*GANITA,Vol.74(2), 2024, 201-209*

# FOURIER SERIES OF GENERALIZED H-FUNCTION

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**Abstract**

The aim of this paper is to evaluate two integrals involving generalization of the H-function and employ them to obtain two Fourier Series for the generalization of the H-function. Some Fourier series for MeijerâĂŹs G-function, MacRobert function are included as particular cases.

2010 *Mathematics subject classification:* 33C60, 42B05, 42C10.
*Keywords and phrases:* H-Function, G-Function ,Fourier Series.

## 1. Introduction

An extension of the H-function of several complex variables introduced by H.M. Srivastava and R. Panda [9]. The function is represented and defined as

$$H[z_1, z_2, ..., z_r] =$$

$$H^{M,N;m_1,n_1;...;m_r,n_r}_{P,Q;p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix}(a_j;\alpha_{1j},...,\alpha_{rj})_{(1,P)};(c_{1j};C_{1j})_{(1,p_1)};...;(c_{rj};C_{rj})_{(1,p_r)} \\ \\ (b_j;\beta_{1j},...,\beta_{rj})_{(1,Q)};(d_{1j};D_{1j})_{(1,q_1)};...;(d_{rj};D_{rj})_{(1,q_r)}\end{matrix}\right.\right] \tag{1.1}$$

$$= \frac{1}{2\pi i}\int_{\lambda_1}..\int_{\lambda_r}\phi_1(s_1)....\phi_r(s_r)\psi(s_1,s_2,...,s_r)z_1^{s_1}z_2^{s_2}...z_r^{s_r}dz_1dz_2...dz_r$$

Where $i = (-1)^{\frac{1}{2}}$ and

$$\psi(s_1, s_2, ..., s_r) = \frac{\prod_{j=1}^{M}\Gamma\left(b_j - \sum_{i=1}^{r}\beta_{ij}s_i\right)\prod_{j=1}^{N}\Gamma\left(1 - a_j + \sum_{i=1}^{r}\alpha_{ij}s_i\right)}{\prod_{j=1}^{Q}\Gamma\left(1 - b_j + \sum_{i=1}^{r}\beta_{ij}s_i\right)\prod_{j=N+1}^{P}\Gamma\left(a_j - \sum_{i=1}^{r}\alpha_{ij}s_i\right)} \tag{1.2}$$

$$\phi_i(s_i) = \frac{\prod_{j=1}^{m_i}\Gamma\left(d_{ij} - D_{ij}s_i\right)\prod_{j=1}^{n_i}\Gamma\left(1 - c_{ij} + C_{ij}s_i\right)}{\prod_{j=m_i+1}^{q_i}\Gamma\left(1 - d_{ij} + D_{ij}s_i\right)\prod_{j=n_i+1}^{p_i}\Gamma\left(c_{ij} - C_{ij}s_i\right)} \tag{1.3}$$

$\begin{cases} a_j, [j = 1, 2, ..., P]; c_{ij}, [j = 1, 2, ..., p_i; & \forall i \in \{1, 2, ..., r\}] \\ b_j, [j = 1, 2, ..., Q]; d_{ij}, [j = 1, 2, ..., q_i; & \forall i \in \{1, 2, ..., r\}] \end{cases}$ are complex numbers, and their corresponding related coefficients

$\begin{cases} \alpha_{ij}, [j = 1, 2, ..., P]; C_{ij}, [j = 1, 2, ..., p_i; & \forall i \in \{1, 2, ..., r\}] \\ \beta_{ij}, [j = 1, 2, ..., Q]; D_{ij}, [j = 1, 2, ..., q_i; & \forall i \in \{1, 2, ..., r\}] \end{cases}$ are positive real numbers such that

$$\Delta_i \equiv \sum_{j=1}^{P} \alpha_{ij} + \sum_{j=1}^{p_i} C_{ij} - \sum_{j=1}^{Q} \beta_{ij} - \sum_{j=1}^{q_i} D_{ij} \leq 0 \tag{1.4}$$

$$\delta_i \equiv \sum_{j=N+1}^{P} \alpha_{ij} + \sum_{j=1}^{n_i} C_{ij} - \sum_{j=n_i+1}^{p_i} C_{ij} - \sum_{j=1}^{Q} \beta_{ij} + \sum_{j=1}^{m_i} D_{ij} - \sum_{j=m_i+1}^{q_i} D_{ij} > 0 \tag{1.5}$$

$\forall i \in \{1, 2, ..., r\}$

where $M, N, P, Q; m_i, n_i; p_i, q_i$ are positive integer and restricted by the $0 \leq N \leq P$; $0 \leq M \leq Q$, and $q_i \geq m \geq 0$; $p_i \geq n_i \geq 0$, $\forall i \in \{1, 2, ..., r\}$ and inequalities(1.4) hold for suitably constrained values of the complex variables $z_1, z_2, ..., z_r$.

The multi-variable H-function represented in terms of Melline-Barnes type contour integral (1.1), is convergence absolutely under the conditions (1.5) [H.M. Srivastava and Panda (1976b), p.130 Eq.9(1.3)][6].

when

$$\left|\arg(z_i)\right| < \frac{\delta_i \pi}{2} \qquad \forall i \in \{1, 2, ..., r\}$$

the points $z_i = 0$, $i = 1, 2, .., r$ and many exceptional parameter values, being tacitly excluded.

## 2. Preliminaries

From the table of integrals (Gradshteyin and Ryzhik, 2007;397, equation 5,12) we are required the following formulae in our proof:

$$\int_0^\pi (\sin\theta)^{2\lambda} \sin(2n+1)\theta d\theta = \frac{(-1)^n \sqrt{\pi}\Gamma\left(\frac{1}{2}+\lambda\right)\Gamma(\lambda+1)}{\Gamma\left(\frac{1}{2}+\lambda-n\right)\Gamma\left(\frac{3}{2}+\lambda+n\right)}, \qquad \text{for } R(\lambda) > -\frac{1}{2} \tag{2.1}$$

$$\int_0^\pi (\sin\theta)^{2\lambda} \cos 2n\theta d\theta = \frac{(-1)^n \sqrt{\pi}\Gamma\left(\frac{1}{2}+\lambda\right)\Gamma(\lambda+1)}{\Gamma(1+\lambda-n)\Gamma(1+\lambda+n)}, \qquad \text{for } R(\lambda) > -\frac{1}{2} \tag{2.2}$$

Equation (2.1) and (2.2) also can be find out by formula (Erdelyi A., p.12 eq.29)[4]

$$\int_0^\pi (\sin\theta)^{2\lambda} e^{2im\theta} d\theta = \frac{\pi}{2^{2\lambda}} \frac{\Gamma(1+2\lambda)e^{im\theta}}{\Gamma(1+\lambda+m)\Gamma(1+\lambda-m)}, \quad \text{for } R(\lambda) > -1 \tag{2.3}$$

## 3. Main Result

In this section we obtain certain integral with the help of (2.1) and (2.2).

### 3.1. First Integral:

$$\int_0^{\pi} (\sin\theta)^{2\lambda} \sin(2n+1)\theta\, H^{M,N;m_1,n_1;\dots;m_r,n_r}_{P,Q;p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix} z_1 \sin^{2u_1}\theta \\ \vdots \\ z_r \sin^{2u_r}\theta \end{matrix}\middle| \begin{matrix} (a_j;\alpha_{1j},\dots,\alpha_{rj})_{(1,P)};(c_{1j};C_{1j})_{(1,p_1)};\dots;(c_{rj};C_{rj})_{(1,p_r)} \\ (b_j:\beta_{1j},\dots,\beta_{rj}t)_{(1,Q)};(d_{1j};D_{1j})_{(1,q_1)};\dots;(d_{rj};D_{rj})_{(1,q_r)} \end{matrix}\right] d\theta$$

$$= (-1)^n \sqrt{\pi}\, H^{M,N+2;m_1,n_1;\dots;m_r,n_r}_{P+2,Q+2;p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} \left(-\lambda+\frac{1}{2};u_{ij},\dots,u_{rj}\right);\left(-\lambda;u_{ij},\dots,u_{rj}\right);\left(a_j;\alpha_{1j},\dots,\alpha_{rj}\right)_{(3,P+2)}\left(c_{1j};C_{1j}\right)_{(1,p_1)};\dots;\left(c_{rj};C_{rj}\right)_{(1,p_r)} \\ \left(b_j;\beta_{ij},\dots,\beta_{rj}\right)_{(1,Q)};\left(-\lambda\pm\frac{1}{2}\pm n;u_{ij},\dots,u_{rj}\right);\left(d_{1j};D_{1j}\right)_{(1,q_1)};\dots;\left(d_{rj};D_{rj}\right)_{(1,q_r)} \end{matrix}\right] \tag{3.1}$$

Where $u_{1j}, u_{2j}, \dots, u_{rj}$ are positive numbers, $n = 0, 1, 2, ..$ and convergence for

$$\Delta_i \equiv \sum_{j=1}^{P}\alpha_{ij} + \sum_{j=1}^{p_i} C_{ij} - \sum_{j=1}^{Q}\beta_{ij} - \sum_{j=1}^{q_i} D_{ij} \le 0$$

$$\delta_i \equiv \sum_{j=N+1}^{P}\alpha_{ij} + \sum_{j=1}^{n_i} C_{ij} - \sum_{j=n_i+1}^{p_i} C_{ij} - \sum_{j=1}^{Q}\beta_{ij} + \sum_{j=1}^{m_i} D_{ij} - \sum_{j=m_i+1}^{q_i} D_{ij} > 0$$

and $\left|\arg z_i\right| < \frac{\pi\delta_i}{2}; \forall\, i \in \{1, 2, \dots, r\}$ where $\delta_i$ defined in (1.5),

$$\mathrm{Re}\left(\Delta_i + \sum_{i=1}^{r} u_{ij}\frac{d_{ij}}{D_{ij}}\right) > \frac{1}{2}\,(1 \le j \le m_k)$$

### 3.2. Second Integral:

$$\int_0^{\pi} (\sin\theta)^{2\lambda} \cos(2n\theta) H^{M,N;m_1,n_1;\dots;m_r,n_r}_{P,Q;p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix} z_1 \sin^{2u_1}\theta \\ \vdots \\ z_r \sin^{2u_r}\theta \end{matrix}\middle| \begin{matrix} \left(a_j;\alpha_{1j},\dots,\alpha_{rj}\right)_{(1,P)};\left(c_{1j};C_{1j}\right)_{(1,p_1)};\dots;(c_{rj};C_{rj})_{(1,p_r)} \\ \left(b_j;\beta_{ij},\dots,\beta_{rj}\right)_{(1,Q)};\left(d_{1j};D_{1j}\right)_{(1,q_1)};\dots;(d_{rj};D_{rj})_{(1,q_r)} \end{matrix}\right] d\theta$$

$$= (-1)^n \sqrt{\pi} H^{M,N+2;m_1,n_1;\dots;m_r,n_r}_{P+2,Q+2;p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} \left(\frac{1}{2}-\lambda;u_{ij},\dots,u_{rj}\right);\left(-\lambda;u_{ij},\dots,u_{rj}\right);(a_j;\alpha_{ij},\dots,\alpha_{rj})_{(1,P+2)};(c_{1j};C_{1j})_{(1,p_1};\dots;(c_{rj};C_{rj})_{(1,p_r)} \\ (b_j;\beta_{ij},\dots,\beta_{rj})_{(1,Q+2)};\left(-\lambda\pm n;u_{ij},\dots,u_{rj}\right);(d_{1j};D_{1j})_{(1,q_1)};\dots;(d_{rj};D_{rj})_{(1,q_r)} \end{matrix}\right] \tag{3.2}$$

The conditions for convergence are same as above equation (3.1)

**3.3. Proof of First Integral:** To establish (3.1) we use the integration (2.1) and H-function of r-variables expressing in the Millen-Barnes type contour integral right-hand side of (1.1) and then due to the absolute convergence of the integrals, interchanging the order of integration, we have

$$= \frac{1}{(2\pi i)^r} \int_{\lambda_1} .. \int_{\lambda_r} \phi_1(s_1)...\phi_r(s_r)\, \psi(s_1, s_2, ..., s_r) \int_0^{\pi} \left\{ (\sin\theta)^{2\left(\lambda + \sum_{i=1}^{r} u_r s_r\right)} \sin(2n+1)\theta d\theta \right\}$$
$$z_1{}^{s_1} z_2{}^{s_2} ... z_r{}^{s_r} ds_1 ds_2 ... ds_r$$

Now evaluating the inner integral with the help (2.1)

$$\frac{\sqrt{\pi}(-1)^n}{(2\pi i)^r} \int_{L_1} .. \int_{L_r} \phi_1(s_1)...\phi_r(s_r)\, \psi(s_1, s_2, ..., s_r) \frac{\Gamma\left(\frac{1}{2} + \lambda + \sum_{i=1}^{r} u_r s_r\right) \Gamma\left(\lambda + \sum_{i=1}^{r} u_r s_r + 1\right)}{\Gamma\left(\frac{1}{2} + \lambda + \sum_{i=1}^{r} u_r s_r - n\right) \Gamma\left(\frac{3}{2} + \lambda + \sum_{i=1}^{r} u_r s_r + n\right)}$$
$$z_1^{s_1} z_2^{s_2} ... z_r^{s_r} ds_1 ds_2 ... ds_r$$

By applying (1.1) the we obtain formula (3.1).
The integral (3.1) can established by use the same processor on equation (1.1) and (2.2).

## 4. Particular Cases:

In (3.1) assuming $s \in z^+$ (positive integer), and putting $\beta_{1j} = \beta_{2j} ... = \beta_{rj} = 1$; $(j = 1, 2, ...Q)$, and $(C_{1j})_{(1,p_1)} = ... = (C_{rj})_{(1,p_r)} = (D_{1j})_{(1,q_1)} = ... = (D_{rj})_{(1,q_r)} = 1$; $\forall u_{ij} = 1$, then from equation (1.1)

$$H[z_1, z_2, ..., z_r] = G^{M,N;m_1,n_1;...;m_r,n_r}_{P,Q;p_1,q_1;...;p_r,q_r} \left[ \begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (a_j)_{(1,P)}; (c_{1j})_{(1,p_1)}; ...; (c_{rj})_{(1,p_r)} \\ \\ (b_j)_{1,Q}; (d_{1j})_{(1,q_1)}; ...; (d_{rj})_{(1,q_r)} \end{matrix} \right]$$

And simplifying with the help of (3.1)

$$\int_0^{\pi} \sin^{2\lambda}\theta \sin(2n+1)\theta\, G^{M,N;m_1,n_1;...;m_r,n_r}_{P,Q;p_1,q_1;...;p_r,q_r} \left[ \begin{matrix} z_1 \sin^2\theta \\ \vdots \\ z_r \sin^2\theta \end{matrix} \middle| \begin{matrix} (a_j)_{(1,P)}; (c_{1j})_{(1,p_1)}; ...; (c_{rj})_{(1,p_r)} \\ \\ (b_j)_{(1,Q)}; (d_{1j})_{(1,q_1)}; ...; (d_{rj})_{(1,q_r)} \end{matrix} \right] d\theta$$
$$= (-1)^n \sqrt{\pi}\, G^{0,N+2;m_1,n_1;...m_r,n_r}_{P+2,Q+2;p_1,q_1;...,p_r,q_r} \left[ \begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} \left(\frac{1}{2} - \lambda\right); (-\lambda); (a_j)_{(1,P+2)}; (c_{1j})_{(1,p_1)}; ...; (c_{rj})_{(1,p_r)} \\ \\ (b_j)_{(1,Q+2)}; \left(-\lambda \pm \frac{1}{2} \pm n\right); (d_{ij})_{(1,q_1)}; ...; (d_{rj})_{(1,q_r)} \end{matrix} \right] \tag{4.1}$$

Similarly with the help of (3.1)

$$\int_0^\pi \sin^{2\lambda}\theta\cos(2n\theta)\, G^{M,N;m_1,n_1;...;m_r,n_r}_{P,Q;p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1\sin^2\theta \\ \vdots \\ z_r\sin^2\theta \end{matrix}\middle|\begin{matrix} (a_j)_{(1,P)};(c_{1j})_{(1,p_1)};...;(c_{rj})_{(1,p_r)} \\ \\ (b_j)_{(1,Q)};(d_{1j})_{(1,q_1)};...;(d_{rj})_{(1,q_r)} \end{matrix}\right] d\theta$$
$$= (-1)^n\sqrt{\pi}\, G^{M,N+2;m_1,n_1;...m_r,n_r}_{P+2,Q+2;p_1,q_1;...,p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix} \left(\frac{1}{2}-\lambda\right);(-\lambda);(a_j)_{(1,P+2)};(c_{1j})_{(1,p_1)};...;(c_{rj})_{(1,p_r)} \\ \\ (b_j)_{(1,Q+2)};(-\lambda\pm n);(d_{1j})_{(1,q_1)};...;(d_{rj})_{(1,q_r)} \end{matrix}\right] \tag{4.2}$$

Put $r=1$; $M,N,P$, and $Q$ replacing by $p,1,q+1$, and $p$ respectively and $m_1=n_1=p_1=q_1=0$. Setting the parameters suitably in view of the formula, H-function (1) reduce to MacRoberts function as

$$E\left[\begin{matrix}(a_j,1)_{(1,p)}:z \\ (b_j,1)_{(1,q)}\end{matrix}\right] = H^{p,1}_{q+1,p}\left[z\middle|\begin{matrix}(1,1);(b_j,1)_{2,q+2} \\ (a_j,1)_{(1,p)}\end{matrix}\right] \tag{4.3}$$

From (6) the integral of MacRoberts function gives the result

$$\int_0^\pi (\sin\theta)^{2\lambda}\sin(2n+1)\theta\, E\left[\begin{matrix}(a_j,1)_{(1,p)}:z\sin^{2u}\theta \\ (b_j,1)_{(1,q)}\end{matrix}\right]d\theta$$
$$= (-1)^n\sqrt{\pi}\, H^{p,3}_{q+3,p+2}\left[z\middle|\begin{matrix}\left(\frac{1}{2}-\lambda;u\right);(-\lambda;u);(1,1);(b_j,1)_{(2,q+1)} \\ (\frac{1}{2}+n-\lambda;u);(-\frac{1}{2}-n-\lambda;u);(a_j,1)_{(1,p)}\end{matrix}\right] \tag{4.4}$$

From (7) the integral of MacRoberts function gives the result

$$\int_0^\pi (\sin\theta)^{2\lambda}\cos(2n\theta)\, E\left[\begin{matrix}(a_j,1)_{(1,p)}:z\sin^{2u}\theta \\ (b_j,1)_{(1,q)}\end{matrix}\right]d\theta$$
$$= (-1)^n\sqrt{\pi}\, H^{p,3}_{q+3,p+2}\left[z\middle|\begin{matrix}\left(\frac{1}{2}-\lambda;u\right);(-\lambda;u);(1,1);(b_j,1)_{(2,q+1)} \\ (n-\lambda;u);(-n-\lambda;u);(a_j,1)_{(1,p)}\end{matrix}\right] \tag{4.5}$$

## 5. The Fourier Series of H-function

### Fourier Sine Series of H-function

$$(\sin\theta)^{2\lambda} H^{M,N;m_1,n_1;\ldots;m_r,n_r}_{P,Q;p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \sin^{2u_1}\theta \\ \vdots \\ z_r \sin^{2u_r}\theta \end{matrix}\middle| \begin{matrix} \left(a_j;\alpha_{1j},\ldots,\alpha_{rj}\right)_{(1,P)}; \\ \\ \left(b_j;\beta_{ij},\ldots,\beta_{rj}\right)_{(1,Q)}; \end{matrix}\right.$$

$$\left.\begin{matrix} \left(c_{1j};C_{1j}\right)_{(1,p_1)};\ldots;(c_{rj};C_{rj})_{(1,p_r)} \\ \left(d_{1j};D_{1j}\right)_{(1,q_1)};\ldots;(d_{rj};D_{rj})_{(1,q_r)} \end{matrix}\right] d\theta$$

$$= \sum_{\rho=0}^{\infty} \frac{2(-1)^\rho}{\sqrt{\pi}} \sin(2\rho+1)\theta\, H^{0,N+2;m_1,n_1;\ldots;m_r,n_r}_{P+2,Q+2;p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} \left(-\lambda+\frac{1}{2};u_{1j},\ldots,u_{rj}\right); \\ \\ (b_j;\beta_{1j},\ldots,\beta_{rj})_{(1,Q+2)}; \end{matrix}\right.$$

$$\left.\begin{matrix} \left(-\lambda;u_{1j},\ldots,u_{rj}\right);(a_j;\alpha_{1j},\ldots,\alpha_{rj})_{(1,P+2)};(c_{1j};C_{1j})_{(1,p_1)};\ldots;(c_{rj};C_{rj})_{(1,p_r} \\ \left(-\lambda\pm\frac{1}{2}\pm\rho;u_{1j},\ldots,u_{rj}\right);(d_{1j};D_{1j})_{(1,q_1)};\ldots;(d_{rj};D_{rj})_{(1,q_r)} \end{matrix}\right] \tag{5.1}$$

$R(2\lambda) \geq 0,\ 0 \leq \theta \leq \pi$

### Fourier Cosine Series of H-function

$$(\sin\theta)^{2\lambda} H^{M,N;m_1,n_1;\ldots;m_r,n_r}_{P,Q;p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \sin^{2u_1}\theta \\ \vdots \\ z_r \sin^{2u_r}\theta \end{matrix}\middle| \begin{matrix} \left(a_j;\alpha_{1j},\ldots,\alpha_{rj}\right)_{(1,P)}; \\ \\ \left(b_j;\beta_{ij},\ldots,\beta_{rj}\right)_{(1,Q)}; \end{matrix}\right.$$

$$\left.\begin{matrix} \left(c_{1j};C_{1j}\right)_{(1,p_1)};\ldots;(c_{rj};C_{rj})_{(1,p_r)} \\ \left(d_{1j};D_{1j}\right)_{(1,q_1)};\ldots;(d_{rj};D_{rj})_{(1,q_r)} \end{matrix}\right] d\theta = \frac{(-1)^\rho}{\sqrt{\pi}} H^{M,N+2;m_1,n_1;\ldots;m_r,n_r}_{P+2,Q+2;p_1,q_1;\ldots;p_r,q_r}$$

$$\left[\begin{matrix} z_1 \\ \\ z_r \end{matrix}\middle| \begin{matrix} \left(\frac{1}{2}-\lambda;u_{1j},\ldots,u_{rj}\right);(a_j;\alpha_{1j},\ldots,\alpha_{rj})_{(1,P)};(c_{1j};C_{1j})_{(1,p_1)};\ldots;(c_{rj};C_{rj})_{(1,p_r} \\ \\ (b_j;\beta_{1j},\ldots,\beta_{rj})_{(1,Q+2)};\left(-\lambda;u_{1j},\ldots,u_{rj}\right);(d_{1j};D_{1j})_{(1,q_1)};\ldots;(d_{rj};D_{rj})_{(1,q_r)} \end{matrix}\right]$$

$$+ \sum_{\rho=1}^{\infty} \frac{2(-1)^\rho}{\sqrt{\pi}} \cos\rho\theta\, H^{M,N+2;m_1,n_1;\ldots;m_r,n_r}_{P+2,Q+2;p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \\ z_r \end{matrix}\middle| \begin{matrix} \left(\frac{1}{2}-\lambda;u_{1j},\ldots,u_{rj}\right); \\ \\ (b_j;\beta_{1j},\ldots,\beta_{rj})_{(1,Q+2)}; \end{matrix}\right.$$

$$\left.\begin{matrix} \left(-\lambda;u_{1j},\ldots,u_{rj}\right);(a_j;\alpha_{1j},\ldots,\alpha_{rj})_{(1,P)};(c_{1j};C_{1j})_{(1,p_1)};\ldots;(c_{rj};C_{rj})_{(1,p_r} \\ \left(-\lambda\pm\rho;u_{1j},\ldots,u_{rj}\right);(d_{1j};D_{1j})_{(1,q_1)};\ldots;(d_{rj};D_{rj})_{(1,q_r)} \end{matrix}\right] \tag{5.2}$$

$R(2\lambda) \geq 0, 0 \leq \theta \leq \pi;$

Where $u_{1j}, u_{2j}, ..., u_{rj}$ are positive numbers, and

$$\Delta_i \equiv \sum_{j=1}^{P} \alpha_{ij} + \sum_{j=1}^{p_i} C_{ij} - \sum_{j=1}^{Q} \beta_{ij} - \sum_{j=1}^{q_i} D_{ij} \leq 0$$

$$\delta_i \equiv \sum_{j=N+1}^{P} \alpha_{ij} + \sum_{j=1}^{n_i} C_{ij} - \sum_{j=n_i+1}^{p_i} C_{ij} - \sum_{j=1}^{Q} \beta_{ij} + \sum_{j=1}^{m_i} D_{ij} - \sum_{j=m_i+1}^{q_i} D_{ij} > 0$$

$\left|\arg z_i\right| < \frac{\pi\delta_i}{2} \qquad \forall i \in \{1, 2, ..., r\}$

**5.1. Proof:** To prove (5.1) Let

$$F(\theta) = (\sin\theta)^{2\lambda} H^{0,N;m_1,n_1;...;m_r,n_r}_{P,Q;p_1,q_1;...;p_r,q_r} \left[ \begin{matrix} z_1 \sin^{2u_1}\theta \\ \vdots \\ z_r \sin^{2u_r}\theta \end{matrix} \left| \begin{matrix} (a_j;\alpha_{1j},...,\alpha_{rj})_{(1,P)}; \\ \\ (b_j;\beta_{1j},...,\beta_{rj})_{(1,Q)}; \end{matrix} \right. \right.$$
$$\left. \begin{matrix} (c_{1j};C_{1j})_{(1,p_1)};...;(c_{rj};C_{rj})_{(1,p_r} \\ \\ (d_{1j};D_{1j})_{(1,q_1)};...;(d_{rj};D_{rj})_{(1,q_r)} \end{matrix} \right] = \sum_{\rho=0}^{\infty} A_\rho \sin(2\rho+1)\theta \tag{5.4}$$

The equation (5.1) is valid, because function $f(\theta)$ is continuous and bounded variation in the interval $(0,\pi)$, when $R(2\lambda) \geq 0$.
Multiplying by $\sin(2n+1)\theta$ in both sides of (5.1) and integrating between limits 0 to $\pi$ with respect to $\theta$ we get

$$\int_0^\pi (\sin\theta)^{2\lambda} \sin(2n+1)\, H^{0,N;m_1,n_1;...;m_r,n_r}_{P,Q;p_1,q_1;...;p_r,q_r} \left[ \begin{matrix} z_1 \sin^{2u_1}\theta \\ \vdots \\ z_r \sin^{2u_r}\theta \end{matrix} \left| \begin{matrix} (a_j;\alpha_{1j},...,\alpha_{rj})_{(1,P)}; \\ \\ (b_j;\beta_{ij},...,\beta_{rj})_{(1,Q)}; \end{matrix} \right. \right.$$
$$\left. \begin{matrix} (c_{1j};C_{1j})_{(1,p_1)};...;(c_{rj};C_{rj})_{(1,p_r} \\ \\ (d_{1j};D_{1j})_{(1,q_1)};...;(d_{rj};D_{rj})_{(1,q_r)} \end{matrix} \right] d\theta = \sum_{\rho=0}^{\infty} A_\rho \int_0^\pi \sin(2n+1)\sin(2\rho+1)\theta d\theta$$

now using the orthogonality property of trigonometric sine function and equation (3.1) and the, we find

$$A_\rho = \frac{(-1)^n 2}{\sqrt{\pi}} H^{M,N+2;m_1,n_1;...;m_r,n_r}_{P+2,Q+2;p_1,q_1;...;p_r,q_r} \left[ \begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \left| \begin{matrix} \left(-\lambda+\frac{1}{2};u_{1j},..,u_{rj}\right);\left(-\lambda;u_{1j},..,u_{rj}\right); \\ \\ (b_j;\beta_{1j},...,\beta_{rj})_{(1,Q+2)};\left(-\lambda \pm \frac{1}{2} \pm \rho;u_{1j},..,u_{rj}\right); \end{matrix} \right. \right.$$
$$\left. \begin{matrix} (a_j;\alpha_{1j},...,\alpha_{rj})_{(1,P)};(c_{1j};C_{1j})_{(1,p_1)};...;(c_{rj};C_{rj})_{(1,p_r)} \\ (d_{1j};D_{1j})_{(1,q_1)};...;(d_{rj};D_{rj})_{(1,q_r)} \end{matrix} \right] \tag{5.5}$$

From equation and (5.3) and (5.5) the result (5.1) is obtained.
To prove (5.2) let

$$f(\theta)=(\sin\theta)^{2\lambda}\,H^{M,N;m_1,n_1;...;m_r,n_r}_{P,Q;p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1\sin^{2u_1}\theta\\ \vdots\\ z_r\sin^{2u_r}\theta\end{matrix}\middle|\begin{matrix}(a_j;\alpha_{1j},...,\alpha_{rj})_{(1,P)};\\ \\ (b_j;\beta_{1j},...,\beta_{rj})_{(1,Q)};\end{matrix}\right.$$

$$\left.\begin{matrix}(c_{1j};C_{1j})_{(1,p_1)};...;(c_{rj};C_{rj})_{(1,p_r)}\\ \\ (d_{1j};D_{1j})_{(1,q_1)};...;(d_{rj};D_{rj})_{(1,q_r)}\end{matrix}\right]=\frac{B_0}{2}+\sum_{\rho=1}^{\infty}B_\rho\cos\rho\theta \tag{5.6}$$

Multiplying by $\cos n\theta$ in both side of equation (5.6) and integrating between limits from 0 to $\pi$ with respect to $\theta$ we find

$$\int_0^\pi \cos\theta\,(\sin\theta)^{2\lambda}\,H^{0,N;m_1,n_1;...;m_r,n_r}_{P,Q;p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1\sin^{2u_1}\theta\\ \vdots\\ z_r\sin^{2u_r}\theta\end{matrix}\middle|\begin{matrix}(a_j;\alpha_{1j},...,\alpha_{rj})_{(1,P)};\\ \\ (b_j;\beta_{1j},...,\beta_{rj})_{(1,Q)};\end{matrix}\right.$$

$$\left.\begin{matrix}(c_{1j};C_{ij})_{(1,p_1)};...;(c_{rj};C_{rj})_{(1,p_r)}\\ (d_{1j};D_{1j})_{(1,q_1)};...;(d_{rj};D_{rj})_{(1,q_r)}\end{matrix}\right]=\int_0^\pi\cos n\theta\left[\frac{B_0}{2}+\sum_{r=1}^{\infty}B_\rho\cos\rho\theta\right]d\theta \tag{5.7}$$

now using the orthogonality property of trigonometric cosine function and equation (3.1), we get

$$B_\rho=\frac{(-1)^\rho 2}{\sqrt{\pi}}H^{0,N+2;m_1,n_1;...;m_r,n_r}_{P+2,Q+2;p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}\left(\frac{1}{2}-\lambda;u_{ij},...,u_{rj}\right);\\ \\ (b_j;\beta_{1j},...,\beta_{rj})_{(1,Q+2)};\end{matrix}\right.$$

$$\left.\begin{matrix}(-\lambda;u_{ij},...,u_{rj});(a_j;\alpha_{1j},...,\alpha_{rj})_{(1,P+2)};(c_{1j};C_{1j})_{(1,p_1)};...;(c_{rj};C_{rj})_{(1,p_r)}\\ \left(-\lambda\pm\rho;u_{ij},...,u_{rj}\right);(d_{1j};D_{1j})_{(1,q_1)};...;(d_{rj};D_{rj})_{(1,q_r)}\end{matrix}\right]$$

From equation (5.7) and (5.8), the formula (5.2) is obtained.

## 6. Particular Cases of Fourier Series of H-function:

In (5.1) and (5.2) assuming $s\in z^+$(positive integer), putting $\alpha_{1j}=1=\alpha_{2j}=...\alpha_{rj}$, $(j=1,2,..,P)$; $\beta_{1j}=1=\beta_{2j}...=\beta_{rj}$, $(j=1,2,...Q)$ and $(C_{1j_{(1,p_1)}}=...=(C_{rj})_{(1,p_r)}=(D_{1j})_{(1,q_1)}=...=(D_{rj})_{(1,q_r)}=1$; $\forall u_{ij}=1$, two Fourier series for G-function of r-variable is obtained.

$$(\sin\theta)^{2\lambda}\,G^{M,N;m_1,n_1;...;m_r,n_r}_{P,Q;p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1\sin^2\theta\\ \vdots\\ z_r\sin^2\theta\end{matrix}\middle|\begin{matrix}(a_j)_{(1,P)};(c_{1j})_{(1,p_1)};...;(c_{rj})_{(1,p_r)}\\ \\ (b_j)_{(1,Q)};(d_{ij})_{(1,q_1)};...;(d_{rj})_{(1,q_r)}\end{matrix}\right]=\sum_{\rho=0}^{\infty}\frac{2(-1)^\rho}{\sqrt{\pi}}\times$$

$$\sin(2\rho+1)\theta G^{M,N+2;m_1,n_1;...;m_r,n_r}_{P+2,Q+2;p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}\left(-\lambda+\frac{1}{2}\right);(-\lambda);(a_j)_{(1,P+2)};(c_{1j})_{(1,p_1)};...;(c_{rj})_{(1,p_r)}\\ \\ (b_j)_{(1,Q+2)};\left(-\lambda\pm\frac{1}{2}\pm\rho\right);(d_{ij})_{(1,q_1)};...;(d_{rj})_{(1,q_r)}\end{matrix}\right] \tag{6.1}$$

and other is

$$(\sin\theta)^{2\lambda}\, G^{M,N;m_1,n_1;...;m_r,n_r}_{P,Q;p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1\sin^2\theta \\ \vdots \\ z_r\sin^2\theta \end{matrix}\left|\begin{matrix} (a_j)_{(1,P)};(c_{1j})_{(1,p_1)};...;(c_{rj(1,p_r)} \\ \\ (b_j)_{(1,Q)};(d_{ij})_{(1,q_1)};...;(d_{rj})_{(1,q_r)} \end{matrix}\right.\right] = \frac{B_0}{2} + \sum_{\rho=1}^{\infty} \frac{2(-1)^{\rho}}{\sqrt{\pi}} \times$$

$$\cos\rho\theta G^{G,N+2;m_1,n_1;...;m_r,n_r}_{P+2,Q+2;p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix} \left(\frac{1}{2}-\lambda\right);(-\lambda)(a_j)_{(1,P+2)}(c_{1j})_{(1,p_1)};...;(c_{rj})_{(1,p_r)} \\ \\ (b_j)_{(1,Q+2)};(-\lambda\pm\rho);(d_{1j})_{(1,q_1)};...;(d_{rj})_{1,q_r} \end{matrix}\right.\right] \tag{6.2}$$

**Conflicts of Interest:** The authors declare no conflict of interest.

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