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MIXED FRAME OPERATORS ON HILBERT SPACES

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Abstract

Different aspects of mixed frame operators on separable Hilbert spaces are discussed. Special attention is given to the concrete spaces $L^2(\mathbb{R})$ and $l^2(\mathbb{Z})$. Also we prove that every surjective map between two separable Hilbert spaces are mixed frame operators.

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1. Introduction

In present day signal processing and time frequency analysis, frame theory has received remarkable research attention because of its wide range applications. This mathematical theory, first initiated by D.Gabor [5] in his *Theory of Communication*, formulated a fundamental approach to signal decomposition in terms of elementary signals. The notion of frames for Hilbert spaces were formally introduced by Duffin and Schaeffer in 1952 [2]. In 1980's, the crucial works of Janssen made it an independent topic of mathematical investigations [7].

Among different classes of frames, Gabor frames (also known as Weyl-Heisenberg frames) has acquired considerable research interest. These frames are very specific as they are being generated by the action of special kind of unitary operators on a single vector(called *window function*) in the space concerned [3, 4]. Associated to each frame, there is a positive invertible operator called the frame operator and these operators are very important tool in frame theory [9, 10]. Mixed frame operators between two separable Hilbert spaces are also received considerable research attention in literature [1]. In this article we analyze certain properties of such operators.

This manuscript is arranged in the following sequel. We begin with some basic definitions and results which are essential for the present work (See Section 2). Section 3 mainly focused on various aspects of mixed frame operators. Through out in this article $\mathcal{H}$ denotes a separable Hilbert space and our basic references for Hilbert space frame theory are [1, 6].

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2. Preliminaries

A countable collection of elements $\{\omega_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is a frame if $A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, \omega_k\rangle|^2 \leq B\|f\|^2, \ \forall f \in \mathcal{H}$ and for some positive constants A, B, which are called *frame bounds*. If $A = B$, the frame is called a *tight frame*. A tight frame is called a *Parseval frame or normalized tight frame* if $A = B = 1$. Whenever a sequence $\{\omega_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ satisfies $\sum_{k=1}^{\infty} |\langle f, \omega_k\rangle|^2 \leq B\|f\|^2, \ \forall f \in \mathcal{H}$ and for some constant $B > 0$, then $\{\omega_k\}_{k=1}^{\infty}$ is said to be a *Bessel sequence* or a *semi-frame sequence* [1] and moreover, it is said to be a *frame sequence*, if it is a frame for $\overline{span}\{\omega_k\}_{k=1}^{\infty}$. A frame is called an *exact frame*, if its frame sequence is minimal.

For a frame $\{\omega_k\}_{k=1}^{\infty}$ for a Hilbert space $\mathcal{H}$, the map $T : l^2(\mathbb{N}) \to \mathcal{H}$ defined by $T(c_k)_{k\in\mathbb{N}} = \sum_{k=1}^{\infty} c_k\omega_k$ is bounded linear and is called *synthesis operator(pre-frame operator)*. Adjoint of this operator $T^* : \mathcal{H} \to l^2(\mathbb{N})$ given by $T^*(f) = \{\langle f, \omega_k\rangle\}_{k\in\mathbb{N}}$ is called *analysis operator*. The map $S : \mathcal{H} \to \mathcal{H}$ defined by, $Sf = TT^*(f) = \sum_{k=1}^{\infty}\langle f, \omega_k\rangle\omega_k,\ f \in \mathcal{H}$ is a bounded linear, invertible, positive operator on $\mathcal{H}$, called the *frame operator* of the frame $\{\omega_k\}_{k=1}^{\infty}$. The frame operator of a tight frame is a scalar multiple of the identity operator and that of a Parseval frame is the identity operator. For a frame $\{\omega_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ with frame operator S, the frame $\{\tilde{\omega}_k\}_{k\in\mathbb{N}} = \{S^{-1}\omega_k\}_{k\in\mathbb{N}}$ is known as the *(canonical) dual frame* of $\{\omega_k\}_{k=1}^{\infty}$. A frame $\{\omega_k\}_{k=1}^{\infty}$ together with its dual frame $\{\tilde{\omega}_k\}_{k\in\mathbb{N}}$ gives the following *frame decomposition* by which every element in $\mathcal{H}$ has a representation as a superposition of the frame elements. That is, for all $f \in \mathcal{H}$, $f = \sum_{k=1}^{\infty}\langle f, S^{-1}\omega_k\rangle\omega_k$ and $f = \sum_{k=1}^{\infty}\langle f, \omega_k\rangle S^{-1}\omega_k$.

For Bessel sequences $\{u_k\}_{k\in\mathbb{Z}}$ and $\{v_k\}_{k\in\mathbb{Z}}$ in Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, we can have a bounded linear operator $M : \mathcal{H} \to \mathcal{K}$ given by $M(x) = \sum_{k\in\mathbb{Z}}\langle x, u_k\rangle v_k$, where the series defining M converges for all $x \in \mathcal{H}$. The operator M is called the *mixed semi-frame operator* associated with the Bessel sequences $\{u_k\}$ and $\{v_k\}$ [1].

Among different categories of frames, a kind of structured frames called Gabor frames (or Weil-Heisenberg frames) are defined as follows. Gabor system in $L^2(\mathbb{R})$ has the form $\{e^{2\pi imbx}g(x - na)\}_{m,n\in\mathbb{Z}}$ for some parameters $a, b > 0$ and a given non zero function $g \in L^2(\mathbb{R})$. Using the translation operators T_a and the modulation operators E_b we can denote a Gabor system by $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. Where the modulation operators E_{mb} and translation operators T_{na} on $L^2(\mathbb{R})$ are given by $E_{mb}g(x) = e^{2\pi imbx}g(x)$ and $T_{na}g(x) = g(x - na)$ for $x \in \mathbb{R}$.

Riesz basis for a Hilbert space $\mathcal{H}$ is a family of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator. A Riesz basis is actually a basis, in fact , one can see that Riesz basis is a basis satisfying some extra conditions [1].

3. Mixed Frame operators

In this section we discuss the formal definition and some nice observations of mixed frame operators.

Definition 3.1. [1] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ are two separable Hilbert spaces with frames $\{\omega_k\}_{k\in\mathbb{N}}$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ respectively. If T and U are the respective synthesis operators then the map $M : \mathcal{H}_1 \to \mathcal{H}_2$ defined by for all $f \in \mathcal{H}_1$, $M(f) = (UT^*)(f) = \sum_{k=1}^{\infty} \langle f, \omega_k \rangle \lambda_k$ is called the *mixed* $(\{\omega_k\}, \{\lambda_k\})$ *frame operator*.

The convergence of the above summation is guaranteed as being the composition of an analysis and a synthesis operators. A simple and direct calculation shows that the mixed frame operator is a bounded linear operator. Moreover the adjoint of the mixed frame operator M is the mixed frame operator $M^* : \mathcal{H}_2 \to \mathcal{H}_1$ given by $M^*(g) = (TU^*)(g) = \sum_{k=1}^{\infty} \langle g, \lambda_k \rangle \omega_k$. Next result from [1] is very useful in our discussion.

Proposition 3.2. *If $\{f_k\}_{k\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$, then there exists a unique sequence $\{g_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}$, such that $f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k$ for all $f \in \mathcal{H}$. The sequence $\{g_k\}_{k\in\mathbb{N}}$ is also a Riesz basis, further $\{f_k\}_{k\in\mathbb{N}}$ and $\{g_k\}_{k\in\mathbb{N}}$ are biorthogonal sequences in $\mathcal{H}$.*

It is interesting to investigate the peculiarities of mixed frame operators with respect to the properties of frames involved in the definition. Following Proposition is a step in this direction.

Proposition 3.3. *If $\{\omega_k\}_{k\in\mathbb{N}}$ is a Riesz basis or exact frame for $\mathcal{H}_1$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ is any frame for $\mathcal{H}_2$, then the mixed $(\{\omega_k\}, \{\lambda_k\})$ frame operator M is a surjective linear map from $\mathcal{H}_1$ to $\mathcal{H}_2$.*

Proof. Suppose that $\{\omega_k\}_{k\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}_1$, then by proposition 3.2, there is a unique sequence $\{\xi_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}_1$ which is again a Riesz basis and biorthogonal to $\{\omega_k\}_{k\in\mathbb{N}}$. Also assume that S is the frame operator of the frame $\{\lambda_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}_2$. Then for $g \in \mathcal{H}_2$, we have the expression $g = \sum_{k=1}^{\infty} \langle g, S^{-1}\lambda_k \rangle \lambda_k$, where $\{S^{-1}\lambda_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}_2$ is the canonical dual frame of the frame $\{\lambda_k\}_{k\in\mathbb{N}}$. Define $f_0 = \sum_{k=1}^{\infty} \langle g, S^{-1}\lambda_k \rangle \xi_k$. Clearly the element f_0 is in $\mathcal{H}_1$ and it is easy to see that

$$\begin{aligned}
M(f_0) &= \sum_{n=1}^{\infty}\langle f_0, \omega_n\rangle\lambda_n \\
&= \sum_{n=1}^{\infty}\langle\sum_{k=1}^{\infty}\langle g, S^{-1}\lambda_k\rangle\xi_k, \omega_n\rangle\lambda_n \\
&= \sum_{n=1}^{\infty}\langle g, S^{-1}\lambda_n\rangle\lambda_n, \text{ since } (\omega_k) \text{ and } (\xi_k) \text{ are biorthogonal} \\
&= g, \text{ proving that } M \text{ is surjective as desired.}
\end{aligned}$$

Now assume that $\{\omega_k\}_{k\in\mathbb{N}}$ is an exact frame for $\mathcal{H}_1$ with frame operator S_1, then it is a known fact that, dual frame $\{S_1^{-1}\omega_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}_1$ is biorthogonal to $\{\omega_k\}_{k\in\mathbb{N}}$ [1]. Let $\{\lambda_k\}_{k\in\mathbb{N}}$ is any frame for $\mathcal{H}_2$ with frame operator S_2. For any $g \in \mathcal{H}_2$, we have $g = \sum_{k=1}^{\infty}\langle g, S_2^{-1}\lambda_k\rangle\lambda_k$, where $\{S_2^{-1}\lambda_k\}_{k\in\mathbb{N}}$ is the canonical dual frame of the frame $\{\lambda_k\}_{k\in\mathbb{N}}$. Define $f_0 = \sum_{k=1}^{\infty}\langle g, S_2^{-1}\lambda_k\rangle S_1^{-1}\omega_k$, clearly the element f_0 is in $\mathcal{H}_1$ and following calculation shows that M is surjective.

$$\begin{aligned}
M(f_0) &= \sum_{n=1}^{\infty}\langle f_0, \omega_n\rangle\lambda_n \\
&= \sum_{n=1}^{\infty}\langle\sum_{k=1}^{\infty}\langle g, S_2^{-1}\lambda_k\rangle S_1^{-1}\omega_k, \omega_n\rangle\lambda_n \\
&= \sum_{n=1}^{\infty}\langle g, S_2^{-1}\lambda_n\rangle\lambda_n = g, \quad \text{since } \{\omega_k\} \text{ and } \{S_1^{-1}\omega_k\} \text{ are biorthogonal}
\end{aligned}$$

□

Corollary 3.4. *If $\{\omega_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}_1$, then the mixed $(\{\omega_k\}, \{\lambda_k\})$ frame operator M is a surjective linear map from $\mathcal{H}_1$ to $\mathcal{H}_2$.*

Proof. An orthonormal basis $\{\omega_k\}_{k\in\mathbb{N}}$ for $\mathcal{H}_1$ is a Riesz basis for $\mathcal{H}_1$ having biorthogonal sequence $\{\omega_k\}_{k\in\mathbb{N}}$ itself. Hence from the proof of above Proposition result easily follows. □

One interesting observation is that, the biorthogonal sequence $\{\xi_k\}_{k\in\mathbb{N}}$ corresponding to the Riesz basis $\{\omega_k\}_{k\in\mathbb{N}}$ for $\mathcal{H}_1$, maps into the frame $\{\lambda_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}_2$ under the mixed frame operator M. For, $(M\xi_k) = \sum_{n=1}^{\infty}\langle\xi_k, \omega_n\rangle\lambda_n = \lambda_k$, for each $k = 1, 2, 3, \ldots$. By similar argument we conclude that, image of an orthonormal basis $\{\omega_k\}_{k\in\mathbb{N}}$ for $\mathcal{H}_1$ under the map M is the frame $\{\lambda_k\}_{k\in\mathbb{N}}$ for $\mathcal{H}_2$. This fact says us that every frame for $\mathcal{H}_2$

is the image of a Riesz basis for $\mathcal{H}_1$ under a surjective map. In other words, the frame $\{\lambda_k\}_{k\in\mathbb{N}}$ for $\mathcal{H}_2$ is the image of a Riesz basis for $\mathcal{H}_1$ under the mixed frame operator $M : \mathcal{H}_1 \to \mathcal{H}_2$.

Assume that $\{\omega_k\}_{k\in\mathbb{N}}$ is a frame for $\mathcal{H}_1$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}_2$, then for $f, g \in \mathcal{H}_1$, $M(f) = M(g)$, gives $\sum_{k=1}^{\infty}\langle f, \omega_k\rangle\lambda_k = \sum_{k=1}^{\infty}\langle g, \omega_k\rangle\lambda_k$, that is, $\sum_{k=1}^{\infty}\langle f - g, \omega_k\rangle\lambda_k = 0$. Since $\{\lambda_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}_2$ is an orthonormal basis it gives $\langle f - g, \omega_k\rangle = 0$, for all $k = 1, 2, 3, \ldots$. Since $\{\omega_k\}_{k\in\mathbb{N}}$ is a frame and hence complete in $\mathcal{H}_1$ we get, $f - g = 0$, proving that M is injective. Hence we have the following remark.

Remark 3.5. *If $\{\omega_k\}_{k\in\mathbb{N}}$ is any frame for $\mathcal{H}_1$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}_2$, then the mixed frame operator $M : \mathcal{H}_1 \to \mathcal{H}_2$ is injective.*

Thus, if $\{\omega_k\}_{k\in\mathbb{N}}$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ are orthonormal basis for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, a straight computation shows that the mixed frame operator $M : \mathcal{H}_1 \to \mathcal{H}_2$ is a bijective linear isometry and is a well known linear algebra result. Following result from [3] helps us to give another nice observation about the mixed frame operator.

Lemma 3.6. *If $\{u_k\}_{k\in\mathbb{N}}$ is a Bessel sequence in $\mathcal{H}$ with pre-frame operator S then, for any $V \in B(\mathcal{H})$, $\{Vu_k\}_{k\in\mathbb{N}}$ is also a Bessel sequence in $\mathcal{H}$ with VSV^* as its associated pre-frame operator.*

Proposition 3.7. *A bounded linear operator on a separable Hilbert space $\mathcal{H}$ is a pre-frame operator if and only if it is positive.*

Proof. It is well known that a pre-frame operator is positive. For the converse, assume that A is a positive operator on $\mathcal{H}$ with square root $B = A^{\frac{1}{2}}$. If $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in $\mathcal{H}$, then the Bessel sequence $\{Be_k\}_{k\in\mathbb{N}}$ has the pre-frame operator $BIB^* = A$, by above lemma. □

Remark 3.8. *Let $M : \mathcal{H}_1 \to \mathcal{H}_2$ be the mixed frame operator as in definition 3.1. Then a direct calculation and use of Proposition 3.7 proves that, there exists Bessel sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$ having M^*M and MM^* as their pre-frame operators.*

Proposition 3.9. *Let $M : \mathcal{H}_1 \to \mathcal{H}_2$ is a mixed frame operator corresponding to the frames $\{\omega_k\}_{k\in\mathbb{N}}$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. If V is any surjective map on $\mathcal{H}_2$, then $VM : \mathcal{H}_1 \to \mathcal{H}_2$ is also a mixed frame operator.*

Proof. Let $\{\omega_k\}_{k\in\mathbb{N}}$ and $\{\lambda_k\}_{k\in\mathbb{N}}$ are frames for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively with mixed frame operator $M : \mathcal{H}_1 \to \mathcal{H}_2$. If V is any surjective map on $\mathcal{H}_2$, then the mixed frame operator corresponding to the frames $\{\omega_k\}_{k\in\mathbb{N}}$ and $\{V\lambda_k\}_{k\in\mathbb{N}}$ is nothing but VM. □

Assume that $\{\omega_k\}$ is a frame in $\mathcal{H}_1$ with frame operator S_1 and $\{\lambda_k\}$ is a frame in $\mathcal{H}_2$ with frame operator S_2. Then the respective canonical dual frames are $\{S_1^{-1}\omega_k\}$ and $\{S_2^{-1}\lambda_k\}$. If we define the mixed frame operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ by $A(f) = \sum_{k=1}^{\infty}\langle f, \omega_k\rangle S_2^{-1}\lambda_k$ then it is easy to see that $A = S_2^{-1}M_1$, where $M_1 : \mathcal{H}_1 \to \mathcal{H}_2$ is the mixed frame operator given by $M_1(f) = \sum_{k=1}^{\infty}\langle f, \omega_k\rangle\lambda_k$. So that $A^* = M_1^* S_2^{-1}$ is the mixed frame operator given by $A^*(g) = \sum_{k=1}^{\infty}\langle g, S_2^{-1}\lambda_k\rangle\omega_k = M_1^* S_2^{-1}(g)$, for all $g \in \mathcal{H}_2$. Further if we consider the operator A^*A on $\mathcal{H}_1$ then $A^*A = M_1^*(S_2^{-1})^2 M_1$ will be a pre-frame operator on $\mathcal{H}_1$ by Proposition 3.7.

Similarly, if we define the mixed frame operator $B : \mathcal{H}_1 \to \mathcal{H}_2$ by $B(f) = \sum_{k=1}^{\infty}\langle f, S_1^{-1}\omega_k\rangle\lambda_k$ then again it is easy to see that $B = M_1 S_1^{-1}$, so that B^* is the mixed frame operator given by $B^*(g) = \sum_{k=1}^{\infty}\langle g, \lambda_k\rangle S_1^{-1}\omega_k = S_1^{-1}M_1^*(g)$, for all $g \in \mathcal{H}_2$. Here also the operator BB^* on $\mathcal{H}_2$ will be a pre-frame operator.

Let us consider the most useful concrete spaces $L^2(\mathbb{R})$ and $l^2(\mathbb{Z})$ as special cases to make some nice findings. The existence of a Riesz basis for $L^2(\mathbb{R})$ corresponding to each parameter $a, b > 0$ with $ab \leq 1$ is guaranteed in [1]. A Gabor frame for the discrete space $l^2(\mathbb{Z})$ is of the form $\{E_{\frac{m}{M}}T_{nN}g : m = 0, 1, 2, ..., M-1, n \in \mathbb{Z}\}$, where M, N are positive integers with $\frac{N}{M} \leq 1$. Now we state a result connecting these two frames, and the proof follows from the paragraph just after corollary 3.4.

Proposition 3.10. *Every Gabor frame $\{E_{\frac{m}{M}}T_{nN}g : m = 0, 1, 2, ..., M-1, n \in \mathbb{Z}\}$ for $l^2(\mathbb{Z})$ is the image of a Riesz basis for $L^2(\mathbb{R})$ under a suitable surjective map.*

Here is an interesting connection between mixed frame operators and invertible operators from $L^2(\mathbb{R})$ into $\mathcal{H}$, which says that every invertible operator from $L^2(\mathbb{R})$ can be approximated by two frames. One can see that the same result holds in the finite space $l^2(\mathbb{Z}_N)$ and in the sequence space $l^2(\mathbb{Z})$ also.

Proposition 3.11. *Every invertible operator $B : L^2(\mathbb{R}) \to \mathcal{H}$ can be identified as a mixed frame operator.*

Proof. Let $B : L^2(\mathbb{R}) \to \mathcal{H}$ is a bounded invertible map. Obviously, B maps any given Gabor frame $\mathcal{G} = \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ to a frame $B(\mathcal{G}) = B\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in $\mathcal{H}$. Let M be the mixed frame operator defined by $Mf = \sum_{m,n\in\mathbb{Z}}\langle f, E_{mb}T_{na}g\rangle B(E_{mb}T_{na}g)$.

Then for all $f \in L^2(\mathbb{R})$,

$$\begin{aligned} Mf &= \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle B(E_{mb}T_{na}g) \\ &= B \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}g \\ &= BS(f), \text{ where } S \text{ is the frame operator of } \mathcal{G}. \end{aligned}$$

Thus, $M = BS$. Now, by choosing $\mathcal{G}$ as a Parseval Gabor frame for $L^2(\mathbb{R})$, we obtain $S = I_{L^2(\mathbb{R})}$ so that $B : L^2(\mathbb{R}) \to \mathcal{H}$ is precisely a mixed frame operator. □

It should be noted that, converse of above theorem is not true. That is every mixed frame operator need not be invertible. Following example illustrates this fact.

Example 3.12. *Consider the sequence space $l^2(\mathbb{Z})$ with respect to the usual inner product and the standard orthonormal basis $\{e_k\}_{k=1}^{\infty}$ in it. The mixed frame operator M on $l^2(\mathbb{Z})$ corresponding to the frames $\{e_k\}_{k=1}^{\infty} = \{e_1, e_2, e_3, \dots\}$ and $\{u_k\}_{k=1}^{\infty} = \{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \dots\}$, defined by $M(x) = \sum_{k=1}^{\infty} \langle x, e_k\rangle u_k$ for all $x \in l^2(\mathbb{Z})$, is not invertible because it is not injective. A direct calculation shows that $M(e_2) = M(e_3)$ even if $e_2 \neq e_3$.*

Here is an interesting relation between mixed frame operators and surjective operators. It says that every surjective map between two separable Hilbert spaces can be viewed as a mixed frame operator. One can easily see that the same result holds in finite dimensional space also.

Theorem 3.13. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ are two separable Hilbert spaces and $B : \mathcal{H}_1 \to \mathcal{H}_2$ be a surjective map, then B is a mixed frame operator.*

Proof. Assume that $B : \mathcal{H}_1 \to \mathcal{H}_2$ is a surjective map. Consider an orthonormal basis $\{u_k : k \in \mathbb{Z}\}$ for $\mathcal{H}_1$. Since B is surjective, it is true that, $\{Bu_k : k \in \mathbb{Z}\}$ is a frame for $\mathcal{H}_2$.

Let M be the $(\{u_k\}, \{Bu_k\})$ mixed frame operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, defined by $Mx = \sum_{k\in\mathbb{Z}} \langle x, u_k\rangle Bu_k$, for all $x \in \mathcal{H}_1$ so that

$$\begin{aligned} Mx &= \sum_{k\in\mathbb{Z}} \langle x, u_k\rangle Bu_k \\ &= B\sum_{k\in\mathbb{Z}} \langle x, u_k\rangle u_k \\ &= BI(x) = Bx, \text{ where } I \text{ is the identity operator of } \mathcal{H}_1. \end{aligned}$$

□

It is clear that Proposition 3.11 is a special case of this theorem. Further, it should be noted that, converse of above theorem need not be true. That is every mixed frame operator may not be surjective. Following example illustrates this fact.

Example 3.14. *Consider the sequence space $l^2(\mathbb{Z})$ with respect to the usual inner product and the standard orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for it. Let $B : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be the left shift operator, it is known that B is surjective and its adjoint is the right shift operator which is not surjective [8]. Hence $\{Be_k\}_{k=1}^{\infty}$ is a frame for $l^2(\mathbb{Z})$. The mixed frame operator M on $l^2(\mathbb{Z})$ corresponding to the frames $\{Be_k\}_{k=1}^{\infty}$ and $\{e_k\}_{k=1}^{\infty}$ defined by, for all $x \in l^2(\mathbb{Z})$, $M(x) = \sum_{k=1}^{\infty}\langle x, Be_k\rangle e_k = \sum_{k=1}^{\infty}\langle B^* x, e_k\rangle e_k = B^* x$ is nothing but the right shift operator, proving the claim.*

Another nice observation is given here.

Theorem 3.15. *If $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ and $\{E_{mb}T_{na}h : m, n \in \mathbb{Z}\}$ are any two Gabor frames for $L^2(\mathbb{R})$ with window functions g and h respectively, then the mixed frame operator $M : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $M(f) = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}h$ commutes with $E_{mb}T_{na}$ for all $m, n \in \mathbb{Z}$.*

Proof. Assume the hypothesis of the proposition, then for each $m', n' \in \mathbb{Z}$,

$$\begin{aligned}
M(E_{m'b}T_{n'a}g) &= \sum_{m,n\in\mathbb{Z}} \langle E_{m'b}T_{n'a}g, E_{mb}T_{na}g\rangle E_{mb}T_{na}h \\
&= \sum_{m,n\in\mathbb{Z}} \langle T_{n'a}g, E_{(m-m')b}T_{na}g\rangle E_{mb}T_{na}h \\
&= \sum_{m,n\in\mathbb{Z}} \langle g, T_{-n'a}E_{(m-m')b}T_{na}g\rangle E_{mb}T_{na}h \\
&= \sum_{m,n\in\mathbb{Z}} \langle g, e^{2\pi i n' a(m-m')b} E_{(m-m')b}T_{(n-n')a}h\rangle E_{mb}T_{na}h
\end{aligned}$$

using the change of variables, $m'' = m - m'$ and $n'' = n - n'$ we get,

$$\begin{aligned}
M(E_{m'b}T_{n'a}g) &= \sum_{m'',n''\in\mathbb{Z}} \langle g, e^{2\pi i n' a m'' b} E_{m''b}T_{n''a}g\rangle E_{(m''+m')b}T_{(n''+n')a}h \\
&= \sum_{m'',n''\in\mathbb{Z}} \langle g, e^{2\pi i n' a m'' b} E_{m''b}T_{n''a}g\rangle E_{m'b}E_{m''b}T_{n'a}T_{n''a}h \\
&= \sum_{m'',n''\in\mathbb{Z}} \langle g, E_{m''b}T_{n''a}g\rangle E_{m'b}T_{n'a}E_{m''b}T_{n''a}h \\
&= E_{m'b}T_{n'a} \sum_{m'',n''\in\mathbb{Z}} \langle g, E_{m''b}T_{n''a}g\rangle E_{m''b}T_{n''a}h \\
&= E_{m'b}T_{n'a}M(g), \quad \text{from the definition of } V.
\end{aligned}$$

That is $M(E_{m'b}T_{n'a}g) = E_{m'b}T_{n'a}M(g)$, for all $m', n' \in \mathbb{Z}$. Hence we conclude that M commutes with $E_{mb}T_{na}$. □

Conflicts of Interest: The authors declare no conflict of interest.

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