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FUZZY IDEALS, COMPLEMENTED $\Gamma-$ SUBSEMIRING, DORRAH EXTENSION AND STRUCTURE THEOREM FOR FUZZY IDEALS OF $\Gamma-$ SEMIRINGS

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Abstract

The main objective of this paper is to derive some semiring theoretical results of fuzzy ideals and $k-$ fuzzy ideals in the framework of $\Gamma-$semirings. We introduce the concept of Dorroh extension, complemented $\Gamma-$semiring, and find an inclusion preserving bijection between the set of fuzzy ideals of a $\Gamma-$semiring R and set of all fuzzy ideals of $\Gamma_{n,n}-$ semiring $R_{n,n}$. Moreover, the fuzzy ideal of R is a $k-$ fuzzy ideal if and only if the corresponding fuzzy ideal of $R_{n,n}$ is a $k-$ fuzzy ideal.

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1. Introduction

In the mathematical universe all around us, semirings are everywhere. The first mathematical structure we come across is the semiring formed by the set of nonnegative integers. Semirings were first considered explicitly by Vandiver[16] in connection with the axiomatization of the arithmetic of the natural numbers. Many scholars have investigated semirings, either independently or as part of an effort to branch out from ring theory or semigroup theory, or in connection with applications. As a generalization of the ring, the notion of a $\Gamma-$ ring was introduced by Nobusawa [10] in 1964. The notion of $\Gamma-$ semigroup was introduced by Sen[13] in 1981 as a generalization of $\Gamma-$ groups. In 1995, the concept of $\Gamma-$semiring was studied by Rao as a generalization of ring and $\Gamma-$ ring[11].

Uncertainty may develop due to incomplete knowledge of the issues, incompletely reliable information, or the receiving of information from multiple sources. A good mathematical method for addressing the uncertainty brought on by ambiguity is fuzzy set theory. L.A. Zadeh introduced the fuzzy set theory [17] in his research publications in 1965. When dealing with uncertainty in particular areas of theoretical computer science, fuzzy set theory is very helpful and decision making. In 1972, Rosenfield [12] was the first who apply the concept of the fuzzy set to group structures and introduced the notion of fuzzy subgroups and ideals of a semigroup. Later, Kuich[8] extended this concept and investigated the concepts of fuzzy ideals, bi-ideals, and semiprime ideals of semigroups. In 1993, Ahsan et.al[1] studied the fuzzy semirings and semimodules. Subsequently, Kim et.al [7] introduced the fuzzy $k-$ideals of semirings, and Ghosh [5] investigated and established some results relating to fuzzy k-ideals. In 2011, Dutta and Goswami [3] studied "Operations on fuzzy ideals of $\Gamma-$ semirings".

It is worth noting here that the study of fuzzification of $\Gamma-$ semirings with ideals, $k-$ ideals, prime ideals, etc. was focused on by many authors in different aspects. The main aim of this study is to generalize some fundamental results of fuzzy ideals and $k-$ fuzzy ideals in $\Gamma-$ semirings proved in [2, 6] and to show their applications. It is seen that the use of fuzzy sets is more convenient in real-life problems than ordinary sets, and so it is important in the case of algebraic structures. As a result, an effort has been made to further examine the $\Gamma-$ semirings structure in a fuzzy situation. Accordingly, there is a scope and area of study for applying some fuzzy $\Gamma-$ semiring notions to challenges in information science and decision-making. These concepts include different types of fuzzy ideals such as semiprime fuzzy ideals, prime fuzzy ideals, and primary fuzzy ideals.

2. Preliminaries and Examples

Recall from [6, 11] that if $(R, +)$ and $(\Gamma, +)$ be two commutative semigroups then R is called a $\Gamma-$ semiring if there exists a mapping $R \times \Gamma \times R \to R$ denoted by $x\alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$ satisfying(i) $x\alpha(y + z) = x\alpha y + x\alpha z$. (ii) $(y + z)\alpha x = y\alpha x + z\alpha x$. (iii) $x(\alpha + \beta)z = x\alpha z + x\beta z$. (iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Let A and B be semirings and $R = Hom(A, B)$ and $\Gamma = Hom(B, A)$ denote the sets of homomorphisms from A to B and B to A respectively. Then R is a $\Gamma-$ semiring with operations of pointwise addition and composition of mappings. Further, let M be a $\Gamma-$ ring and let R be the set of ideals of M. Define addition in the natural way and if $A, B \in R$, $\gamma \in \Gamma$, let $A\gamma B$ denote the ideal generated by $\{x\gamma y | x, y \in M\}$. Then R is a $\Gamma-$ semiring. A $\Gamma-$ semiring R is said to be commutative if $x\gamma y = y\gamma x$ for all $x, y \in R$ and for all $\gamma \in \Gamma$. A $\Gamma-$ semiring R is said to have a zero element if $0\gamma x = 0 = x\gamma 0$ and $x + 0 = x = 0 + x$ for all $x \in R$ and $\gamma \in \Gamma$. R is said to have an identity element if there exists $\gamma \in \Gamma$ such that $1\gamma x = x = x\gamma 1$ for all $x \in R$. An element x of a $\Gamma-$ semiring R is said to be multiplicative $\Gamma-$ idempotent if there exists $\gamma \in \Gamma$ such that $x = x\gamma x$. If every element of R is multiplicative $\Gamma-$ idempotent then R is called multiplicative $\Gamma-$ idempotent $\Gamma-$ semiring. A non empty subset S of a $\Gamma-$ semiring R is said to be a sub $\Gamma-$ semiring of R if $(S, +)$ is a sub semigroup of $(R, +)$ and $x\gamma y \in S$ for all $x, y \in S$ and $\gamma \in \Gamma$.

Let X be a non-empty set. A mapping μ : X $\to$ [0,1] is called a fuzzy subset of X. A fuzzy subset μ of R is a function μ: $R \to [0, 1]$, for a fixed $0 \leq t \leq 1$, the set $\mu_t = \{x \in R \mid \mu(x) \geq t\}$ is called level subset of μ. For $t \in [0, 1)$, the subset $\mu_{[t]} = \{x \in X \mid \mu(x) > t\}$ is called the strong level subset of the fuzzy subset μ.

Let S be a $\Gamma-$ semigroup and λ be a fuzzy subset of S then λ is called a fuzzy $\Gamma-$ subsemigroup of S if $\lambda(x\alpha y) \geq min(\lambda(x), \lambda(y))$ for all x,y $\in$ S and for all $\alpha \in \Gamma$. If

$x \in X$ and $r \in (0, 1]$ then a fuzzy point x_r of X is a fuzzy subset of X, defined by

$$x_r(y) = \begin{cases} r & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

If A is a fuzzy subset of a non-empty set X and $x_r \in X$ is a fuzzy point then $x_r \in A$ means $x_r \subseteq A$. For the definitions of fuzzy ideals, the sum of fuzzy ideals, and the product of fuzzy ideals one may refer to [3].

Throughout this paper, R will denote a $\Gamma-$ semiring with zero element 0 and identity element 1 unless otherwise stated.

3. Operations on Fuzzy Ideals of Γ− Semirings

In this section, we prove characteristic criterion, and level set criterion and study fuzzy level ideals and strong fuzzy level ideals. Finally, we study the effect of homomorphism of $\Gamma-$ semirings on fuzzy ideals and fuzzy $k-$ ideals.

Remark 3.1. *All the results proved in this paper are true for all left ideals, right ideals, and ideals (fuzzy left ideals, fuzzy right ideals, and fuzzy ideals) of a $\Gamma-$ semiring R. So, here we will prove all the results only for left ideals(fuzzy left ideals).*

Definition 3.2. Let A be a non-empty subset of a $\Gamma-$ semiring R. The characteristic function of A is a fuzzy subset of R and is defined as

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Theorem 3.3. *Let R be a $\Gamma-$ semiring and I a non-empty subset of R and χ_I be the characteristic function of I. Then I is a left ideal of R if and only if χ_I is a fuzzy left ideal of R.*

The following result can easily be proved by using the similar arguments of [[2], proposition 0.2.11].

Lemma 3.4. *Let X be a non empty set and x_r, y_s be any two fuzzy points in X then $x_r\Gamma y_s = (x\alpha y)_{min(r,s)}$, for all $\alpha \in \Gamma$.*

Theorem 3.5. *Let R be a $\Gamma-$ semiring. A non-empty fuzzy subset λ of R is a fuzzy left ideal if and only if for any $r, s \in (0, 1]$,*

(i) *$x_r + y_s \in \lambda$ where x_r, $y_s \in \lambda$ and*
(ii) *$x_r\Gamma\lambda \subseteq \lambda$ $\quad (\lambda\Gamma x_r \subseteq \lambda)$ whenever $x_r \in R$.*

Proof. (i) Let λ be a fuzzy ideal of R and $x_r, y_s \in \lambda$. For any $z \in R$, $(x_r + y_s)(z) = \sup_{z=u+v} [min\{x_r(u), y_s(v)\}$, if $z = u + v$ for some $u, v \in R] \leq \sup_{z=u+v} \{min\{\lambda(u), \lambda(v)\}$, $\leq \sup_{z=u+v} \lambda(u + v) = \lambda(z)$. If $(x_r + y_s)(z) = 0$, then obviously $(x_r + y_s)(z) \leq \lambda(z)$. Hence, $x_r + y_s \in \lambda$.

(ii) Let $u, v \in R$, $\alpha \in \Gamma$.
$(x_r\Gamma\lambda)(z) = \sup_{z=u\alpha v}[\{min[x_r(u), \lambda(v)]\}, if z = u\alpha v$ for some $u, v \in R$ and $\alpha \in \Gamma]$
$\leq \sup_{z=u\alpha v}[min\{\lambda(u), \lambda(u\alpha v)\}] \leq \sup_{z=u\alpha v}\{\lambda(u\alpha v)\} = \lambda(z)$. Again, if $(x_r\Gamma\lambda)(z) = 0$, then obviously $(x_r\Gamma\lambda)(z) \leq \mu(z)$. Hence, $x_r\Gamma\lambda \subseteq \lambda$.
Conversely, assume that conditions (i) and (ii) are satisfied. Let $x_{\lambda(x))}, y_{\lambda(y)} \in \lambda$, then clearly $x_{\lambda(x)} + y_{\lambda(y)} \in \lambda$. Further, $\lambda(x + y) \geq (x_{\lambda(x)} + y_{\lambda(y)})(x + y) = \sup_{x+y=u+v}[min[x_{\lambda(x)}(u), y_{\lambda(y)}(v)]] \geq min[x_{\lambda(x)}(x), y_{\lambda(y)}(y)] = min[\lambda(x), \lambda(y)]$.
Again, $(x_1\Gamma\lambda) \subseteq \lambda$, for $r = 1$, so for $\alpha \in \Gamma$, $\lambda(x\alpha y) \geq (x_1\Gamma\lambda)(x\alpha y) = \sup_{x\alpha y=u\alpha v}[min[x_1(u), \lambda(v)] \geq min[1, \lambda(y)] = \lambda(y)$. Thus, λ is a fuzzy left ideal of R.

□

The proof of the following theorems is quite easy, so we omit their proofs.

Theorem 3.6. *Let R be a $\Gamma-$ semiring. Let λ be a fuzzy left ideal of R and $x, y \in R$. Then $\lambda(x) \geq \lambda(y)$ whenever $x \in< y >$ where $< y >$ denotes the ideal of R generated by y.*

Theorem 3.7. *Let R be a $\Gamma-$ semiring.*

(i) *Let λ be a fuzzy left ideal of R. Then $\lambda_0 = \{x \in R \mid \lambda(x) = \lambda(0)\}$ is a left ideal of R.*
(ii) *Let I be an ideal of R and $a \leq b \neq 0$ be any two elements in $[0, 1]$. Then the fuzzy subset λ of R, defined by*

$$\lambda(x) = \begin{cases} b & if\ x \in I \\ a & if\ otherwise \end{cases}$$

is a fuzzy ideal of R.

Theorem 3.8. *Let R be a $\Gamma-$semiring and λ be a non-empty fuzzy subset of R. Then λ is a fuzzy left ideal of R if and only if $\lambda_t' s$ are left ideals of R for all $t \in [0, 1]$, where $\lambda_t = \{x \in R \mid \lambda(x) \geq t\}$.*

Remark 3.9. *Let R be a $\Gamma-$semiring and λ a fuzzy left ideal of R. Then the ideals $\lambda_t' s$ are called the level left ideals of λ where $t \in [0, 1]$.*

One can easily prove the upcoming three theorems by using similar arguments of the results in [2] with necessary variations.

Theorem 3.10. *Let R be a $\Gamma-$ semiring and If λ is a fuzzy left ideal of R and $t_1 > t_2$ then $\lambda_{t_1} \subseteq \lambda_{t_2}$. In particular , if $t_1, t_2 \in [0, 1]$, then $t_1 > t_2$ if and only if $\lambda_{t_1} \subset \lambda_{t2}$.*

Theorem 3.11. *Let R be a $\Gamma-$ semiring. Let λ be a fuzzy left ideal of R with $t_2 > t_1$,$t_1, t_2 \in [0, 1]$. Then $\lambda_{t_1} = \lambda_{t_2}$ if and only if there is no $x \in R$ such that $t_1 \leq \lambda(x) < t_2$.*

Theorem 3.12. *Let R be a* Γ- *semiring. Then any left ideal of R can be a level left ideal of some fuzzy left ideal of R.*

Now, we turn our attention to the strong level subsets and homomorphisms of $\Gamma-$ semirings and prove the following results.

Theorem 3.13. *Let R be a* $\Gamma-$ *semiring and* λ *a fuzzy left ideal of R. Then every strong level subset* $\lambda_{[t]}$ *is a left ideal of R for each* $t \in [0, 1)$*, where* $\lambda_{[t]} = \{x \in R \mid \lambda(x) > t\}$.

Proof. Let λ be a fuzzy left ideal of R. Let $t \in [0, 1)$ and $x, y \in R$. Assume that $x, y \in \lambda_{[t]}$. Then $\lambda(x) > t$ and $\lambda(y) > t$. Therefore, $\lambda(x + y) \geq min[\lambda(x), \lambda(y)] > t$. So $x + y \in \lambda_{[t]}$. Again let $r \in R, \alpha \in \Gamma$ and $x \in \lambda_{[t]}$. Then $\lambda(x) > t$. Now $\lambda(r\alpha x) \geq \lambda(x) > t$. This implies that $r\alpha x \in \lambda_{[t]}$. Thus, $\lambda_{[t]}$ is a left ideal of R. □

Remark 3.14. *Let R be a* $\Gamma-$*semiring and* λ *a fuzzy left ideal of R. Then the ideals* $\lambda_{[t]}'s$ *are called the strong level left ideals of* λ *where* $t \in [0, 1]$.

Definition 3.15. Let R be a $\Gamma-$ semiring. A fuzzy ideal μ of R is said to be k-fuzzy ideal if $\mu(x) \geq min\{\mu(x + y), \mu(y)\}$ for all $x, y \in R$.
Or

A fuzzy ideal μ of R is said to be k-fuzzy ideal of R if $\mu(x + y) \geq \lambda$, $\mu(y) \geq \lambda$ then $\mu(x) \geq \lambda$ for all $x, y \in R$, $\lambda \in [0, 1]$.

Example 3.16. *Let* λ, μ *be fuzzy subsets of a* $\Gamma-$ *semiring* $\mathbb{N}$*, set of all non negative integers defined by*

$$\lambda(x) = \begin{cases} 1/3 & \textit{if x is odd} \\ 1/2 & \textit{if x is even} \\ 0 & \textit{if x =0} \end{cases}$$

and

$$\mu(x) = \begin{cases} 1 \quad if \quad x > 7 \\ 1/2 \quad if \quad 5 < x < 7 \\ 0 \quad if \quad 0 \leq x < 5 \end{cases}$$

Then λ *is a k−fuzzy ideal of* $\mathbb{N}$ *and* μ *is a fuzzy ideal of* $\mathbb{N}$*, but not a k− fuzzy ideal of* $\mathbb{N}$.

Definition 3.17. Let R_1 and R_2 be two $\Gamma-$ semirings. Then $f : R_1 \to R_2$ is called a $\Gamma-$homomorphism if $f(x + y) = f(x) + f(y)$ and $f(x\alpha y) = f(x)\alpha f(y)$ for all $x, y \in R_1$ and $\alpha \in \Gamma$. If f is both one-one and onto then f is a $\Gamma-$ isomorphism.

Theorem 3.18. *Let* $f : R \to S$ *be a homomorphism of* $\Gamma-$*semirings R and S.*

(i) *If* λ *is a fuzzy left ideal of S then* $f^{-1}(\lambda)$ *is a fuzzy left ideal of R.*
(ii) *If f is a surjective homomorphism and* μ *is a fuzzy left ideal of R then* $f(\mu)$ *is a fuzzy left ideal of S.*

Proof. (i) Let λ be a fuzzy left ideal of R. Since $(f^{-1}(\lambda))(0) = \lambda(f(0)) = \lambda(0') \geq \lambda(x') \neq 0$, for some $x' \in S, 0' \in S, 0 \in R$, this implies that $f^{-1}(\lambda)$ is non empty. Now, for any $r, s \in R, \alpha \in \Gamma$. $(f^{-1}(\lambda))(r+s) = \lambda(f(r+s)) = \lambda(f(r)+f(s)) \geq min[\lambda(f(r)), \lambda(f(s))] = min[(f^{-1}(\lambda)(r)), (f^{-1}(\lambda)(s)]$. Again, for $\alpha \in \Gamma$ $f^{-1}(\lambda)(r\alpha s) = \lambda(f(r\alpha s)) = \lambda(f(r)\alpha f(s)) \geq \lambda(f(s)) = f^{-1}(\lambda)(s)$. Hence, $f^{-1}(\lambda)$ is a fuzzy left ideal of R.

(ii) Assume that μ is a fuzzy left ideal of R. Since $(f(\mu))(x') = \sup_{f(x)=x'} \{\mu(x)\}$, for $x' \in S$ and μ is non empty, it follows that $f(\mu)$ is non-empty. For any x', y' in S, $(f(\mu))(x'+y')) = \sup_{f(z)=x'+y'} \mu(z) \geq \sup_{f(x)=x', f(y)=y'} \mu(x+y) \geq \sup_{f(x)=x', f(y)=y'} min[\mu(x), \mu(y)]$ $= min[\sup_{f(x)=x'} \mu(x), \sup_{f(y)=y'} \mu(y)] = min[(f(\mu))(x')),$ $(f(\mu))(y')]$. Again, $(f(\mu))(x'\alpha y') = \sup_{f(z)=x'\alpha y'} \mu(z) \geq \sup_{f(x)=x', f(y)=y'} \mu(x'\alpha y') \geq \sup_{f(y)=y'} \mu(y) = (f(\mu)(y')$ for $\alpha \in \Gamma$. Hence, $f(\mu)$is a fuzzy left ideal of R.

□

Theorem 3.19. *Let $f : R \to S$ be a homomorphism of $\Gamma-$semirings R and S. If λ is a $k-$ fuzzy left ideal of S then $f^{-1}(\lambda)$ is a $k-$ fuzzy left ideal of R.*

4. Complemented Γ− Subsemiring, Dorrah extension and Structure Theorem of Γ− Semirings

The idea behind the results proved in this section was first brought to the author's attention during the study of the fuzzy ideals of semirings given by Biswas [2] and ideals in $\Gamma-$ semirings [15]. Now, we characterize these ideas to examine the $\Gamma-$ semirings structure in fuzzy situations.

Let us define the set of all fuzzy ideals of R by $F_I(R)$ and is given as $F_I(R) = \{\mu \in$ fuzzy ideal$(R) \mid x \in R, \mu(x) \neq 0\}$

Definition 4.1. A $\Gamma-$ semiring R is said to have a strong identity element if for all $x \in R$, $1\alpha x = x = x\alpha 1$ for all $\alpha \in \Gamma$.

Theorem 4.2. *Let R be a $\Gamma-$ semiring with strong identity and λ a left fuzzy ideal of R. Then $e\Gamma\lambda = \lambda\Gamma e = \lambda$, e is a left fuzzy ideal of R defined by $e(x) = 1$ for all $x \in R$.*

Remark 4.3. *The fuzzy ideal e of a $\Gamma-$ semiring R is the multiplicative identity of the collection of all fuzzy ideals denoted by $F_I(R)$.*

Definition 4.4. A $\Gamma-$ semiring R is centreless if and only if $x+y = 0$ implies that $x = y = 0$.

Definition 4.5. An element x of a $\Gamma-$ semiring R is infinite if and only if $x + r = x = r + x$ for all $r \in R$.

Definition 4.6. A $\Gamma-$ semiring R with zero element is simple if and only if $x+1 = 1 = 1 + x$ for all $x \in R$.

Or
A $\Gamma-$ semiring R with zero elements is simple if and only if 1 is infinite.

THEOREM 4.7. *Let λ and μ be two fuzzy left ideals of a $\Gamma-$ semiring R. Then $\lambda \subseteq \lambda + \mu$.*

REMARK 4.8. *By theorem 4.7 , $e \subseteq e + \lambda$, for any fuzzy left ideal λ and e of R. Also, $(e + \lambda)(x) = \sup_{x=y+z}[\min\{e(y), \lambda(z)\}, \ y, \ z \ \in R] = \sup_{x=y+z}[\min\{1, \lambda(z)\}, y, \ z \ \in R] \leq 1 = e(x)$ for all $x \in R$. Therefore, $e + \lambda \subseteq e$. Hence, $e + \lambda = e$ for all fuzzy left ideals λ of R. Thus, e is an infinite element of $F_I(R)$.*

THEOREM 4.9. *Let R be a $\Gamma-$ semiring and λ be a fuzzy left ideal of R. Then $\theta + \lambda = \lambda = \lambda + \theta$, where θ is a fuzzy ideal of R defined by*

$$\theta(x) = \begin{cases} 1 & if\ x = 0 \\ 0 & if\ x \neq 0. \end{cases}$$

REMARK 4.10. *The fuzzy set θ of R is the additive identity of all fuzzy ideals of R denoted by $F_I(R)$.*

The proofs of the following results are straightforward so we omit it.

THEOREM 4.11. *Let R be a $\Gamma-$ semiring. Then the set of all fuzzy left ideals of R is centreless.*

THEOREM 4.12. *Let R be a commutative $\Gamma-$semiring and λ, μ two fuzzy left ideals of R satisfying $\lambda + \mu = e$. Then $\lambda\Gamma\mu = \lambda \cap \mu$.*

The following result proved in ([3], Proposition 3.11) which shows that $'\circ'$ distributes over $'\oplus'$ from both sides.

PROPOSITION 4.13. *[3] Let $\mu_1, \mu_2, \mu_3 \in FLI(S)[FRI(S), FI(S)]$. Then*

(i) $\mu_1 \circ (\mu_2 \oplus \mu_3) = (\mu_1 \circ \mu_2) \oplus (\mu_1 \circ \mu_3)$.
(ii) $(\mu_2 \oplus \mu_3) \circ \mu_1) = (\mu_2 \circ \mu_1) \oplus (\mu_3 \circ \mu_1)$*, where $FLI(S)[FRI(S), FI(S)]$ denotes respectively the set of fuzzy left ideals, the set fuzzy right ideals and the set of fuzzy ideals of the $\Gamma-$semiring S.*

DEFINITION 4.14. Let R be a multiplicatively $\Gamma-$ idempotent $\Gamma-$ semiring and μ be a fuzzy ideal of R. Then μ multiplicatively $\Gamma-$ idempotent if $(\mu\Gamma\mu)(x) = sup[min\{\mu(x), \mu(x)\}, \ x \in R] = \mu(x)$.

THEOREM 4.15. *Let R be a commutative and multiplicatively $\Gamma-$ idempotent $\Gamma-$ semiring. If $(F_I(R), +, \cap)$ is a $\Gamma-$ semiring then $(F_I(R), +, \cap)$ is simple and multiplicatively $\Gamma-$ idempotent .*

To prove the following theorem, we will use the concept of Proposition 4.13.

THEOREM 4.16. *Let R be a commutative and multiplicatively $\Gamma-$ idempotent $\Gamma-$ semiring. Then $(F_I(R), +, \cap)$ is a $\Gamma-$ semiring if and only if $(F_I(R), \cap, +)$ is a $\Gamma-$ semiring.*

Proof. Assume that $(F_I(R), +, \cap)$ is a $\Gamma-$ semiring. Then $(F_I(R), +, \cap)$ is simple. Let $\mu_1, \mu_2, \mu_3 \in F_I(R)$ and e multiplicative identity of $F_I(R)$. Then $(\mu_1 + \mu_2) \cap (\mu_1 + \mu_3) = ((\mu_1 \cap \mu_1) + (\mu_2 \cap \mu_1)) + ((\mu_1 \cap \mu_3) + (\mu_2 \cap \mu_3) = (\mu_1 + (\mu_2 \cap \mu_1)) + ((\mu_1 \cap \mu_3 + (\mu_2 \cap \mu_3)) = ((\mu_2 \cap \mu_1) + \mu_1) + ((\mu_1 \cap \mu_3) + (\mu_2 \cap \mu_3)) = ((\mu_2 \cap \mu_1) + (e \cap \mu_1)) + ((\mu_1 \cap \mu_3) + (\mu_2 \cap \mu_3)) = ((\mu_2 + e) \cap \mu_1) + ((\mu_1 \cap \mu_3) + (\mu_2 \cap \mu_3)) = (e \cap \mu_1) + ((\mu_1 \cap \mu_3) + (\mu_2 \cap \mu_3)) = (\mu_1 \cap e) + (\mu_1 \cap \mu_3) + (\mu_2 \cap \mu_3) = (\mu_1 \cap (e + \mu_3)) + (\mu_2 + \mu_3) = (\mu_1 \cap e) + (\mu_2 \cap \mu_3) = \mu_1 + (\mu_2 \cap \mu_3)$. Similarly, $(\mu_2 \cap \mu_3) + \mu_1 = (\mu_2 + \mu_1) \cap (\mu_3 + \mu_1)$. Thus, $(F_I(R), \cap, +)$ is a $\Gamma-$ semiring. Conversely, let $(F_I(R), \cap, +)$ is a $\Gamma-$ semiring. Now e is the multiplicative identity of the $\Gamma-$ semiring $(F_I(R), \cap, +)$. So $e \cap \mu = e$ for all $e \in F_I(R)$. Therefore, $(F_I(R), \cap, +)$ is simple. Hence, with same analogy of above part we have $\mu_1 \cap (\mu_2 + \mu_3) = (\mu_1 \cap \mu_2) + (\mu_1 \cap \mu_3)$ and $(\mu_2 + \mu_3 \cap \mu_1) = (\mu_2 \cap \mu_1) + (\mu_3 \cap \mu_1)$. Thus, $(F_I(R), +, \cap)$ is a $\Gamma-$ semiring. □

As an application, now we have the following results which are proved in [3].

Theorem 4.17. *Let R be a $\Gamma-$ semiring. Then $(F_I(R), +, \cap)$ is a complete lattice.*

Theorem 4.18. *Let R be a $\Gamma-$ semiring.Then the lattice $(F_I(R), +, \cap)$ is modular if each of its members is a $k-$ fuzzy ideal.*

Complemented $\Gamma-$ subsemiring

Recall the following definitions from [14].

Definition 4.19. [14] Let x, y be elements of a $\Gamma-$ semiring R then x is Γ-interior y denoted by $x\nabla y$ if and only if there exist an element $z \in R, \alpha \in \Gamma$ such that $x\alpha z = z\alpha x = 0$, and $z + y = 1$.

Definition 4.20. [14] An element x is complemented if and only if $x\nabla x$. That is there exists an element $y \in R, \alpha \in \Gamma$ such that $x\alpha y = y\alpha x = 0$ and $x + y = 1$.This element $y \in R$ is the complement of R.We will denote the complement of x by $x^{\perp}$.

Theorem 4.21. *Let R be a commutative Γ-semiring with strong identity and $F_I(R) = \{\mu \in fuzzy\ ideal(R) \mid x \in R, \mu(x) \neq 0$ implies that $x\nabla y$ for some $y \in R\}$.Then $F_I(R)$ is a $\Gamma-$ subsemiring of R.*

Proof. Let

$$\theta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Clearly, θ is a fuzzy ideal of R. Let $a \in R$. Since $0\nabla 0$, $a\nabla 1$ and $\theta(0) \neq 0$, $e(a) \neq 0$, $e(1) \neq 0$, so $e, \theta \in F_I(R)$. Let μ_1, $\mu_2 \in F_I(R)$ and $(\mu_1 + \mu_2)(x) \neq 0$, for some $x \in R$. Then $\sup\limits_{x=u+v} [min(\mu_1(u), \mu_2(v))] \neq 0$. So there exist $a, a' \in R$ such that $x = a + a'$ and $min(\mu_1(a), \mu_2(a')) \neq 0$. This implies that $\mu_1(a) \neq 0, \mu_2(b) \neq 0$. Now, there exist b, $b' \in R$ such that $a\nabla b$ and $a'\nabla b'$ for which $\mu_1(b) \neq 0, \mu_2(b') \neq 0$. Since $a\nabla b$ and $a'\nabla b'$ so there exist $c, c' \in R, \alpha \in \Gamma$ such that $a\alpha c = a'\alpha c' = 0$ and $b + c = b' + c' = 1$. Now, $(a + a')\alpha(c\beta c') = a\alpha(c\alpha c') + a'\alpha(c\alpha c') = (a\alpha c)\beta c' + (a'\alpha c')\beta c = 0$ for all $\alpha, \beta \in \Gamma$ and $(b + c)\alpha(b' + c') = 1\alpha 1 = 1$ for all $\alpha \in \Gamma$. This implies that $b\alpha b' + b\alpha c' + c\alpha b' + c\alpha c' = 1$.

Now, $(\mu_1+\mu_2)(b\alpha c'+c\alpha b'+b\alpha b') \geq min\{\mu_1(b\alpha c'), \mu_2(c\alpha b'+b\alpha b')\} \geq min(\mu_1(b), \mu_2(b'))$ $\neq 0$. Thus, $(\mu_1 + \mu_2)(x) \neq 0$ implies that $(a + a')\nabla(b\alpha c' + c\alpha b' + b\alpha b')$. This implies that $(\mu_1 + \mu_2)(b\alpha c' + c\alpha b' + b\alpha b') \neq 0$. Therefore, $\mu_1 + \mu_2 \in S$. Further, let $a \in S$ and $(\mu_1\Gamma\mu_2)(a) \neq 0$. Since $\mu_1\Gamma\mu_2 = \mu_1 \cap \mu_2$ so $(\mu_1 \cap \mu_2)(a) \neq 0$. Therefore, $\mu_1(a) \neq 0$ and $\mu_2(a) \neq 0$. Thus, there exist $b, b' \in R$ such that $a\nabla b$ and $a\nabla b'$ and $\mu_1(b) \neq 0$ and $\mu_2(b') \neq 0$. Since, $a\nabla b$ and $a\nabla b'$ so there exists $c, c' \in R$ such that $a\beta c = a\beta c' = 0$ and $b + c = b' + c' = 1$, for all $\beta \in \Gamma$. Now, $(b + c)\alpha(b' + c') = b\alpha b' + b\alpha c' + c\alpha b' + c\alpha c'$ $= 1\alpha 1 = 1$ for all $\alpha \in \Gamma$ and $a\beta(b\alpha c' + c\alpha b' + c\alpha c') = (a\beta c')\alpha b + (a\beta c)\alpha b' + (a\beta c)\alpha c'$ $= 0$. Again, $(\mu_1\Gamma\mu_2)(b\alpha b') = [sup\{\inf_{1\leq i\leq n}(\mu_1(p_i), \mu_2(q_i))\}(b\alpha b') \mid b\alpha b' = \sum_{i=1}^{n} p_i\alpha q_i]$ $\geq min(\mu_1(b), (\mu_2(b')) \neq 0$, p_i, $q_i \in R, n \in Z^+$. Thus, $(\mu_1\Gamma\mu_2)(a) \neq 0$ implies that $a\nabla b\alpha b'$ for some $b\alpha b' \in R$ satisfying $(\mu_1\Gamma\mu_2)(b\alpha b') \neq 0$. Hence, $\mu_1\Gamma\mu_2 \in F_I(R)$ and so $F_I(R)$ is a sub Γ− semiring of R. □

Dorrah extension

Following Golan[4], we now analogously define Dorrah extension for Γ− semirings as follows:

Let R be a Γ− semiring without multiplicative identity and $\mathbb{N}$ be the set of all non-negative integers. Let $S = R \times \mathbb{N}$. We define operations on $R \times \mathbb{N}$ as follows: $(x_1, n_1) + (x_2, n_2) = (x_1 + x_2, n_1 + n_)$ and $(x_1, n_1)\alpha(x_2, n_2) = (n_1x_2 + n_2x_1 + x_1\alpha x_2, n_1n_2)$ for all $(x_1, n_1), (x_2, n_2) \in R \times \mathbb{N}$ and some $\alpha \in \Gamma$. Then $(S, +, .)$ can be easily varified to be a Γ−semiring with multiplicative identity $(0, 1)$ called the Dorroh extension of R by $\mathbb{N}$.

Now, we will study the fuzzy ideals and fuzzy k− ideals of the Dorrah extension of a Γ−semiring R by the set $\mathbb{N}$ of non-negative integers.

Theorem 4.22. *Let R be a Γ−semiring without multiplicative identity. Let λ be a non empty fuzzy subset of R and μ a fuzzy subset of the Dorrah extension $R \times \mathbb{N}$ of R by $\mathbb{N}$, defined by*

$$\mu(x, n) = \begin{cases} 0 \text{ if } n \neq 0 \\ \lambda(x) \text{ if } n = 0 \end{cases}$$

Then λ is a fuzzy left ideal of R if and only if μ is a fuzzy left ideal of $R \times \mathbb{N}$.

Proof. Let λ be a fuzzy left ideal of R. Let (x_1, n_1), $(x_2, n_2) \in R \times \mathbb{N}$ and $\alpha \in \Gamma$.
Case I If $n_1 \neq 0, n_2 \neq 0$ then $\mu[(x_1, n_1) + (x_2, n_2)] \geq min[\mu(x_1, n_1) + \mu(x_2, n_2)]$.
Case II If $n_1 = 0, n_2 = 0$ then $n_1 + n_2 = 0$. Therefore, $\mu[(x_1, n_1) + (x_2, n_2)] = \mu[(x_1 + x_2, n_1 + n_2)] = \lambda(x_1 + x_2) \geq min[\lambda(x_1), \lambda(x_2)] = min[\mu(x_1, n_1) + \mu(x_2, n_2)]$. Thus, $\mu[(x_1, n_1) + (x_2, n_2)] \geq min[\mu(x_1, n_1) + \mu(x_2, n_2)]$.
Case III If $n_1 \neq 0$ and $n_2 \neq 0$ then $n_1n_2 \neq 0$. Thus, $\mu[(x_1, n_1)\alpha(x_2, n_2)] = \mu[(n_1x_2 + n_2x_1 + x_1\alpha x_2, n_1n_2)] = 0 = \mu[(x_2, n_2)]$.
Case IV If $n_1 = 0$ and $n_2 \neq 0$ then $n_1n_2 = 0$. So, $\mu[(x_1, n_1)\alpha(x_2, n_2)] = \lambda[(n_2x_1 + x_1\alpha x_2)]. \geq 0 = \mu[(x_2, n_2)]$.

Case V If $n_1 \neq 0$ and $n_2 = 0$ then $n_1 n_2 = 0$. Therefore, $\mu[(x_1, n_1)\alpha(x_2, n_2)] = \lambda[(n_1 x_2 + x_1 \alpha x_2)]. \geq min[\lambda(n_1 x_2), \lambda(x_1 \alpha x_2))] \geq \lambda(x_2) = \mu[(x_2, n_2)]$.
Case VI If $n_1 = 0$ and $n_2 = 0$ then $\mu[(x_1, n_1)\alpha(x_2, n_2)] = \lambda(x_1 \alpha x_2) \geq \lambda(x_2) = \mu[(x_2, n_2)]$. Thus, μ is a fuzzy left ideal of $R \times \mathbb{N}$.
Conversely, assume that μ is a fuzzy left ideal of $R \times \mathbb{N}$. For any $x, y \in R, \alpha \in \Gamma$, we have $\lambda(x + y) = \mu(x + y, 0) = \mu[(x, 0) + (y, 0)] \geq min[\mu(x, 0), \mu(y, 0)] = min[\lambda(x), \lambda(y)]$. Again, $\lambda(x\alpha y) = \mu(x\alpha y, 0) = \mu[(x, 0)\alpha(y, 0)] \geq \mu(y, 0) = \lambda(y)$. Thus, λ is a fuzzy left ideal of R. □

The proof of the following theorem is simple and straightforward.

Theorem 4.23. *Let R be a $\Gamma-$ semiring without multiplicative identity. Let λ be a non-empty fuzzy subset of R and μ a fuzzy subset of Dorrah extension $R \times \mathbb{N}$, defined by*

$$\mu(x, n) = \begin{cases} 0 \ if \ n \neq 0 \\ \lambda(x) \ if \ n = 0 \end{cases}$$

Then λ is a fuzzy left $k-$ideal of R if and only if μ is a fuzzy left $k-$ ideal of $R \times \mathbb{N}$.

Following J. Luh[9], we now introduce the notion of a matrix in $\Gamma-$ semirings. Let S be an additive abelian semigroup with identity element 0. Let $S_{m,n}$ the additive abelian semigroup with the identity, of all $m \times n$ matrices over the semigroup S. For $1 \leq i \leq m, 1 \leq j \leq n$ and $x \in S$, let xE_{ij} denote the matrix having x at i^{th} row and j^{th} column and 0 elsewhere.

Note that

$$xE_{ij}yE_{kl} = \begin{cases} xyE_{il} & \text{if } j = k \\ 0 & \text{if otherwise} \end{cases}$$

for all $x, y \in S$.
Let R be a $\Gamma-$ semiring. Let $R_{m,n}$ and $\Gamma_{n,m}$ be additive abelian semigroups with identity elements. For $(a_{ij}), (b_{ij}) \in R_{m,n}$ and $(\gamma_{ij}) \in \Gamma_{n,m}$, we define $(a_{ij})(\gamma_{ij})(b_{ij}) = (c_{ij})$,where $c_{ij} = \sum_{k=1}^{m} \sum_{h=1}^{n} (a_{ih}\gamma_{hk}b_{kj})$. Then $R_{m,n}$ forms $\Gamma_{n,m}-$ semiring.

Remark 4.24. *If R is a commutative $\Gamma-$ semiring, then $R_{m,n}$ need not be a commutative $\Gamma_{n,m}-$semiring.*

The following theorem is proved in [15].

Theorem 4.25. *Let R be a commutative $\Gamma-$semiring and $R_{m,n}$ be a $\Gamma_{n,m}-$ semiring such that $T : R \to R_{m,n}$ be an onto homomorphism. If A is an ideal of R then $T(A) = \{(a_{ij}) \in R_{m,n} \mid a_{ij} \in A$ for all $1 \leq i \leq m, 1 \leq j \leq n\}$ is an ideal of $R_{m,n}$.*

We now introduce a more general result in terms of fuzzification which is the generalization of Theorem 4.25.

Theorem 4.26. *Let R be a $\Gamma-$ semiring and $R_{m,n}$ be a $\Gamma_{n,m}-$semiring such that $\psi : R \to R_{m,n}$ be an onto homomorphism. If μ is a fuzzy ideal of R then the fuzzy subset $\sigma = \psi(\mu)$ of $R_{m,n}$ defined by $\sigma([a_ij]) = min\{\mu(a_{ij}) \mid a_{ij} \in [a_{ij}], 1 \leq i \leq m, 1 \leq j \leq n\}$ is a fuzzy ideal of $R_{m,n}$.*

Proof. Let μ be a fuzzy ideal of R and σ ba a fuzzy subset of $R_{m,n}$. Let $[a_{ij}], [b_{ij}] \in R_{m,n}$. Now, $\sigma([a_{ij}]+[b_{ij}]) = min\{\mu(a_{ij}+b_{ij}) \mid a_{ij}+b_{ij} \in [a_{ij}+b_{ij}], 1 \leq i \leq m, 1 \leq j \leq n\}$ $\geq min[min\{\mu(a_{ij}), \mu(b_{ij})\} \mid a_{ij} \in [a_{ij}], b_{ij} \in [b_{ij}], 1 \leq i \leq m, 1 \leq j \leq n\}]$ $= min[min\{\mu(a_{ij}) \mid a_{ij} \in [a_{ij}]\}, min\{\mu(b_{ij}) \mid b_{ij} \in [b_{ij}], 1 \leq i \leq m, 1 \leq j \leq n\}]$ $= min[\sigma([a_{ij}]), \sigma([b_{ij}])$ for all $[a_{ij}], [b_{ij}] \in R_{m,n}$. Let $[c_{ij}] \in R_{m,n}, [\alpha_{ji}] \in \Gamma_{n,m}$, Then, $\sigma([a_{ij}][\alpha_{ji}][b_{ij}]) = \sigma([d_{ij}])$, where $d_{ij} = \sum_{k=1}^{n}\sum_{h=1}^{n} a_{ih}\alpha_{hk}b_{kj}, 1 \leq i \leq m, 1 \leq j \leq n\}$. Therefore, $\sigma([a_{ij}][\alpha_{ji}][b_{ij}]) = min[\mu(d_{ij}) \mid d_{ij} \in [d_{ij}], 1 \leq i \leq m, 1 \leq j \leq n]$ $\geq min[\mu(b_{ij}) \mid b_{ij} \in [b_{ij}], 1 \leq i \leq m, 1 \leq j \leq n] = \sigma([b_{ij}])$ for all $[a_{ij}], [b_{ij}] \in R_{m,n}$ and $= [\alpha_{ji}] \in \Gamma_{n,m}$. Similarly, $\sigma([a_{ij}][\alpha_{ji}][b_{ij}) \geq \sigma([a_{ij}])$ for all $[a_{ij}], [b_{ij}] \in R_{m,n}$, $[\alpha_{ji}] \in \Gamma_{n,m}$. Also, $\sigma(0) = 1$. Hence, σ is a fuzzy ideal of $R_{m,n}$. □

Finally, the following theorem describes the structure of inclusion preserving bijection between the set of all fuzzy ideals of R and set of all ideals of $R_{n,n}$ which is a generalization of theorem 4.8 proved in [15].

Structure Theorem

Theorem 4.27. *Let R be a $\Gamma-$semring and $R_{n,n}$ be a $\Gamma_{n,n}-$ semiring, n is a positive integer then there exists an inclusion preserving bijection between the set of all fuzzy ideals of R and the set of all fuzzy ideals of $R_{n,n}$. Moreover, a fuzzy ideal of R is a k-fuzzy ideal if and only if the corresponding fuzzy ideal of $R_{n,n}$ is a $k-$ fuzzy ideal.*

Proof. Let $F_I(R)$ denote the set of all fuzzy ideals of R and $F_I(R_{n,n})$ be the set of all fuzzy ideals of $R_{n,n}$. Let U be the multiplicative identity of $R_{n,n}$. Let μ and σ be fuzzy ideals of R and $R_{n,n}$ respectively. Let us define a mapping $f : F_I(R) \to F_I(R_{n,n})$ such that $f(\mu) = \sigma([a_ij]) = min\{\mu(a_{ij}) \mid a_{ij} \in [a_{ij}], 1 \leq i \leq m, 1 \leq j \leq n\}$. Let $\mu_1, \mu_2 \in F_I(R)$, be such that $\mu_1 \neq \mu_2$. Then there exists $r \in R$ such that $\mu_1(r) \neq \mu_2(r)$. Let U_{hk} denote the matrix $[a_{ij}]$ of $R_{n,n}$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) = (h, k) \\ 0 & \text{if } otherwise, \end{cases}$$

for all $1 \leq h \leq n, 1 \leq k \leq n$. Now $\sigma(U_{hk}) = \mu_1(r) \neq \mu_2(r) = \sigma_2(U_{hk})$. This proves that f is injective. Moreover, if $\mu_1 \subseteq \mu_2$ then $(f(\mu_2)([a_{ij}]) = \sigma_2([a_{ij}]) = min\{\mu_2(a_{ij}) \mid a_{ij} \in [a_{ij}], 1 \leq i \leq n, 1 \leq j \leq n]\} \geq min\{\mu_1(a_{ij}) \mid a_{ij} \in [a_{ij}], 1 \leq i \leq n, 1 \leq j \leq n\}$ $=\sigma_1([a_{ij}]) = (f(\mu_1))([a_{ij}])$, for all $[a_{ij}] \in R_{n,n}$. Thus, $f(\mu_1) \subseteq f(\mu_2)$. So, f is order-preserving. Let λ be a fuzzy ideal of $R_{n,n}$ and τ be a fuzzy subset of R defined by $\tau(a) = \lambda(a\alpha U)$ for all $a \in R, \alpha \in \Gamma$ and U being the identity element of $R_{n,n}$. Now, $\tau(a + b) = \lambda((a + b)\alpha U) = \lambda(a\alpha U + b\alpha U) \geq min[\lambda(a\alpha U), \lambda(b\alpha U)] = min\{\tau(a), \tau(b)\}$

for all $a, b \in R$, $\alpha \in \Gamma$. Again, $\tau(a\alpha b) = \lambda((a\alpha b)\alpha U) = \lambda((a\alpha U)\alpha(b\alpha U)) \geq \lambda(b\alpha U) = \tau(b)$ for all $a, b \in R$, $\alpha \in \Gamma$. Similarly $\tau(a\alpha b) \geq \tau(a)$ for all $a, b \in R$. Thus, τ is a fuzzy ideal of R.

Now,
$$\begin{aligned}\tau_1([a_{ij}]) &= min\{\tau(a_{ij}) \mid a_{ij} \in [a_{ij}], 1 \leq i, j \leq n\}, \tau_1 \in F_I(R_{n,n})\\ &= min\{\lambda(a_{ij}\Delta U) \mid a_{ij} \in [a_{ij}] 1 \leq i, j \leq n, \Delta \subseteq \Gamma_{n,n}\},\\ &= min\{\lambda(\sum_{k=1}^{n} U_{ki}\Delta U_{hk} \vartriangle U_{jk}) \mid 1 \leq i, j \leq n\}, \text{ where } U_{hk} = [a_{ij}].\\ &\geq min[min\{\lambda(U_{1i}\Delta U_{hk}\Delta U_{j1}, \lambda(U_{2i}\Delta U_{hk}\Delta U_{j2}, ..., \lambda(U_{ni}\Delta U_{hk}\Delta U_{jn}\},\\ &\qquad 1 \leq i, j \leq n]\\ &\geq \lambda(U_{hk})\\ &= \lambda([a_{ij}]).\end{aligned}$$

Thus, $\lambda \subseteq \tau_1$. Again, $\lambda([a_{ij}]) = \lambda\{\sum a_{ij}\Delta U \mid a_{ij} \in [a_{ij}], 1 \leq i, j \leq n\} \leq min\{\lambda(a_{ij}\Delta U) \mid a_{ij} \in [a_{ij}], 1 \leq i, j \leq n\} = min\{\tau(a_{ij}) \mid a_{ij} \in [a_{ij}], 1 \leq i, j \leq n\} = \tau_1([a_{ij}])$ Thus, $\tau_1 \subseteq \lambda$. Therefore, $\tau_1 = \lambda$, that is, $f(\tau) = \tau_1 = \lambda$. Thus, f is surjective. Hence, f is an inclusion-preserving bijection from R to $R_{n,n}$. Further, let μ be a $k-$ fuzzy ideal of R.
Now,
$$\begin{aligned}\sigma([a_{ij}]) &= min\{\mu(a_{ij}) \mid a_{ij} \in []a_{ij}, 1 \leq i, j \leq n\}\\ &\geq min[min\{\mu(a_{ij} + b_{ij}), \mu(b_{ij})\} \mid a_{ij} \in [a_{ij}], a_{ij} + b_{ij} \in [a_{ij}] + [b_{ij}],\\ &\qquad 1 \leq i, j \leq n\}]\\ &= min[min\{\mu(a_{ij} + b_{ij}) \mid a_{ij} + b_{ij} \in [a_{ij}] + [b_{ij}],\\ &\qquad 1 \leq i, j \leq n\}, min\{\mu(b_{ij}) \mid b_{ij} \in [b_{ij}], 1 \leq i, j \leq n\}]\\ &= min\{\sigma([a_{ij} + b_{ij}]), \sigma([b_{ij}]) \text{ for all } [a_{ij}], [b_{ij}] \in R_{n,n}\}.\end{aligned}$$

Thus, σ is $k-$ fuzzy ideal of $R_{n,n}$.
Conversely, let λ be a $k-$ fuzzy ideal of $R_{n,n}$. Now, $\tau(a) = \lambda(a\Delta U) \geq min\{(\lambda(a\Delta U + b\Delta U), \lambda(b\Delta U)\} = min\{\tau(a + b), \tau(b)\}$, for all $a, b \in R$. Hence, τ is a $k-$ fuzzy ideal of R. □

Conclusion: In this paper, first we study some basic results like characteristic criterion, level set criterion, fuzzy level ideals, and strong fuzzy level ideals. Further, we study the effect of homomorphism of $\Gamma-$ semirings on fuzzy ideals and fuzzy $k-$ ideals. Then we introduce the concept of complementation, Dorrah extension and find applications in terms of complete lattices and modular law for $\Gamma-$ semiring $(F_I(R), +, \cap)$ (Theorem 4.17 and Theorem 4.18). Finally, an inclusion preserving bijection between the set of fuzzy ideals of a $\Gamma-$semiring R and set of all fuzzy ideals of $\Gamma_{n,n}-$ semiring $R_{n,n}$ is developed and find that the fuzzy ideal of R is a $k-$ fuzzy ideal if and only if the corresponding fuzzy ideal of $R_{n,n}$ is a $k-$ fuzzy ideal.

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