|x - x_0| \leq h$ for each Y_0 satisfying $|Y_0 - y_0| \leq \delta$. If ϕ be the unique solution of (1.1) for $Y_0 = y_0$ and $\overline{\phi}$ is the unique solution of (1.1) for $Y_0 = \overline{y_0}$, where y_0 and $\overline{y_0}$ satisfy $|\overline{y_0} - y_0| = \delta_1 \leq \delta$, then $|\overline{\phi}(x) - \phi(x)| \leq \delta_1 E_{\alpha,1}(Kh^\alpha)$ on $|x - x_0| \leq h$.

Proof. Here $f(x,y)$ is a continuous function and satisfies Lipschitz condition w.r.t. y in the domain D. Therefore there exists a number $K > 0$ such that

$$|f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2|, \tag{3.2}$$

in D.

Now by Picard's Iterative method for fractional order differential equation [15], we have

$$\lim_{n\to\infty} \phi_n(x) = \phi(x), |x - x_0| \leq h,$$

where

$$\phi_n(x) = y_0 + {}_{x_0}I_x^{\alpha} f(t, \phi_{n-1}(t)), (n = 1.2, 3, \cdots),$$

and $\phi_0(x) = y_0$.

Similarly $\lim_{n\to\infty} \overline{\phi_n}(x) = \overline{\phi}(x)$, in $|x - x_0| \leq h$,

where

$$\overline{\phi_n}(x) = \overline{y}_0 + {}_{x_0}I_x^{\alpha} f\left(t, \overline{\phi}_{n-1}(t)\right), (n = 1.2, 3, \cdots),$$

and $\overline{\phi}_0(x) = \overline{y}_0$.

Let x lies in $[x_0, x_0 + h]$.

Now

$$\begin{aligned}
|\overline{\phi}_1(x) - \phi_1(x)| &= |\overline{y}_0 + {}_{x_0}I_x^{\alpha} f(t, \overline{y}_0) - y_0 - {}_{x_0}I_x^{\alpha} f(t, y_0)| \\
&= |\overline{y}_0 - y_0| + |{}_{x_0}I_x^{\alpha}[(t, \overline{y}_0) - f(t, y_0)]| \\
&\leq \delta_1 + K {}_{x_0}I_x^{\alpha}|\overline{y}_0 - y_0| \\
&\leq \delta_1 + K\delta_1 \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, dt \\
&= \delta_1 + K\delta_1 \frac{(x-x_0)^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned}$$

$$|\overline{\phi}_1(x) - \phi_1(x)| \leq \delta_1 + K\delta_1 \frac{(x-x_0)^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.3}$$

Now

$$\begin{aligned}
|\overline{\phi}_2(x) - \phi_2(x)| &= |\overline{y}_0 + {}_{x_0}I_x^{\alpha} f\left(t, \overline{\phi}_1\right) - y_0 - {}_{x_0}I_x^{\alpha} f(t, \phi_1)| \\
&= |\overline{y}_0 - y_0| + |{}_{x_0}I_x^{\alpha}\left[\left(t, \overline{\phi}_1\right) - f(t, \phi_1)\right]| \\
&\leq \delta_1 + K {}_{x_0}I_x^{\alpha}|\overline{\phi}_1 - \phi_1| \\
&\leq \delta_1 + K\delta_1 \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, dt + \frac{K^2\delta_1}{\Gamma(\alpha+1)\Gamma(\alpha)} \cdot \\
&\int_0^x (x-t)^{\alpha-1}(t-x_0)^{\alpha}\, dt.
\end{aligned}$$

If we use the substitution $x - t = z(x - x_0)$ (similar substitution used in [15]), then the above inequality becomes

$$|\overline{\phi}_2(x) - \phi_2(x)| \le \delta_1 + K\delta_1 \frac{(x-x_0)^\alpha}{\Gamma(\alpha+1)} + \frac{K^2\delta_1}{\Gamma(2\alpha+1)}\left[(x-x_0)^\alpha\right]^2. \tag{3.4}$$

Similarly we can write

$$|\overline{\phi}_3(x) - \phi_3(x)| \le \delta_1 + K\delta_1 \frac{(x-x_0)^\alpha}{\Gamma(\alpha+1)} + \frac{K^2\delta_1}{\Gamma(2\alpha+1)}\left[(x-x_0)^\alpha\right]^2 + \frac{K^3\delta_1}{\Gamma(3\alpha+1)}\left[(x-x_0)^\alpha\right]^3. \tag{3.5}$$

In general

$$|\overline{\phi}_n(x) - \phi_n(x)| \le \delta_1 + K\delta_1 \frac{(x-x_0)^\alpha}{\Gamma(\alpha+1)} + \frac{K^2\delta_1}{\Gamma(2\alpha+1)}\left[(x-x_0)^\alpha\right]^2 + \frac{K^3\delta_1}{\Gamma(3\alpha+1)}\left[(x-x_0)^\alpha\right]^3 + \cdots + \frac{K^n\delta_1}{\Gamma(n\alpha+1)}\left[(x-x_0)^\alpha\right]^n,$$

$$|\overline{\phi}_n(x) - \phi_n(x)| \le \delta_1 \sum_{j=0}^{n} \frac{K^j\left[(x-x_0)^\alpha\right]^j}{\Gamma(j\alpha+1)}, \forall x \in [x_0, x_0+h].$$

By applying similar argument in the interval $[x_0 - h, x_0]$, we have

$$|\overline{\phi}_n(x) - \phi_n(x)| \le \delta_1 \sum_{j=0}^{n} \frac{K^j\left[|(x-x_0)^\alpha|\right]^j}{\Gamma(j\alpha+1)} \le \sum_{j=0}^{n} \frac{K^j(h^\alpha)^j}{\Gamma(j\alpha+1)}, \forall x \in \{x : |x-x_0| \le h\}.$$

By taking limit $n \to \infty$, we have

$$|\overline{\phi}(x) - \phi(x)| \le \delta_1 E_{\alpha,1}(Kh^\alpha), \forall x \in \{x : |x-x_0| \le h\}. \tag{3.6}$$

Thus this theorem geometrically describes that if any two solutions $\phi(x)$ and $\overline{\phi}(x)$ are initially close to each other, then they will be close to each other for all values of x in $|x - x_0| \le h$. In the next theorem, we have shown how the solution of (1.1) depends on the right hand side function $f(x, y)$. □

THEOREM 3.2. Suppose $f(x, y)$ be a continuous function and satisfies Lipschitz condition w.r.t. y in some domain D of xy-plane. Let $F(x, y)$ be a continuous function that satisfy

$$|F(x,y) - f(x,y)| \le \epsilon \quad \forall (x,y) \in D.$$

For any fixed point (x_0, y_0) in D, let ϕ be the solution of the initial value problem (1.1), ψ be the solution of the initial value problem

$$\begin{cases} {}^{C}_{x_0}D^\alpha_x y(x) = F(x,y) \\ y(x_0) = y_0, \end{cases} \tag{3.7}$$

and $(x, \phi(x)), (x, \psi(x)) \in D$. Then

$$|\phi(x) - \psi(x)| \leq \frac{\epsilon}{K}\left[E_{\alpha,1}(Kh^{\alpha})\right], |x - x_0| \leq h.$$

Proof. Suppose that $\overline{\phi}_0(x) = \psi(x)$ and consider a sequence $\{\overline{\phi}_n\}$ defined by

$$\overline{\phi}_n(x) = y_0 + {}_{x_0}I_x^{\alpha} f\left(t, \overline{\phi}_{n-1}(t)\right), |x - x_0| \leq h, (n = 1, 2, \cdots).$$

Therefore

$$\overline{\phi}(x) = \lim_{n\to\infty} \overline{\phi}_n(x)$$

is a solution of (1.1) and $\overline{\phi}(x_0) = y_0$ in $|x - x_0| \leq h$.
As the right-hand side function $f(x, y)$ of (1.1) is continuous and satisfies Lipschitz condition, therefore (1.1) has a unique solution. Hence $\overline{\phi}(x) = \phi(x)$ on $|x - x_o| \leq h$. This implies that

$$\lim_{n\to\infty} \overline{\phi}_n(x) = \phi(x).$$

Also $\psi(x)$ is the solution of the initial value problem (3.7), therefore

$$\psi(x) = y_0 + {}_{x_0}I_x^{\alpha} F(t, \psi(t)) \text{ in } |x - x_0| \leq h.$$

Let x belongs to $[x_0, x_0 + h]$.
Now

$$\begin{aligned}
|\overline{\phi}_1(x) - \psi(x)| &= |y_0 + {}_{x_0}I_x^{\alpha} f(t, y_0) - y_0 - {}_{x_0}I_x^{\alpha} F(t, \psi(t))| \\
&\leq |{}_{x_0}I_x^{\alpha}[f(t, y_0) - F(t, \psi(t))]| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x - t)^{\alpha-1} |f(t, y_0) - F(t, \psi(t))| dt \\
&\leq \frac{1}{\Gamma(\alpha + 1)} (x - x_0)^{\alpha}.
\end{aligned}$$

Therefore we have

$$|\overline{\phi}_1(x) - \psi(x)| \leq \frac{1}{\Gamma(\alpha)} (x - x_0)^{\alpha}. \tag{3.8}$$

Let $F(x,y) = f(x,y) + \delta(x,y)$. Now

$$\begin{aligned}
|\overline{\phi}_2(x) - \psi(x)| &= |y_0 + {}_{x_0}I_x^{\alpha} f\left(t, \overline{\phi}_1(t)\right) - y_0 - {}_{x_0}I_x^{\alpha} F(t, \psi(t))| \\
&\le |{}_{x_0}I_x^{\alpha}[f\left(t, \overline{\phi}_1(t)\right) - F(t, \psi(t))]| \\
&\le \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} |f\left(t, \overline{\phi}_1(t)\right) - F(t, \psi(t))| dt \\
&\le \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} |f\left(t, \overline{\phi}_1(t)\right) - f(t, \psi(t)) - \delta(t, \psi(t))| dt \\
&\le \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} |f\left(t, \overline{\phi}_1(t)\right) - f(t, \psi(t))| dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} |\delta(t, \psi(t))| dt \\
&\le \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} |\overline{\phi}_1(t) - \psi(t)| dt + \frac{\epsilon}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} dt \\
&\le \frac{K}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} \frac{\epsilon (t-x_0)^{\alpha}}{\Gamma(\alpha+1)} dt + \frac{\epsilon}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} dt \\
&\le \frac{\epsilon K}{\Gamma(\alpha)\Gamma(\alpha+1)} \int_{x_0}^{x} (x-t)^{\alpha-1} (t-x_0)^{\alpha} dt + \frac{\epsilon}{\Gamma(\alpha+1)} (x-x_0)^{\alpha}.
\end{aligned}$$

After using the substitution $x - t = z(x - x_0)$ above inequality becomes

$$|\overline{\phi}_2(x) - \psi(x)| \le \epsilon \frac{(x-x_0)^{\alpha}}{\Gamma(\alpha+1)} + \epsilon K \frac{[(x-x_0)^{\alpha}]^2}{\Gamma(2\alpha+1)}, \text{ in } [x_0, x_0+h]. \tag{3.9}$$

Similarly we can write

$$\begin{aligned}
|\overline{\phi}_3(x) - \psi(x)| \le \epsilon \frac{(x-x_0)^{\alpha}}{\Gamma(\alpha+1)} &+ \epsilon K \frac{[(x-x_0)^{\alpha}]^2}{\Gamma(2\alpha+1)} \\
&+ \epsilon K^2 \frac{[(x-x_0)^{\alpha}]^3}{\Gamma(3\alpha+1)}, x \in [x_0, x_0+h].
\end{aligned} \tag{3.10}$$

By continuing in this way in general we can write

$$\begin{aligned}
|\overline{\phi}_n(x) - \psi(x)| \le \epsilon \frac{(x-x_0)^{\alpha}}{\Gamma(\alpha+1)} + \epsilon K \frac{[(x-x_0)^{\alpha}]^2}{\Gamma(2\alpha+1)} &+ \epsilon K^2 \frac{[(x-x_0)^{\alpha}]^3}{\Gamma(3\alpha+1)} \\
+ \cdots + \epsilon K^{n-1} \frac{[(x-x_0)^{\alpha}]^n}{\Gamma(n\alpha+1)}, x \in [x_0, x_0+h].
\end{aligned} \tag{3.11}$$

This implies that

$$|\overline{\phi}_n(x) - \psi(x)| \le \frac{\epsilon}{K} \sum_{j=1}^{n} \frac{[K(x-x_0)^{\alpha}]^j}{\Gamma(j\alpha+1)}, x \in [x_0, x_0+h]. \tag{3.12}$$

Now by applying similar argument in $[x_0 - h, x_0]$, we have

$$|\overline{\phi}_n(x) - \psi(x)| \le \frac{\epsilon}{K} \sum_{j=1}^{n} \frac{[Kh^\alpha]^j}{\Gamma(j\alpha + 1)}, x \in \{x : |x - x_0| \le h\}. \tag{3.13}$$

Now letting $n \to \infty$, we have

$$|\overline{\phi}(x) - \psi(x)| \le \frac{\epsilon}{K} \sum_{j=1}^{\infty} \frac{[Kh^\alpha]^j}{\Gamma(j\alpha + 1)}.$$

Therefore

$$|\overline{\phi}(x) - \psi(x)| \le \frac{\epsilon}{K}[E_{\alpha,1}(Kh^\alpha) - 1]. \tag{3.14}$$

Thus if we take ϵ sufficiently small, then the difference between ϕ and ψ will be arbitrarily small on $|x - x_0| \le h$. □

4. Examples

In this section, we present two illustrative examples that vividly showcase the correlation between the solution and both the initial condition and the right-hand side function $f(x, y)$. Through these examples, we provide concrete evidence of how even slight variations in the initial condition and the function $f(x, y)$ can lead to significant changes in the resulting solution. These examples serve to highlight the importance of understanding the sensitivity of the solution with respect to the given initial conditions and the underlying right-hand side function.

In the first example, we consider the differential equation

$${}^{C}_{x_0}D^{\alpha}_{x}y(x) = y,$$

along with the initial conditions

$$y_0 = 1,$$

and

$$\overline{y}_0 = \frac{1}{2}.$$

In Figure 1 we use Matlab to plot the graph for the initial condition $y_0 = 1$ represented by blue curves, and for the initial condition $\overline{y}_0 = \frac{1}{2}$ represented by green curves. As depicted in the figure, after a few iterations, the curves start to coincide, indicating convergence. This convergence leads the sequences ϕ_n and $\overline{\phi}_n$ to approach the unique solutions ϕ and $\overline{\phi}$respectively, for initial conditions $y_0 = 1$ and $\overline{y}_0 = \frac{1}{2}$ as mentioned in [15]. The graph clearly illustrates that solutions ϕ and $\overline{\phi}$ which are initially close, remain close for subsequent values of x. In the second example we have slightly changed the right-hand side function and retained the initial condition $y(0) = 1$. Here we have taken

$$f(x, y) = y,$$

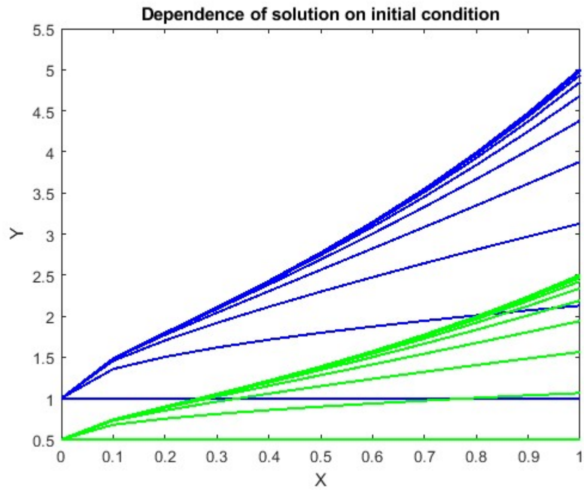

FIGURE 1. Graphs of ${}^{C}_{x_0}D^{\frac{1}{2}}_{x}y\,(x) = y$, for initial condition $\phi\,(0) = 1$ and $\overline{\phi}\,(0) = \frac{1}{2}$ in [0, 1].

and

$$F\,(x, y) = y + 1.$$

In Figure2 blue curves are representing the approximate solutions of

$$\begin{cases} {}^{C}_{x_0}D^{\frac{1}{2}}_{x}y(x) = y \\ y\,(x_0) = 1, \end{cases}$$

and green curves are representing the approximate solutions of

$$\begin{cases} {}^{C}_{x_0}D^{\frac{1}{2}}_{x}y(x) = y + 1 \\ y\,(x_0) = 1. \end{cases}$$

From Figure 2 it is clear that a slight change in right-hand side function resulted in slight change in the solution.

5. Discussions

In situations where analytic methods are not feasible for solving fractional order differential equations, numerical methods offer valuable assistance in obtaining approximate solutions. One such method is Picard's iterative method for fractional differential equations. In this research, we demonstrate how the solution to the initial value problem 1.1 is contingent upon both the initial condition and the right-hand side function. Theorem 3.1 establishes that if two solutions, ϕ and $\overline{\phi}$, differ slightly in their initial values, then the values of the solutions will vary by an arbitrarily small

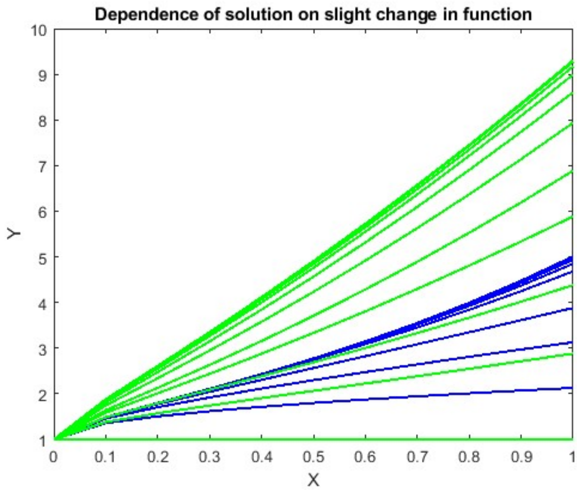

FIGURE 2. Blue curves for ${}^{C}_{x_0}D^{\frac{1}{2}}_{x}y(x) = y$ and green curves for ${}^{C}_{x_0}D^{\frac{1}{2}}_{x}y(x) = y + 1$ for initial condition $y(0) = 1$ in $[0, 1]$.

amount for subsequent values of x. Geometrically, Figure 1 visually depicts how initially close curves remain close for successive values of x, validating the theorem's assertion. Likewise, in theorem 3.2, we prove that the difference between the solutions ψ and ϕ is small if the parameter ϵ is sufficiently small. Through these theorems and corresponding illustrations, we provide a comprehensive understanding of the dependency of the solutions on initial conditions and the underlying functions.

6. Material and Methods

Our approach involved using MATLAB software to draw solutions for Picard's technique, demonstrating how the answers rely on the underlying functions and beginning circumstances.

Author contributions:
*Conceptualisation:*J. Mohan, A. Sood ; *Software:* A. Kumar ; *Writing-Original Draft:* J. Mohan, A. Sood

Conflicts of Interest: The authors declare no conflict of interest.

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