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ON THE STRUCTURE OF SETS OF ELEMENTS OF ε-APPROXIMATION AND ε-FARTHEST POINTS

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Abstract

Given a non-empty subset G of a metric space (X, d), $x \in X$ and $\varepsilon > 0$, we say that $g_o \in G$ is an ε-approximation to x if $d(x, g_o) \leq d(x, g) + \varepsilon$ for all $g \in G$. Analogously, for a given non-empty bounded subset K of a metric space (X, d), $x \in X$ and $\varepsilon > 0$, an element $k_o \in K$ is called an $\varepsilon-$farthest point to x if $d(x, k_o) \geq d(x, k) - \varepsilon$ for all $k \in K$. Let $P_{G,\varepsilon}(x) = \{g_o \in G : d(x, g_o) \leq d(x, G) + \varepsilon\}$ and $F_{K,\varepsilon}(x) = \{k_o \in K : d(x, k_o) \geq \delta(x, K) - \varepsilon\}$, where $d(x, G) \equiv \inf\{d(x, g) : g \in G\}$ and $\delta(x, K) \equiv \sup\{d(x, k) : k \in K\}$. The paper mainly deals with the structure of these two sets and the related sets. A relationship between elements of $\varepsilon-$approximation and $\varepsilon-$farthest points is also given in the paper. The underlying spaces are metric spaces, convex metric spaces and metric linear spaces.

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1. Introduction

For a non-empty subset G of a metric space (X, d), $x \in X$ and $\varepsilon > 0$, we say that $g_o \in G$ is an $\varepsilon-$approximation or good approximation to x if $d(x, g_o) \leq d(x, G) + \varepsilon$ where $d(x, G) \equiv \inf\{d(x, g) : g \in G\}$ (for $\varepsilon = 0$, such a g_o is a best approximation to x in G). The set of all $\varepsilon-$approximations to x in G is denoted by $P_{G,\varepsilon}(x)$. Analogously, for a given non-empty bounded subset K of a metric space (X, d), $x \in X$ and $\varepsilon > 0$, an element $k_o \in K$ is called an $\varepsilon-$farthest point or nearly farthest point to x if $d(x, k_o) \geq \delta(x, K) - \varepsilon$, where $\delta(x, K) \equiv \sup\{d(x, k) : k \in K\}$ (for $\varepsilon = 0$, such a k_o is a farthest point to x in K). The set of all $\varepsilon-$farthest points to x in K is denoted by $F_{K,\varepsilon}(x)$. The two sets $P_{G,\varepsilon}(x)$ and $F_{K,\varepsilon}(x)$ are always non-empty as the elements of $\varepsilon-$approximations and $\varepsilon-$farthest points always exist.

R. C. Buck([2]) was first to introduce and discuss the notion of $\varepsilon-$approximation and subsequently, the study was taken up in [8], [9], [10], [13], [14] and in references cited therein. The notion of $\varepsilon-$farthest points was introduced and discussed by Narang [6] and subsequently, taken up in [3], [6] , [7] and in references cited therein. In this paper, we discuss the structure of the sets of elements of $\varepsilon-$approximations, $\varepsilon-$farthest points and related sets. We also discuss a relationship between elements of $\varepsilon-$approximation and $\varepsilon-$farthest points. The underlying spaces are metric spaces, convex metric spaces and metric linear spaces.

2. Notations, Definitions and Preliminaries

Let (X, d) be a metric space, $x \in X$, $r \in \mathbb{R}^+$ and $\varepsilon > 0$. Let $B[x, r+\varepsilon] \equiv \{y \in X : d(x, y) \le r+\varepsilon\}$, $B^c[x, r+\varepsilon] \equiv \{y \in X : d(x, y) \ge r+\varepsilon\}$, $[x, y] \equiv \{z \in X : d(x, z) + d(z, y) = d(x, y)\} \equiv$ the line segment joining x and y, $[x, y, ...\rangle \equiv \{z \in X : d(x, y) + d(y, z) = d(x, z)\} \equiv$half ray starting from x and passing through y i.e. it is the union of segments $[x, z]$, where $[x, y] \subseteq [x, z]$.

Definition 2.1. Let A be a subset of a metric space (X, d) and $\varepsilon > 0$. A point $a_o \in A$ is said to be an $\varepsilon-$ **approximation** for $x \in X$ if $d(x, a_o) \le d(x, a) + \varepsilon$ for all $a \in A$ i.e. $d(x, a_o) \le d(x, A) + \varepsilon$. For $x \in X$, the set of all $\varepsilon-$ approximations of x in A is denoted by $P_{A,\varepsilon}(x)$ i.e.

$$\begin{aligned} P_{A,\varepsilon}(x) &= \{a_o \in A : d(x, a_o) \le d(x, a) + \varepsilon \text{ for all } a \in A\} \\ &= \{a_o \in A : d(x, a_o) \le d(x, A) + \varepsilon\} \\ &= A \cap B[x, d(x, A) + \varepsilon]. \end{aligned}$$

Also, $P_A(x) = \bigcap_{\varepsilon>0} P_{A,\varepsilon}(x)$, where

$$\begin{aligned} P_A(x) &= A \cap B[x, d(x, A)] \\ &= A \cap S[x, d(x, A)]. \end{aligned}$$

In normed linear spaces $P_A(x) = \partial A \cap S[x, d(x, A)]$ but it need not be true in metric spaces(see [14]) p-381).

Definition 2.2. A metric space (X, d) is called a **metric linear space** if (i) X is a linear space (ii) addition and scalar multiplication in X are continuous, and (iii) d is translation invariant i.e. $d(x+z, y+z) = d(x, y)$ for all $x, y, z \in X$.

Every normed linear space is a metric linear space but there are plenty of metric linear spaces which are not normed linear spaces (see [11]).

For a linear subspace A of a metric linear space (X, d), we define

$$P^{-1}_{A,\varepsilon}(0) = \{x \in X : 0 \in P_{A,\varepsilon}(x)\}.$$

Similarly, $P^{-1}_{A,\varepsilon}(a_o) = \{x \in X : a_o \in P_{A,\varepsilon}(x)\}$.

Definition 2.3. Suppose (X, d) is a metric linear space, $x, y \in X$ and $\varepsilon > 0$. We say that x is ε- **orthogonal** to y, $x \perp_\varepsilon y$ if

$$d(x, 0) \le d(x + \alpha y, 0) + \varepsilon$$

for every scalar $\alpha \in \mathbb{R}$. For a subset W of X, we define

$$W^\perp_\varepsilon = \{x \in X : x \perp_\varepsilon w \text{ for every } w \in W\}.$$

We call $W^\perp_\varepsilon$ to be $\varepsilon-$complemented orthogonal set to W.

The following results follow from the definitions :

1. $P_{A,\varepsilon}(x)$ is a bounded set in a metric space.
2. For a convex set A in a Takahashi convex metric space (X, d, W) (see [15]), the set $P_{A,\varepsilon}(x)$ is convex.
   In metric linear spaces, we have
3. $a_o \in P_{A,\varepsilon}(x) \Leftrightarrow 0 \in P_{A,\varepsilon}(x - a_o)$,
4. $a_o \in A$ is an ε-approximation to $x \in X$ if and only if $(x - a_o) \perp_\varepsilon A$ i.e. $(x - a_o) \in \hat{A}_\varepsilon \equiv \{x \in X : x \perp_\varepsilon A\}$,
5. $\bigcap_{\varepsilon>0} \hat{A}_\varepsilon = \hat{A}$, where $\hat{A} = \{x \in X : d(x, 0) = d(x, A)\}$,
6. $P_{A,\varepsilon}(x) = A \cap (x - \hat{A}_\varepsilon)$.

Definition 2.4. Let (X, d) be a metric space, $x \in X$, A a bounded subset of X and $\varepsilon > 0$. An element $a_o \in A$ is called an **ε-farthest point** or **nearly farthest point up to** ε of A from x if $d(x, a_o) \geq \delta(x, A) - \varepsilon$. The set of all ε-farthest points of A from x is denoted by $F_{A,\varepsilon}(x)$.

Definition 2.5. A metric space (X, d) is said to be

1. **externally convex** [4] if for every two distinct elements $x, y \in X$ with $d(x, y) = \lambda$, and $r > \lambda$ there exists a $z \in X$ such that $d(x, y) + d(y, z) = d(x, z) = r$.
2. **Convex Space or M-Space** [4] if for every two distinct points $x, y \in X$ and for every $0 \leq \lambda \leq 1$, there exists a $z \in X$ such that $d(x, z) = (1 - \lambda)d(x, y)$ and $d(z, y) = \lambda d(x, y)$.

Every normed linear space is a convex metric space but there are many convex metric spaces which are not normed linear spaces (see [4], [15]). A normed linear space need not be externally convex and externally convex metric space need not be a normed linear space(see [4]).

3. On $\varepsilon-$Approximation and $\varepsilon-$Farthest Points

The following theorem gives an interesting relationship between elements of $\varepsilon-$approximation and $\varepsilon-$farthest points:

Theorem 3.1. *Let W be a non-empty bounded closed subset of a convex metric space (X, d) which is also externally convex and $\varepsilon > 0$. If $z \in F_{W,\varepsilon}(x)$, then $z \in P_{W,\varepsilon}(x\prime)$ for every $x\prime \in [x, z, ...\rangle \setminus [x, z]$.*

Proof. Suppose $z \in F_{W,\varepsilon}(x)$, then for every $y \in W$,

$$\begin{aligned} d(z, x) &\geq \delta(x, W) - \varepsilon \\ &\geq d(x, y) - \varepsilon. \end{aligned}$$

Suppose $x\prime \in [x, z, ...\rangle \setminus [x, z]$ be arbitrary. Consider

$$\begin{aligned} d(x\prime, z) &= d(x\prime, x) - d(x, z) \\ &\leq d(x\prime, y) + d(y, x) - d(x, y) + \varepsilon \text{ for every } y \in W \\ &= d(x\prime, y) + \varepsilon \text{ for every } y \in W. \end{aligned}$$

This implies $d(x\prime, z) \leq d(x\prime, W) + \varepsilon$, i.e. $z \in P_{W,\varepsilon}(x\prime)$. □

For $\varepsilon = 0$, we obtain Proposition 3.1 of [12].

Theorem 3.2. *If W is a bounded subset of an externally convex metric space (X, d), $g_o \in F_{W,\varepsilon}(x_o)$ for $x_o \in X$ and $\varepsilon > 0$, then $g_o \in F_{W,\varepsilon}(x_\lambda)$, where $x_\lambda \in [g_o, x_o, ...\rangle$.*

Proof. Let $g_o \in F_{W,\varepsilon}(x_o)$. Consider

$$\begin{aligned} d(x_\lambda, g_o) &= d(x_\lambda, x_o) + d(x_o, g_o) \text{ for } x_\lambda \in [g_o, x_o, ...\rangle \\ &\geq d(x_\lambda, x_o) + \delta(x_o, W) - \varepsilon. \end{aligned}$$

This gives

$$\begin{aligned} d(x_\lambda, g_o) &\geq d(x_\lambda, x_o) + d(x_o, g) - \varepsilon \text{ for every } g \in W \\ &= d(x_\lambda, g) - \varepsilon \text{ for every } g \in W, \end{aligned}$$

and so, $d(x_\lambda, g_o) \geq \delta(x_\lambda, W) - \varepsilon$ i.e. $g_o \in F_{W,\varepsilon}(x_\lambda)$. □

Theorem 3.3. *If A is non-empty subset of a metric space (X, d), $x \in X$ and $\varepsilon > 0$, then $a_o \in P_{A,\varepsilon}(x)$ if and only if $a \in B^c[x, d(x, a_o) - \varepsilon]$ for each $a \in A$.*

Proof. Let $a_o \in P_{A,\varepsilon}(x)$. If there were an $a_1 \in A$ such that $a_1 \notin B^c[x, d(x, a_o) - \varepsilon]$ then $d(x, a_1) < d(x, a_o) - \varepsilon$ or $d(x, a_o) - \varepsilon > d(x, a_1) \geq \inf_{a \in A} d(x, a) = d(x, A)$. This implies $d(x, a_o) > d(x, A) + \varepsilon$, which is a contradiction.

Conversely, suppose $a \in B^c[x, d(x, a_o) - \varepsilon]$ for each $a \in A$. If $a_o \notin P_{A,\varepsilon}(x)$, then $d(a_o, x) > d(x, A) + \varepsilon$. Therefore, there is some $a_1 \in A$ such that $d(a_o, x) - \varepsilon > d(x, a_1)$ i.e. $d(x, a_1) < d(a_o, x) - \varepsilon$ and so $a_1 \notin B^c[x, d(x, a_o) - \varepsilon]$), which is a contradiction. □

Theorem 3.4. *If A is non-empty bounded subset of a metric space (X, d), $x \in X$ and $\varepsilon > 0$, then $a_o \in F_{A,\varepsilon}(x)$ if and only if $a \in B[x, d(x, a_o) + \varepsilon]$ for each $a \in A$.*

Proof. Let $a_o \in F_{A,\varepsilon}(x)$. If there were an element $a_1 \in A$ such that $a_1 \notin B[x, d(x, a_o) + \varepsilon]$, then $d(a_o, x) + \varepsilon < d(a_1, x) \leq \delta(A, x)$ or $d(a_o, x) < \delta(A, x) - \varepsilon$, which is a contradiction.

Conversely, suppose $a \in B[x, d(x, a_o) + \varepsilon]$ for each $a \in A$. If $a_o \notin F_{A,\varepsilon}(x)$, then $d(a_o, x) < \delta(A, x) - \varepsilon$, so by the definition of $\delta(A, x)$, there is some $a_1 \in A$, such that $d(a_o, x) + \varepsilon < d(a_1, x)$ i.e. $a_1 \notin B[x, d(x, a_o) + \varepsilon]$, which is a contradiction. □

THEOREM 3.5. *Suppose (X, d) be a metric space, W is a non-empty subset of X, $x \in X$ and $\varepsilon > 0$. If $\{r_n\}$ is a sequence of real numbers converging to $d(x, W)$ and $r_n > d(x, W)$, then*

$$P_{W,\varepsilon}(x) = \bigcap_{n=1}^{\infty} B[x, r_n + \varepsilon] \cap W$$

and

$$P_{W,\varepsilon}(x) = B[x, d(x, d(x, W) + \varepsilon)] \cap W.$$

PROOF. Suppose $z \in P_{W,\varepsilon}(x)$, then $d(x, z) \leq d(z, W) + \varepsilon$. Therefore $d(x, z) \leq r_n + \varepsilon$ for every n, that is, $z \in B[x, r_n + \varepsilon] \cap W$ for all n i.e. $z \in \bigcap_{n=1}^{\infty} B[x, d(x, r_n + \varepsilon)] \cap W$.

If $z \in \bigcap_{n=1}^{\infty} B[x, r_n + \varepsilon] \cap W$, then $d(x, z) \leq r_n + \varepsilon$. Since $r_n \to d(x, W)$ as $n \to \infty$, we have $d(x, z) \leq d(x, W) + \varepsilon$. Therefore $z \in P_{W,\varepsilon}(x)$.

By the definition it is clear that

$$P_{W,\varepsilon}(x) = B[x, d(x, d(x, W) + \varepsilon)] \cap W.$$

□

Taking $\varepsilon = 0$, we obtain

COROLLARY 3.6. *Let W be a non-empty subset of a metric space (X, d) and $< r_n >$ a decreasing sequence of positive real numbers converging to $d(x, W)$ and $r_n > d(x, W)$, then*

$$P_W(x) = \bigcap_{n=1}^{\infty} B[x, r_n] \cap W$$

and

$$P_W(x) = B[x, d(x, W)] \cap W.$$

THEOREM 3.7. *Suppose (X, d) is a metric space and W a nonempty subset of X, $w_o \in W$ and $\varepsilon > 0$. Then, there exists a sequence $\{r_n\}_{n \geq 1}$ such that $r_n \geq d(z, W)$ and $P^{-1}_{W,\varepsilon}(w_o) \subseteq \bigcap_{n=1}^{\infty} B[w_o, r_n + \varepsilon]$.*

PROOF. Suppose $z \in P^{-1}_{W,\varepsilon}(w_o)$, then $d(z, w_o) \leq d(z, W) + \varepsilon$. Set $r_n = d(z, W) + \frac{1}{n}$. Then $d(z, w_o) \leq r_n + \varepsilon$ for every n i.e. $z \in B[w_o, r_n + \varepsilon]$ for every n i.e. $z \in \bigcap_{n=1}^{\infty} B[w_o, r_n + \varepsilon]$.

Conversely, $z \in \bigcap_{n=1}^{\infty} B[w_o, r_n + \varepsilon]$ implies that $z \in B[w_o, r_n + \varepsilon]$ for every n. So, $d(z, w_o) \leq r_n + \varepsilon$ for every n. This implies $d(z, w_o) \leq d(z, W) + \frac{1}{n} + \varepsilon$ for every n and so $d(z, w_o) \leq d(z, W) + \varepsilon$, and this gives $z \in P^{-1}_{W,\varepsilon}(w_o)$. □

THEOREM 3.8. *Suppose (X, d) is a metric space. W is a non-empty bounded subset of X, $x \in X$ and $\varepsilon > 0$. Then for $x \in X$ and $w_o \in W$, if $w_o \in F_W(x)$, then*

$$F^{-1}_{W,\varepsilon}(w_o) \subseteq B^c[w_o, \delta(x, W) - \varepsilon]$$

Proof. We know that if $z \in F^{-1}_{W,\varepsilon}(w_o)$, then $d(z, w_o) \geq \delta(z, W) - \varepsilon$. Set $r_n = \delta(z, W) - \frac{1}{n}$ for every $n \geq 1$. Then $z \in B^c[w_o, r_n - \varepsilon]$ for every $n \geq 1$.

Conversely, let $x \in \bigcap_{n=1}^{\infty} B^c[w_o, r_n - \varepsilon]$. This implies that $x \notin B[w_o, r_n - \varepsilon]$ for any n. Therefore $d(x, w_o) > r_n - \varepsilon$ for every n, and so $d(x, w_o) > \delta(x, W) - \frac{1}{n} - \varepsilon$ for every n, this gives $d(x, w_o) \geq \delta(x, W) - \varepsilon$, and so $x \in F^{-1}_{W,\varepsilon}(w_o)$.

By the definition it is clear that

$$F_{W,\varepsilon}(x) = B^c[x, \delta(x, W) - \varepsilon] \cap W.$$

□

Theorem 3.9. *Let (X, d) be a metric linear space, W a subspace of X, $w_o \in W$ and $\varepsilon > 0$. Then*

$$X = W + P^{-1}_{W,\varepsilon}(w_o).$$

Proof. Let $x \in X$. Since the set $P_{W,\varepsilon}(x)$ is non-empty, there exists $y \in W$ such that $y \in P_{W,\varepsilon}(x)$ i.e. $d(x, y) \leq d(x, W) + \varepsilon = d(x, y + W) + \varepsilon$, which implies $d(x - y, 0) \leq d(x - y, W) + \varepsilon$, this gives $d(x - y + w_o, w_o) \leq d(x - y + w_o, W) + \varepsilon$. This implies $w_o \in P_{W,\varepsilon}(x - y + w_o)$ i.e. $x - y + w_o \in P^{-1}_{W,\varepsilon}(w_o)$. Put $u = x - y + w_o$, then $x = y - w_o + u \in W + P^{-1}_{W,\varepsilon}(w_o)$. Hence the result. □

Theorem 3.10. *If W is a linear subspace of a metric linear space (X, d), then $P^{-1}_{W,\varepsilon}(0) = W^{\perp}_{\varepsilon}$. Therefore for $w_o \in W$, we have $P^{-1}_{W,\varepsilon}(w_o) = w_o + W^{\perp}_{\varepsilon}$.*

Proof. It is clear that $P^{-1}_{W,\varepsilon}(w_o) = w_o + P^{-1}_{W,\varepsilon}(0)$. Now

$$\begin{aligned} x \in P^{-1}_{W,\varepsilon}(0) &\Leftrightarrow d(x, 0) \leq d(x, W) + \varepsilon \\ &\Leftrightarrow d(x, 0) \leq d(x, \alpha y) \text{ for all } y \in W \\ &\Leftrightarrow x \in W^{\perp}_{\varepsilon}. \end{aligned}$$

$$\begin{aligned} x \in P^{-1}_{W,\varepsilon}(w_o) &\Leftrightarrow w_o \in P_{W,\varepsilon}(x) \\ &\Leftrightarrow d(x, w_o) \leq d(x, W) + \varepsilon \\ &\Leftrightarrow d(x - w_o, 0) \leq d(x, W) + \varepsilon. \end{aligned} \tag{3.1}$$

$$\begin{aligned} x \in w_o + P^{-1}_{W,\varepsilon}(0) &\Leftrightarrow x - w_o \in P^{-1}_{W,\varepsilon}(0) \\ &\Leftrightarrow 0 \in P_{W,\varepsilon}(x - w_o) \\ &\Leftrightarrow d(x - w_o, 0) \leq d(x, W) + \varepsilon. \end{aligned} \tag{3.2}$$

(3.1) and (3.2) imply

$$\begin{aligned} P^{-1}_{W,\varepsilon}(w_o) &= w_o + P^{-1}_{W,\varepsilon}(0) \\ &= w_o + W^{\perp}_{\varepsilon}. \end{aligned}$$

□

Remark 3.11. *Theorems 3.1, 3.2, 3.5-3.10 were proved for normed linear spaces in* [5].

Theorem 3.12. *Let (X, d) be a metric linear space and W a subset of X, $x \in X$ and $\varepsilon > 0$. Then $X = \bigcup_{x \in X} H_{d_x^\varepsilon}$, where $H_{d_x^\varepsilon} = W + B[0, d_x^\varepsilon]$ and $d_x^\varepsilon = d(x, W) + \varepsilon$.*

Proof. We shall prove the result in two parts. (i) $X \subseteq \bigcup_{x \in X} H_{d_x^\varepsilon}$ and (ii) $X \supseteq \bigcup_{x \in X} H_{d_x^\varepsilon}$.
(i) First, let $x \in X$, then there exists a $w_o \in W$ such that $d(x, w_o) \leq d(x, W) + \epsilon$. Now, x can be written as $x = w_o + (x - w_o)$. Also, $d(x - w_o, 0) = d(x, w_o) \leq d(x, W) + \varepsilon = d_x^\varepsilon$ i.e. x is sum of two elements, w_o and $x - w_o$ where $w_o \in W$ and $x - w_o \in B[0, d_x^\varepsilon]$. Therefore $x \in H_{d_x^\varepsilon}$ i.e. $x \in \bigcup_{x \in X} H_{d_x^\varepsilon}$.
(ii) Now, let $y \in \bigcup_{x \in X} H_{d_x^\varepsilon}$ i.e. $y \in H_{d_x^\varepsilon}$ for at least one x, and so $y \in W + B[0, d_x^\varepsilon]$ for at least one x. This implies $y = w\prime + z$ such that $w\prime \in W$ and $z \in B[0, d_x^\epsilon]$. So $y = w\prime + z$ such that $w\prime \in W$ and $d(z, 0) \leq d(x, W) + \varepsilon$ and $z \in X$. Now $w\prime \in W \subseteq X$ and $z \in X$. This implies $y = w\prime + z \in X$. Therefore $\bigcup_{x \in X} H_{d_x^\varepsilon} \subseteq X$.
Hence the result. □

For $\epsilon = 0$, we have

Corollary 3.13. *Let (X, d) be a metric linear space and W a subset of X, $x \in X$. Then $X = \bigcup_{x \in X} H_{d_x}$, where $H_{d_x} = W + B[0, d_x]$ and $d_x = d(x, W)$.*

Theorem 3.14. *Let (X, d) be a metric linear space and W a bounded subset of X, $x \in X$ and $\varepsilon > 0$. Then $X = \bigcup_{x \in X} K_{\delta_x^\varepsilon}$, where $K_{\delta_x^\varepsilon} = W + B^c[0, \delta_x^\varepsilon]$ and $\delta_x^\varepsilon = \delta(x, K) - \varepsilon$.*

Proof. We shall prove the result in two steps: (i) $X \subseteq \bigcup_{x \in X} K_{\delta_x^\varepsilon}$ (ii) $X \supseteq \bigcup_{x \in X} K_{\delta_x^\varepsilon}$.
(i) Firstly, let $x \in X$, then there exists at least one $w_o \in W$ such that $d(x, w_o) \geq \delta(x, W) - \varepsilon$. Now x can be written as $x = w_o + (x - w_o)$. Also, $d(x - w_o, 0) = d(x, w_o) \geq \delta(x, W) - \varepsilon = \delta_x^\varepsilon$ i.e. x is sum of two elements, w_o and $x - w_o$, where $w_o \in W$ and $x - w_o \in B^c[0, \delta_x^\varepsilon]$. This implies $x \in W + B^c[0, \delta_x^\varepsilon]$ i.e. $x \in K_{\delta_x^\varepsilon}$ and so $x \in \bigcup_{x \in X} K_{\delta_x^\varepsilon}$.
(ii) Secondly, let $z \in \bigcup_{x \in X} K_{\delta_x^\varepsilon}$ i.e. $z \in K_{\delta_x^\varepsilon}$ for at least one x, and so $z \in W + B^c[0, \delta_x^\varepsilon]$ for at least one x. This implies $z = w\prime + z\prime$ such that $w\prime \in W$ and $z\prime \in B^c[0, \delta_x^\varepsilon]$. This gives $z = w\prime + z\prime$ such that $w\prime \in W$ and $d(z\prime, 0) \geq \delta_x^\varepsilon$, $z\prime \in X$.

Now $w\prime \in W \subseteq X$ and $z\prime \in X$ implies $z = w\prime + z\prime \in X$ i.e. $\bigcup_{x \in X} K_{\delta_x^\varepsilon} \subseteq X$. □

For $\epsilon = 0$ we obtain

Corollary 3.15. *Let (X, d) be a metric linear space and W a bounded subset of X, $x \in X$. Then $X = \bigcup_{x \in X} K_{\delta_x}$, where $K_{\delta_x} = W + B^c[0, \delta_x]$ and $\delta_x = \delta(x, K)$.*

Remark 3.16. *For normed linear spaces, Theorems 3.12 and 3.14 were proved in* [1].

THEOREM 3.17. *Let K be a non-empty bounded subset of a metric space (X, d), $x \in X$ and $\varepsilon > 0$. If $< r_n >$ is an increasing sequence of positive real numbers converging to $\delta(x, K)$ and $r_n < \delta(x, K)$, then*

$$F_{K,\varepsilon}(x) = \bigcap_{n=1}^{\infty} B^c[x, r_n - \varepsilon] \cap K.$$

PROOF. Suppose $k_o \in F_{K,\varepsilon}(x)$, then $d(x, k_o) \geq \delta(x, K) - \varepsilon$, and so $d(x, k_o) > r_n - \varepsilon$ for all $n \geq 1$. Therefore $k_o \in \bigcap_{n=1}^{\infty} B^c[x, r_n - \varepsilon] \cap K$.

Conversely, suppose $k_o \in \bigcap_{n=1}^{\infty} B^c[x, r_n - \varepsilon] \cap K$, then $d(x, k_o) \geq r_n - \varepsilon$ for all $n \geq 1$ and $k_o \in K$. This in the limiting case gives $d(x, k_o) \geq \delta(x, K)$, i.e. $k_o \in F_{K,\varepsilon}(x)$.

□

For $\varepsilon = 0$ we obtain

COROLLARY 3.18. *Let K be a non-empty bounded subset of a metric space (X, d) and $< r_n >$ an increasing sequence of positive real numbers converging to $\delta(x, K)$ and $r_n < \delta(x, K)$, then*

$$F_K(x) = \bigcap_{n=1}^{\infty} B^c[x, r_n] \cap K.$$

THEOREM 3.19. *Suppose K is a non-empty bounded subset of a metric space (X, d), $k_o \in K$ and $\varepsilon > 0$. If there exists a sequence $< r_n >$ such that $r_n < \delta(x, K)$, then $F^{-1}_{K,\varepsilon}(k_o) = \bigcap_{n=1}^{\infty} B^c[k_o, r_n - \varepsilon]$.*

PROOF. Let $x \in F^{-1}_{K,\varepsilon}(k_o)$. Then $k_o \in F_{K,\varepsilon}(x)$ and so $d(x, k_o) \geq \delta(x, K) - \varepsilon$. Set $r_n = \delta(x, K) - \frac{1}{n}$. Then $x \in \bigcap_{n=1}^{\infty} B^c[k_o, r_n - \varepsilon]$.

Conversely, let $x \in \bigcap_{n=1}^{\infty} B^c[k_o, r_n - \varepsilon]$. This implies $x \in B^c[k_o, r_n - \varepsilon]$ for every n i.e. $d(x, k_o) \geq r_n - \varepsilon$ for every n. So $d(x, k_o) \geq \delta(x, K) - \frac{1}{n} - \varepsilon$ for every n by setting $r_n = \delta(x, K) - \frac{1}{n}$. This in the limiting case gives $d(x, k_o) \geq \delta(x, K) - \varepsilon$ i.e. $x \in F^{-1}_{K,\varepsilon}(k_o)$. □

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