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CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH RUSCHEWEYH-GOYAL DERIVATIVE OPERATOR

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Abstract

In our present investigation, we introduce two new subclasses of holomorphic and bi-univalent functions related to Ruscheweyh-Goyal derivative operator in the unit disc $\mathbb{D} = \{z : z \in \mathbb{C}, |z| < 1\}$. Also, we find the bounds for second and third Taylor-Maclaurin coefficient for the two new classes of bi-univalent functions. In additional we highlight a few unique applications of our findings.

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1. Introduction

Let $\mathbb{A}$ denote the collection of all holomorphic functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

which are defined in the open unit disc

$$\mathbb{D} = \{z : z \in \mathbb{C}, |z| < 1\}.$$

The class $\mathcal{S}$ be a subclass of $\mathbb{A}$ consisting a function of the type (1.1) which are normalized with the conditions

$$f(0) = 0, f'(0) = 1.$$

and also univalent in $\mathbb{D}$.
By the Koebe one-quarter theorem [6], every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z, \qquad (z \in \mathbb{D})$$

$$f(f^{-1}(w)) = w, \qquad (|w| < r_0(f), r_0(f) \geq 1/4),$$

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$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \tag{1.2}$$

A function $f(z)$ in the class $\mathbb{A}$ is said to be Bi-univalent function in $\mathbb{D}$ if both f and f^{-1} both are univalent in $\mathbb{D}$. Let $\sum$ denote the class of all bi-univalent functions in $\mathbb{D}$ given by the Taylor-Maclaurin series expansion (1.1). In fact, studies of holomorphic and bi-univalent functions have been resurrected in recent years thanks to Srivastava *et al.*[12]; efforts along these lines were followed by Frasin and Aouf [8]. It is evident that the class $\sum$ is not empty. Now some examples of functions for the class $\sum$ are

$$\frac{z}{(1-z)}, \quad \text{and} \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2}\log\frac{(1+z)}{(1-z)}$$

with the corresponding inverse functions

$$\frac{w}{(1+w)}, \quad \frac{(e^w - 1)}{e^w}, \quad \text{and} \quad \frac{e^{2w}-1}{e^{2w}+1} \quad \text{respectively.}$$

Now some examples of functions are not in class $\sum$ such as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}.$$

Definition 1.1. (Ruscheweyh-Goyal Derivative Operator)
The Ruscheweyh-Goyal derivative operator for the holomorphic univalent functions $f \in \mathbb{A}$ on unit disc $\mathbb{D}$ is defind as [2]

$$\mathbb{J}^{\lambda,\mu}(f(z)) = \frac{\Gamma(\mu-\lambda+\xi+2)}{\Gamma(\mu+1)\Gamma(\xi+2)} z\mathcal{J}_{0,z}^{\lambda,\mu,\xi}\left(z^{\mu-1}f(z)\right) = z + \sum_{n=k}^{\infty} a_n \mathcal{B}^{\lambda,\mu}(n) z^n \tag{1.3}$$

where

$$\mathcal{B}^{\lambda,\mu}(n) = \frac{\Gamma(n+\mu)\Gamma(n+\xi+1)\Gamma(\xi+2+\mu-\lambda)}{\Gamma(1+\mu)\Gamma(n)\Gamma(\xi+2)\Gamma(n+\xi+1+\mu-\lambda)},$$

where λ is order of Saigo fractional derivative for the function $f(z)$ is defined as:

$$\mathcal{J}_{0,z}^{\lambda,\mu,\xi} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\left\{z^{\lambda-\mu}\int_0^z (z-\varsigma)^{-\lambda}{}_2F_1(\mu-\lambda,-\xi;1-\lambda;1-\frac{\varsigma}{2})f(\varsigma)d\varsigma\right\}, \\ \qquad (1>\lambda\geq 0), \\ \frac{d^r}{dz^r}\mathcal{J}_{0,z}^{\lambda-r,\mu,\xi} f(z), (r \leq \lambda < r+1, r \in \mathbb{N}) \end{cases} \tag{1.4}$$

where $f(z) = O(|z|^n), (z \longrightarrow 0, \max\{0, \mu-\xi-1\} - 1 < n)$.
taking $\lambda = \mu$ in definition (1.4), we have

$$\mathcal{J}_{0,z}^{\lambda,\lambda,\xi} f(z) = D_z^{\lambda} f(z), (1 > \lambda \geq 0).$$

For $\lambda = \mu$, Rusecheweyh-Goyal Derivative Operator $\mathbb{J}^{\lambda,\mu}$ reduces to the Ruscheweyh derivative operator D^{λ}.

We need the following Lemma (1.2) due to [6], which will be utilized to establish our main results.

LEMMA 1.2. *[6] If $h \in \mathbb{P}$, then $|c_s| \leq 2$ for each $s \in \mathbb{N}$, where $\mathbb{P}$ is the family of all holomorphic functions h in unit disc $\mathbb{D}$ for which*

$$Re\,[h(z)] > 0, \quad (z \in \mathbb{D})$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \quad \textit{for} \quad z \in \mathbb{D}.$$

2. COEFFICIENT ESTIMATES FOR THE CLASS $\mathcal{K}_{\sum}(\varepsilon, \sigma, \rho, \lambda, \mu; \alpha)$

Throughout this section, we suppose that $\varepsilon \geq 0$, $\rho \geq 1$, $0 < \alpha \leq 1$, $\lambda \in \mathbb{N}_0$ and $\sigma \in \mathbb{C} \setminus \{0\}$.

DEFINITION 2.1. A function $f \in \sum$ defined as (1.1) is said to be in the class $\mathcal{K}_{\sum}(\varepsilon, \sigma, \rho, \lambda, \mu; \alpha)$, if it satisfied the following conditions:

$$\left|\arg\left(1 + \frac{1}{\sigma}\left[(1-\rho)\frac{\mathbb{J}^{\lambda,\mu} f(z)}{z} + \rho\left(\mathbb{J}^{\lambda,\mu} f(z)\right)' + \varepsilon z\left(\mathbb{J}^{\lambda,\mu} f(z)\right)'' - 1\right]\right)\right| < \frac{\alpha\pi}{2} \tag{2.1}$$

and

$$\left|\arg\left(1 + \frac{1}{\sigma}\left[(1-\rho)\frac{\mathbb{J}^{\lambda,\mu} g(w)}{w} + \rho\left(\mathbb{J}^{\lambda,\mu} g(w)\right)' + \varepsilon w\left(\mathbb{J}^{\lambda,\mu} g(w)\right)'' - 1\right]\right)\right| < \frac{\alpha\pi}{2} \tag{2.2}$$

where $f^{-1} = g$ is defined in (1.2) and $z, w \in \mathbb{D}$.

REMARK 2.2. *The family $\mathcal{K}_{\sum}(\varepsilon, \sigma, \rho, \lambda, \mu; \alpha)$ is a generalisation of known classes investigated previously. These are as follows:*
(1) $\lambda = \mu$ the class $\mathcal{K}_{\sum}(\varepsilon, \sigma, \rho, \lambda, \mu; \alpha)$ reduces to the class $Q_{\sum}(\varepsilon, \sigma, \rho, \lambda; \alpha)$ which is considered by Bulut and Wanas [5].
(2) $\lambda = \mu = 0$ and $\rho = \sigma = 1$ the class $\mathcal{K}_{\sum}(\varepsilon, \sigma, \rho, \lambda, \mu; \alpha)$ reduces to the class $\mathcal{H}_{\sum}(\varepsilon, \lambda)$ which was considered by Frasin [7].
(3) for $\lambda = \mu = \varepsilon = 0$ and $\sigma = 1$, the class $\mathcal{K}_{\sum}(\varepsilon, \sigma, \rho, \lambda, \mu; \alpha)$ reduces to the class $\mathcal{B}_{\sum}(\rho, \lambda)$ which was given by Frasin and Aouf [8].
(4) $\lambda = \mu = \varepsilon = 0$ and $\rho = \sigma = 1$, the class $\mathcal{K}_{\sum}(\varepsilon, \sigma, \rho, \lambda, \mu; \alpha)$ reduces to the class $\mathcal{H}^{\alpha}_{\sum}$ which was investigated by Srivastava et al.[12].

THEOREM 2.3. *Let $f \in \mathcal{K}_{\sum}(\varepsilon, \sigma, \rho, \lambda, \mu; \alpha)$ be given by (1.1), then*

$$|a_2| \leq 2\alpha\,|\sigma| \sqrt{\frac{(2+\xi+\mu-\lambda)^2(3+\xi+\mu-\lambda)}{\left|\begin{array}{r}\alpha\sigma(1+\mu)(2+\mu)(2+\xi)(3+\xi)(2+\xi+\mu-\lambda)(1+2\rho+6\varepsilon)\\ +(1-\alpha)(1+\mu)^2(2+\xi)^2(1+\rho+2\varepsilon)^2(3+\xi+\mu-\lambda)\end{array}\right|}}$$

and

$$|a_3| \leq \frac{4\alpha^2\left|\sigma^2\right|(2+\xi+\mu-\lambda)^2}{(1+\mu)^2(2+\xi)^2(1+\rho+2\varepsilon)^2} + \frac{4\alpha\,|\sigma|\,(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}{(1+\mu)(2+\mu)(2+\xi)(3+\xi)(1+2\rho+6\varepsilon)}.$$

Proof. Considering the conditions (2.1) and (2.2), we conclude that

$$1+\frac{1}{\sigma}\left[(1-\rho)\frac{\mathbb{J}^{\lambda,\mu}f(z)}{z}+\rho\left(\mathbb{J}^{\lambda,\mu}f(z)\right)'+\varepsilon z\left(\mathbb{J}^{\lambda,\mu}f(z)\right)''-1\right]=[p(z)]^{\alpha} \tag{2.3}$$

and

$$1+\frac{1}{\sigma}\left[(1-\rho)\frac{\mathbb{J}^{\lambda,\mu}g(w)}{w}+\rho\left(\mathbb{J}^{\lambda,\mu}g(w)\right)'+\varepsilon w\left(\mathbb{J}^{\lambda,\mu}g(w)\right)''-1\right]=[q(w)]^{\alpha}. \tag{2.4}$$

Where $f^{-1}=g$ is defined as (1.2) and $p,q\in\mathbb{P}$ have the following expansion:

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots \tag{2.5}$$

and

$$q(w)=1+q_1w+q_2w^2+q_3w^3+\cdots \tag{2.6}$$

equating the coefficients of (2.3) and (2.4), we get

$$\frac{(1+\mu)(2+\xi)}{\sigma(2+\xi+\mu-\lambda)}(1+\rho+2\varepsilon)a_2=\alpha p_1 \tag{2.7}$$

$$\frac{(1+\mu)(2+\mu)(2+\xi)(3+\xi)}{2\sigma(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}(1+2\rho+6\varepsilon)a_3=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_2^2 \tag{2.8}$$

$$-\frac{(1+\mu)(2+\xi)}{\sigma(2+\xi+\mu-\lambda)}(1+\rho+2\varepsilon)a_2=\alpha q_1 \tag{2.9}$$

$$\frac{(1+\mu)(2+\mu)(2+\xi)(3+\xi)}{2\sigma(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}(1+2\rho+6\varepsilon)(2a_2^2-a_3)=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2 \tag{2.10}$$

From (2.7) and (2.9), we obtain that

$$p_1=-q_1 \tag{2.11}$$

and

$$\frac{2}{\sigma^2}\frac{(1+\mu)^2(2+\mu)^2)}{(2+\xi+\mu-\lambda)^2}(1+\rho+2\varepsilon)^2a_2^2=\alpha^2(p_1^2+q_1^2) \tag{2.12}$$

Using (2.8) and (2.10) together with (2.12), we get

$$\frac{(1+\mu)(2+\mu)(2+\xi)(3+\xi)}{\sigma(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}(1+2\rho+6\varepsilon)a_2^2=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)$$
$$=\alpha(p_2+q_2)+\frac{(\alpha-1)(1+\mu)^2(2+\xi)^2}{\alpha\sigma^2(2+\xi+\mu-\lambda)^2}(1+\rho+2\varepsilon)^2a_2^2$$

Further computations show that

$$a_2^2=\frac{\alpha^2\sigma^2(p_2+q_2)(2+\xi+\mu-\lambda)^2(3+\xi+\mu-\lambda)}{\begin{pmatrix}\alpha\sigma(1+\mu)(2+\mu)(2+\xi)(3+\xi)(2+\xi+\mu-\lambda)(1+2\rho+6\varepsilon)\\ +(1-\alpha)(1+\mu)^2(2+\xi)^2(1+\rho+2\varepsilon)^2(3+\xi+\mu-\lambda)\end{pmatrix}} \tag{2.13}$$

Taking the absolute value of (2.13) and using Lemma 1.2 for the coefficients p_2 and q_2, we have

$$|a_2| = 2\alpha\,|\sigma|\,\sqrt{\frac{(2+\xi+\mu-\lambda)^2(3+\xi+\mu-\lambda)}{\left|\begin{array}{c}\alpha\sigma(1+\mu)(2+\mu)(2+\xi)(3+\xi)(2+\xi+\mu-\lambda)(1+2\rho+6\varepsilon)\\ +(1-\alpha)(1+\mu)^2(2+\xi)^2(1+\rho+2\varepsilon)^2(3+\xi+\mu-\lambda)\end{array}\right|}}$$

To find the bound on $|a_3|$ by subtracting (2.10) from (2.8), yields

$$\frac{(1+\mu)(2+\mu)(2+\xi)(3+\xi)}{\sigma(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}(1+2\rho+6\varepsilon)(a_3-a_2^2) = \alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2) \tag{2.14}$$

Now substitute the value of a_2^2 from (2.12) into (2.14) and using (2.11), we conclude that

$$a_3 = \frac{\alpha^2\sigma^2(p_1^2+q_1^2)(2+\xi+\mu-\lambda)^2}{(1+\mu)^2(2+\xi)^2(1+\rho+2\varepsilon)^2} + \frac{\alpha\sigma(p_2-q_2)(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}{(1+\mu)(2+\mu)(2+\xi)(3+\xi)(1+2\rho+6\varepsilon)} \tag{2.15}$$

Taking the absolute value of (2.15) and using Lemma (1.2) for the coefficient p_1, p_2, q_1 and q_2, we get

$$|a_3| \leq \frac{4\alpha^2\left|\sigma^2\right|(2+\xi+\mu-\lambda)^2}{(1+\mu)^2(2+\xi)^2(1+\rho+2\varepsilon)^2} + \frac{4\alpha\,|\sigma|\,(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}{(1+\mu)(2+\mu)(2+\xi)(3+\xi)(1+2\rho+6\varepsilon)}.$$

□

Remark 2.4. *The results of theorem 2.3 is convert into following notable results, if we choose*

(1) $\lambda = \mu$, then we have the results obtained by Bulut and Wanas [5] Theorem 1.

(2) $\lambda = \mu = 0$ and $\rho = \sigma = 1$, then we have results introduced by Frasin [7] Theorem 2.2.

(3) $\lambda = \mu = \varepsilon = 0$ and $\sigma = 1$, then we have results established by Aouf and Frasin [8] Theorem 2.2 .

(4) $\lambda = \mu = \varepsilon = 0$ and $\rho = \sigma = 1$, then we have results introduced by Srivastava et al. [12] Theorem 1.

3. COEFFICIENT ESTIMATES FOR THE CLASS $\mathcal{K}^*_{\Sigma}(\varepsilon, \sigma, \rho, \lambda, \mu; \beta)$

We suppose that $\varepsilon \geq 0, \rho \geq 1, 0 \leq \beta < 1, \lambda \in \mathbb{N}_0$, and $\sigma \in \mathbb{C} \setminus \{0\}$.

Definition 3.1. A function $f \in \sum$ defined in (1.1) is called in the class $\mathcal{K}^*_{\Sigma}(\varepsilon, \sigma, \rho, \lambda, \mu; \beta)$, if it satisfied the following conditions:

$$Re\left\{1+\frac{1}{\sigma}\left[(1-\rho)\frac{\mathbb{J}^{\lambda,\mu}f(z)}{z}+\rho\left(\mathbb{J}^{\lambda,\mu}f(z)\right)'+\varepsilon z\left(\mathbb{J}^{\lambda,\mu}f(z)\right)''-1\right]\right\} > \beta \tag{3.1}$$

and

$$Re\left\{1+\frac{1}{\sigma}\left[(1-\rho)\frac{\mathbb{J}^{\lambda,\mu}g(w)}{w}+\rho\left(\mathbb{J}^{\lambda,\mu}g(w)\right)'+\varepsilon w\left(\mathbb{J}^{\lambda,\mu}g(w)\right)''-1\right]\right\}>\beta \tag{3.2}$$

where $z, w \in \mathbb{D}$ and $f^{-1} = g$ is defined in (1.2).

REMARK 3.2. *The family $\mathcal{K}^*_{\Sigma}(\varepsilon,\sigma,\rho,\lambda,\mu;\beta)$ is a generalization of known families investigated previously. These are as follows:*
(1) $\lambda = \mu$ the class $\mathcal{K}^*_{\Sigma}(\varepsilon,\sigma,\rho,\lambda,\mu;\beta)$ reduces to the class $\mathcal{Q}^*_{\Sigma}(\varepsilon,\sigma,\rho,\lambda;\beta)$ which is considered by Bulut and Wanas [5].
(2) $\lambda = \mu = 0$ with $\rho = \sigma = 1$ the class $\mathcal{K}^*_{\Sigma}(\varepsilon,\sigma,\rho,\lambda,\mu;\beta)$ reduces to the class $\mathcal{H}^*_{\Sigma}(\varepsilon,\lambda)$ which was considered by Frasin [7].
(3) $\lambda = \mu = \varepsilon = 0$ and $\sigma = 1$, the class $\mathcal{K}^*_{\Sigma}(\varepsilon,\sigma,\rho,\lambda,\mu;\beta)$ reduces to the class $\mathcal{B}^*_{\Sigma}(\rho,\lambda)$ which was introduced by Frasin and Aouf [8].
(4) $\lambda = \mu = \varepsilon = 0$ and $\rho = \sigma = 1$, the class $\mathcal{K}^*_{\Sigma}(\varepsilon,\sigma,\rho,\lambda,\mu;\beta)$ converts to the class $\mathcal{H}^{\alpha}_{\Sigma}$ which was obtained by Srivastava et al. [12].
(5) $\lambda = \mu = 0$ and $\sigma = 1$ the class $\mathcal{K}^*_{\Sigma}(\varepsilon,\sigma,\rho,\lambda,\mu;\beta)$ reduces to the class $\mathcal{N}_{\Sigma}(\varepsilon,\rho;\beta)$ this was given by Bulut [3].

THEOREM 3.3. *Let $f \in \mathcal{K}^*_{\Sigma}(\varepsilon,\sigma,\rho,\lambda,\mu;\alpha)$ be given by (1.1), then*

$$|a_2| \leq \begin{cases} 2\sqrt{\frac{(1-\beta)|\sigma|(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}{(1+\mu)(2+\mu)(2+\xi)(3+\xi)(1+2\rho+6\varepsilon)}}, 0 \leq \beta \leq 1-\frac{(1+\mu)(2+\xi)(3+\xi+\mu-\lambda)(1+\rho+2\varepsilon)^2}{|\sigma|(2+\mu)(2+\xi+\mu-\lambda)(3+\xi)(1+2\rho+6\varepsilon)} \\ \frac{2|\sigma|(1-\beta)(2+\xi+\mu-\lambda)}{(1+\mu)(2+\xi)(1+\rho+2\varepsilon)}, 1-\frac{(1+\mu)(2+\xi)(3+\xi+\mu-\lambda)(1+\rho+2\varepsilon)^2}{|\sigma|(2+\mu)(2+\xi+\mu-\lambda)(3+\xi)(1+2\rho+6\varepsilon)} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \frac{4\,|\sigma|\,(1-\beta)(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}{(1+\mu)(2+\mu)(2+\xi)(3+\xi)(1+2\rho+6\varepsilon)}.$$

PROOF. From the conditions (3.1) and (3.2) we are able to conclude that

$$1+\frac{1}{\sigma}\left[(1-\rho)\frac{\mathbb{J}^{\lambda,\mu}f(z)}{z}+\rho\left(\mathbb{J}^{\lambda,\mu}f(z)\right)'+\varepsilon z\left(\mathbb{J}^{\lambda,\mu}f(z)\right)''-1\right]=\beta+(1-\beta)p(z) \tag{3.3}$$

and

$$1+\frac{1}{\sigma}\left[(1-\rho)\frac{\mathbb{J}^{\lambda,\mu}g(w)}{w}+\rho\left(\mathbb{J}^{\lambda,\mu}g(w)\right)'+\varepsilon w\left(\mathbb{J}^{\lambda,\mu}g(w)\right)''-1\right]=\beta+(1-\beta)q(w) \tag{3.4}$$

Where $p, q \in \mathbb{P}$ and defined as (2.5) and (2.6) respectively.
Now analysing the equivalent coefficients in (3.3) and (3.4), we get

$$\frac{(1+\mu)(2+\xi)}{\sigma(2+\xi+\mu-\lambda)}(1+\rho+2\varepsilon)a_2=(1-\beta)p_1 \tag{3.5}$$

$$\frac{(1+\mu)(2+\mu)(2+\xi)(3+\xi)}{2\sigma(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}(1+2\rho+6\varepsilon)a_3=(1-\beta)p_2 \tag{3.6}$$

$$-\frac{(1+\mu)(2+\xi)}{\sigma(2+\xi+\mu-\lambda)}(1+\rho+2\varepsilon)a_2=(1-\beta)q_1 \tag{3.7}$$

$$\frac{(1+\mu)(2+\mu)(2+\xi)(3+\xi)}{2\sigma(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}(1+2\rho+6\varepsilon)(2a_2^2-a_3)=(1-\beta)q_2 \tag{3.8}$$

In view of (3.5) and (3.7), we conclude that

$$p_1=-q_1 \tag{3.9}$$

and

$$\frac{2}{\sigma^2}\frac{(1+\mu)^2(2+\xi)^2}{(2+\xi+\mu-\lambda)^2}(1+\rho+2\varepsilon)^2a_2^2=(1-\beta)^2(p_1^2+q_1^2) \tag{3.10}$$

Adding (3.6) and (3.8), we have

$$\frac{(1+\mu)(2+\mu)(2+\xi)(3+\xi)}{\sigma(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}(1+2\rho+6\varepsilon)a_2^2=(1-\beta)(p_2+q_2) \tag{3.11}$$

by (3.10) and (3.11), we get

$$a_2^2=\frac{\sigma^2(p_1^2+q_1^2)(2+\xi+\mu-\lambda)^2(1-\beta)^2}{(1+\mu)^2(2+\xi)^2(1+\rho+2\varepsilon)^2}$$

and

$$a_2^2=\frac{\sigma(p_2+q_2)(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)(1-\beta)}{(1+\mu)(2+\mu)(2+\xi)(3+\xi)(1+2\rho+6\varepsilon)}$$

Using Lemma (1.2) to the coefficients p_2 and q_2 respectively, we get

$$|a_2|\le\frac{2\,|\sigma|\,(2+\xi+\mu-\lambda)(1-\beta)}{(1+\mu)(2+\xi)(1+\rho+2\varepsilon)}$$

and

$$|a_2|\le 2\sqrt{\frac{|\sigma|\,(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)(1-\beta)}{(1+\mu)(2+\mu)(2+\xi)(3+\xi)(1+2\rho+6\varepsilon)}}$$

Subtracting (3.8) from (3.6), we get the bound on $|a_3|$

$$\frac{(1+\mu)(2+\mu)(2+\xi)(3+\xi)}{\sigma(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}(1+2\rho+6\varepsilon)(a_3-a_2^2)=(1-\beta)(p_2-q_2)$$

or similarly

$$a_3=a_2^2+\frac{\sigma(p_2-q_2)(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)(1-\beta)}{(1+\mu)(2+\mu)(2+\xi)(3+\xi)(1+2\rho+6\varepsilon)} \tag{3.12}$$

Now substituting the value of a_2^2 from (3.10) and (3.11) into (3.12), we get

$$\begin{aligned} a_3 &= \frac{\sigma^2(p_1^2+q_1^2)(2+\xi+\mu-\lambda)^2(1-\beta)^2}{2(1+\mu)^2(2+\xi)^2(1+\rho+2\varepsilon)^2} \\ &+ \frac{\sigma(p_2-q_2)(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)(1-\beta)}{(1+\mu)(2+\mu)(2+\xi)(3+\xi)(1+2\rho+6\varepsilon)} \end{aligned}$$

and

$$a_3 = \frac{2\sigma p_2(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)(1-\beta)}{(1+\mu)(2+\mu)(2+\xi)(3+\xi)(1+2\rho+6\varepsilon)}$$

Again, using Lemma (1.2) to the coefficients p_1 , q_1 , p_2 and q_2 respectively, it follows that

$$|a_3| \le \frac{4\,|\sigma|^2\,(2+\xi+\mu-\lambda)^2(1-\beta)^2}{(1+\mu)^2(2+\xi)^2(1+\rho+2\varepsilon)^2} + \frac{4\,|\sigma|\,(1-\beta)(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}{(1+\mu)(2+\mu)(2+\xi)(3+\xi)(1+2\rho+6\varepsilon)}$$

and

$$|a_3| \le \frac{4(1-\beta)\,|\sigma|\,(2+\xi+\mu-\lambda)(3+\xi+\mu-\lambda)}{(1+\mu)(2+\mu)(2+\xi)(3+\xi)(1+2\rho+6\varepsilon)}$$

this completes the proof. □

Remark 3.4. *The results of theorem 3.3 is convert into following notable results, if we choose*
(1) $\lambda = \mu$ *then we have the results of family* $\mathcal{K}^*_\Sigma(\varepsilon,\sigma,\rho,\lambda,\mu;\alpha)$ *obtained by Bulut and Wanas [5].*
(2) $\lambda = \mu = 0$ *and* $\rho = \sigma = 1$ *then we have the results obtained by Frasin [7] Theorem 3.2.*
(3) $\lambda = \mu = \varepsilon = 0$ *and* $\sigma = 1$*, then we have the results obtained by Frasin and Aouf [8] Theorem 3.2.*
(4) $\lambda = \mu = \varepsilon = 0$ *and* $\rho = \sigma = 1$*, then we have the results obtained by Srivastava et al.[12] Theorem 2.*
(5) $\lambda = 0$ *and* $\sigma = 1$ *then we have the improvements results obtained by Bulut[3] Theorem 3.4*

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