GANITA,Vol.74(2), 2024, 247-258

ON CERTAIN BASIC HYPERGEOMETRIC SERIES IDENTITIES AND q-EXPONENTIAL OPERATORS

S. AHMAD ALI ✉ and SALONI KUSHVAHA

Abstract

In the present work, an operator approach has been taken to establish certain identities of basic hypergeometric series. Some special cases of our results have also been discussed.

2010 *Mathematics subject classification:* 05A30.
Keywords and phrases: Basic hypergeometric Series, Parameter Augmentation, q-Differential Operator, q-Shift Operator, q-Exponential Operator.

1. Introduction

The Umbral calculus provides a methodology that translates combinatorial and algebraic problems into the language of classical analysis by employing "umbrae" or shadow variables [13]. This formalism simplifies the manipulation of sequences and polynomials, making it particularly useful in the study of hypergeometric series. The power of umbral calculus approach (or operator approach) in the study of both ordinary and basic hypergeometric series has been demonstrated in their works by Andrews [5], Roman [12], Goldman and Rota [7, 8], Wimp [9] and many others. In recent years, a lot of interest has been shown in the operator approach in the study of the basic hypergeometric series and polynomial identities.

Chen and Liu in a series of papers [16, 17] introduced the certain q-exponential operators and showed that how an identity of basic hypergeometric series can be recovered from its special case by augmenting the parameters with the help of exponential operator. This operator approach has also been used to derive new q-series identities and also in finding new proofs of the known q-series identities, see Rota [3], Zhang [1], Chen & Fu [15], Liu [19].

In the present paper, we have used some q-exponential operators to derive certain transformations and summations of basic hypergeometric series. In our work, we use the notations and definitions of [4]. For the convenience of readers, we define a basic hypergeometric series as

$$ {}_{r+1}\phi_r\,(a_1,a_2,...,a_{r+1};b_1,b_2,...,b_r;q,z)=\sum_{n=0}^{\infty}\frac{(a_1,a_2,...,a_{r+1};q)_n}{(q,b_1,b_2,...,b_r;q)_n}z^n, \tag{1.1} $$

where

$$(a;q)_n = (1-a)(1-aq).....(1-aq^{n-1})$$

and $|z| < 1$, $|q| < 1$, $c \neq q^{-n}, n = 1, 2, \ldots$.

Further, in section 3 of this paper, we have used of the following known results.

$$ {}_4\phi_3\,(a,b,c,q^{-n};e,g,h;q,q) = \frac{(g/c, eg/ab;q)_n}{(g, eg/cab;q)_n}\,{}_4\phi_3\left(e/a, e/b, c, q^{-n}; e, cq^{1-n}/g, cq^{1-n}/h; q,q\right). \quad (1.2)$$

(Equ. 3.2, [14])

$$ {}_8\phi_7\left(a, q\sqrt{a}, -q\sqrt{a}, c, d, e, f, q^{-n};\ \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f, aq^{n+1}; q, a^2q^{n+2}/cdef\right) = \frac{(-aq, aq/ef;q)_n}{(aq/e, aq/f;q)_n}\,{}_4\phi_3\,(aq/cd, e, f, q^{-n}; aq/c, aq/d, efq^{-n}/a; q,q). \quad (1.3)$$

(Equ. 3.4.1.5, [10])

$$ {}_{12}\phi_{11}\left(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, k^2q^n/a, q^{-n};\ \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, k/\sqrt{a}, -k/\sqrt{a}, k\sqrt{q/a}, -k\sqrt{q/a}, aq^{1-n}/k, kq^{n+1}; q,q\right) = \frac{(kq, k^2/a^2, -k/\sqrt{a};q)_n}{(k/a, k^2/a, -kq/\sqrt{a};q)_n}\times {}_6\phi_5\left(a, q\sqrt{a}, b, c, d, q^{-n};\ \sqrt{a}, aq/b, aq/c, aq/d, a^2q^{1-n}/k^2; q,q\right), \quad (1.4)$$

where $k = a^2q/bcd$.

(Equ. 4.6, [14])

$$ {}_4\phi_3\left(a^2, aq, c, q^{-n}; a, a^2q/c, c^2q^{1-n}; q,q\right) = \frac{(a^2/c^2, 1/c, -aq/c;q)_n}{(a^2q/c, 1/c^2, -a/c;q)_n}. \quad (1.5)$$

(Equ. 1.6, [14])

2. Exponential Operators

Chen and Liu in a series of papers [16, 17] introduced the exponential operators

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n}{(q;q)_n} q^{\frac{n(n-1)}{2}} \quad (2.1)$$

and

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q;q)_n} \tag{2.2}$$

In above definition, D_q is the usual q-differential operator

$$D_q\{f(a)\} = \frac{1}{a}(f(a) - f(aq))$$

and the operator θ is defined as [12]

$$\theta = \eta^{-1} D_q,$$

where η is q-shift operator defined as

$$\eta\{f(a)\} = f(aq).$$

The exponential operators (2.1) and (2.2) have the following fundamental properties [16, 17]

$$T(dD_q)\left\{\frac{1}{(at;q)_\infty}\right\} = \frac{1}{(at,dt;q)_\infty}. \tag{2.3}$$

$$T(dD_q)\left\{\frac{1}{(as,at;q)_\infty}\right\} = \frac{(adst;q)_\infty}{(as,at,ds,dt;q)_\infty}. \tag{2.4}$$

$$E(d\theta)\{(at;q)_\infty\} = (at,dt;q)_\infty. \tag{2.5}$$

$$E(d\theta)\{(as,at;q)_\infty\} = \frac{(as,at,ds,dt;q)_\infty}{(adst/q;q)_\infty}. \tag{2.6}$$

Zhang & Wang [18] gave the following extensions of these identities

$$T(dD_q)\left\{\frac{(av;q)_\infty}{(as,at,aw;q)_\infty}\right\} = \frac{(av,dv,adstw/v;q)_\infty}{(as,at,aw,ds,dt,dw;q)_\infty}\,{}_3\phi_2\left(v/s, v/t, v/w; av, dv; q, adstw/v\right). \tag{2.7}$$

$$E(d\theta)\left\{\frac{(at,as,aw;q)_\infty}{(av;q)_\infty}\right\} = \frac{(as,at,aw,ds,dw;q)_\infty}{(av,adsw/q;q)_\infty}\,{}_3\phi_2\left(t/v, q/as, q/aw; q/av, q^2/adsw; q, q\right). \tag{2.8}$$

These exponential operators have been used with advantage by Chen and Liu [16, 17] to derive many fundamental results in the theory of basic hypergeometric series. Further, in recent years, these operators have been successfully used in deriving new q-series identities and also providing the alternate proofs of such known identities - see Somashekara et al [2], Saad & Sukhi [6], Ali & Agnihotri [11].

3. Main Results

In this section, we establish the following identities and summations.

$$
\begin{aligned}
&{}_5\phi_4\left(a, 1/a, c, d, q^{-n}; e, g, h, ade; q, q\right)\\
&\qquad = \frac{(g/c, eg; q)_n}{(g, eg/c; q)_n}\,{}_5\phi_4\left(e/a, ae, de, c, q^{-n}; e, ade, cq^{1-n}/g, cq^{1-n}/h; q, q\right). \qquad (3.1)
\end{aligned}
$$

$$
\begin{aligned}
&{}_5\phi_4\left(a, b, c, q^{-n}, edc; e, ecq, dcq, h; q, q\right)\\
&\qquad = \frac{(eq, e^2cq/ab; q)_n}{(ecq, e^2q/ab; q)_n}\,{}_5\phi_4\left(e/a, e/b, c, edc, q^{-n}; d, e, q^{-n}/e, cq^{1-n}/h; q, q\right). \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
&{}_9\phi_8\Big(a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f, q^{-n};\ \sqrt{a}, -\sqrt{a}, abq/cd, aq/c, aq/d, aq/e, aq/f,\\
&\qquad\qquad aq^{n+1}; q, a^2q^{n+2}/cdef\Big)\\
&= \frac{(-aq, aq/ef; q)_n}{(aq/e, aq/f; q)_n}\,{}_5\phi_4\left(aq/cd, bq/cd, e, f, q^{-n}; aq/c, aq/d, abq/cd, efq^{-n}/a; q, q\right). \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
&{}_9\phi_8\Big(a, q\sqrt{a}, -q\sqrt{a}, c, d, e, abe/c, a^2q^n, q^{-n};\ \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e,\\
&\qquad\qquad bq/c, q^{1-n}/a, aq^{n+1}; q, q^2/cde\Big)\\
&= \left(\frac{1}{e}\right)^n \frac{(-aq, ae; q)_n}{(aq/e, a; q)_n}\,{}_5\phi_4\left(aq/cd, e, abe/c, a^2q^n, q^{-n}; ae, be, aq/c, aq/d; q, q\right). \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
&{}_{13}\phi_{12}\Big(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a, ke/a, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq},\\
&\qquad k^2q^n/a, q^{-n};\ \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, kde/a, k/\sqrt{a}, -k/\sqrt{a},\\
&\qquad\qquad k\sqrt{q/a}, -k\sqrt{q/a}, aq^{1-n}/k, kq^{n+1}; q, q\Big)\\
&\qquad\qquad = \frac{(kq, k^2/a^2, -k/\sqrt{a}; q)_n}{(k/a, k^2/a, -kq/\sqrt{a}; q)_n}\times\\
&\quad {}_7\phi_6\left(a, q\sqrt{a}, b, c, d, e, q^{-n};\ \sqrt{a}, aq/b, aq/c, aq/d, kde/a, a^2q^{1-n}/k^2; q, q\right), \qquad (3.5)
\end{aligned}
$$

where $k = a^2q/bcd$.

$$
\begin{aligned}
&{}_{13}\phi_{12}\Big(k, q\sqrt{k}, -q\sqrt{k}, kc/a, kd/a, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, kq/c\sqrt{a},\\
&\qquad c\sqrt{ae}/q, k^2q^n/a, q^{-n}; \sqrt{k}, -\sqrt{k}, aq/c, aq/d, k/\sqrt{a}, -k/\sqrt{a},\\
&\qquad\qquad k\sqrt{q/a}, -k\sqrt{q/a}, c\sqrt{a}, c\sqrt{e}, aq^{1-n}/k, kq^{n+1}; q, q\Big)\\
&\qquad\qquad\qquad = \frac{(kq, k^2/a^2, -k/\sqrt{a}; q)_n}{(k/a, k^2/a, -kq/\sqrt{a}; q)_n}\times\\
&{}_7\phi_6\Big(a, q\sqrt{a}, q\sqrt{a}/c, c\sqrt{ae}/q, c, d, q^{-n}; c\sqrt{a}, \sqrt{a}, \sqrt{e}, aq/c, aq/d,\\
&\qquad\qquad a^2q^{1-n}/k^2; q, q\Big), \quad (3.6)
\end{aligned}
$$

where $k = a\sqrt{a}/d$.

$$
\begin{aligned}
&{}_5\phi_4\Big(b, q\sqrt{b}, c, d, q^{-n}; \sqrt{b}, bq/c, bd/c^2, c^2q^{1-n}; q, q\Big)\\
&\qquad = \frac{(b/c^2, d/c^2, 1/c, -q\sqrt{b}/c; q)_n}{(bq/c, bd/c^2, 1/c^2, -\sqrt{b}/c; q)_n}. \quad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
&{}_5\phi_4\Big(a^2, aq, -c, ad/cq, q^{-n}; a, d, -a^2q/c, c^2q^{1-n}; q, q\Big)\\
&\qquad = \frac{(a^2/c^2, -1/c, aq/c, ad/cq; q)_n}{(-a^2q/c, 1/c^2, a/c, d/c; q)_n}. \quad (3.8)
\end{aligned}
$$

Proof of (3.1): Put $b = 1/a$ in (1.2) and then applying the operator $T(dD_q)$ both sides, we obtain

$$
\begin{aligned}
&\sum_{m=0}^{n} \frac{(1/a, c, q^{-n}; q)_m}{(e, g, h, q; q)_m} q^m T(dD_q)\left\{\frac{1}{(ae, aq^m; q)_\infty}\right\}\\
&\quad = \frac{(g/c, eg; q)_n}{(g, eg/c; q)_n} \sum_{m=0}^{n} \frac{(e/a, c, q^{-n}; q)_m}{(e, cq^{1-n}/g, cq^{1-n}/h, q; q)_m} q^m T(dD_q)\left\{\frac{1}{(a, aeq^m; q)_\infty}\right\}.
\end{aligned}
$$

Next using (2.4), we get

$$
\begin{aligned}
&\sum_{m=0}^{n} \frac{(1/a, c, q^{-n}; q)_m}{(e, g, h, q; q)_m} q^m \left\{\frac{(adeq^m; q)_\infty}{(ae, aq^m, de, dq^m; q)_\infty}\right\}\\
&\quad = \frac{(g/c, eg; q)_n}{(g, eg/c; q)_n} \sum_{m=0}^{n} \frac{(e/a, c, q^{-n}; q)_m}{(e, cq^{1-n}/g, cq^{1-n}/h, q; q)_m} q^m \left\{\frac{(adeq^m; q)_\infty}{(a, aeq^m, d, deq^m; q)_\infty}\right\}.
\end{aligned}
$$

Now multiplying $1/(ade; q)_\infty$ both sides, we get (3.1).

Proof of (3.2): Taking $g = ecq$ in (1.2) and then applying the operator $E(d\theta)$, we get

$$\sum_{m=0}^{n}\frac{(a,b,c,q^{-n};q)_m}{(q,e,h;q)_m}q^m E(d\theta)\left\{(ecq^{1+m},e;q)_\infty\right\}$$
$$=\frac{(eq,e^2cq/ab;q)_n}{(ecq,e^2q/ab;q)_n}\sum_{m=0}^{n}\frac{(e/a,e/b,c,q^{-n};q)_m}{(q,q^{-n}/e,cq^{1-n}/e;q)_m}q^m E(d\theta)\left\{(eq^m,ecq;q)_\infty\right\}.$$

Nest, using (2.6), we get

$$\sum_{m=0}^{n}\frac{(a,b,c,q^{-n};q)_m}{(q,e,h;q)_m}q^m\left\{\frac{(ecq^{1+m},e,dcq^{1+m},d;q)_\infty}{(edcq^m;q)_\infty}\right\}$$
$$=\frac{(eq,e^2cq/ab;q)_n}{(ecq,e^2q/ab;q)_n}\sum_{m=0}^{n}\frac{(e/a,e/b,c,q^{-n};q)_m}{(q,q^{-n}/e,cq^{1-n}/e;q)_m}q^m\left\{\frac{(eq^m,ecq,dq^m,dcq;q)_\infty}{(edcq^m;q)_\infty}\right\}.$$

Now multiplying $(edc;q)_\infty$ both sides, we obtain (3.2).

Proof of (3.3): Rearranging the terms in (1.3) and then applying the operator $T(bD_q)$, we get

$$\sum_{m=0}^{n}\frac{(q\sqrt{a},-q\sqrt{a},c,d,e,f,q^{-n};q)_m}{(\sqrt{a},-\sqrt{a},aq/c,aq/d,aq/e,aq/f,aq^{n+1},q;q)_m}\left(\frac{a^2q^{n+2}}{cdef}\right)^m\times$$
$$T(bD_q)\left\{\frac{1}{(aq/cd,aq^m;q)_\infty}\right\}$$
$$=\frac{(-aq,aq/ef;q)_n}{(aq/e,aq/f;q)_n}\sum_{m=0}^{n}\frac{(e,f,q^{-n};q)_m}{(aq/c,aq/d,efq^{-n}/a,q;q)_m}q^m\times$$
$$T(bD_q)\left\{\frac{1}{(a,aq^{1+m}/cd;q)_\infty}\right\}.$$

The above identity on using (2.4) gives

$$\sum_{m=0}^{n}\frac{(q\sqrt{a},-q\sqrt{a},c,d,e,f,q^{-n};q)_m}{(\sqrt{a},-\sqrt{a},aq/c,aq/d,aq/e,aq/f,aq^{n+1},q;q)_m}\left(\frac{a^2q^{n+2}}{cdef}\right)^m\times$$
$$\left\{\frac{(abq^{m+1}/cd;q)_\infty}{(aq/cd,aq^m,bq/cd,bq^m;q)_\infty}\right\}$$
$$=\frac{(-aq,aq/ef;q)_n}{(aq/e,aq/f;q)_n}\sum_{m=0}^{n}\frac{(e,f,q^{-n};q)_m}{(aq/c,aq/d,efq^{-n}/a,q;q)_m}q^m\times$$
$$\left\{\frac{(abq^{m+1}/cd;q)_\infty}{(a,aq^{1+m}/cd,b,bq^{1+m}/cd;q)_\infty}\right\}.$$

Multiplying both sides by $1/(abq/cd;q)_\infty$, we get (3.3).

Proof of (3.4): Substitute $f = a^2q^n$ in (1.3) and then applying the operator $E(b\theta)$, we have

$$\sum_{m=0}^{n} \frac{(a, q\sqrt{a}, -q\sqrt{a}, c, d, e, a^2q^n, q^{-n}; q)_m}{(q, \sqrt{a}, -\sqrt{a}, aq/d, aq/e, q^{1-n}/a, aq^{n+1}; q)_m} \left(\frac{q^2}{cde}\right)^m \times E(b\theta)\left\{(aq^{m+1}/c, ae; q)_\infty\right\}$$
$$= \left(\frac{1}{e}\right)^n \frac{(-aq, ae; q)_n}{(aq/e, a; q)_n} \sum_{m=0}^{n} \frac{(aq/cd, e, a^2q^n, q^{-n}; q)_m}{(q, aq/c, aq/d; q)_m} q^m E(b\theta)\{(aeq^m, aq/c; q)_\infty\}.$$

Next, applying (2.6), we get

$$\sum_{m=0}^{n} \frac{(a, q\sqrt{a}, -q\sqrt{a}, c, d, e, a^2q^n, q^{-n}; q)_m}{(q, \sqrt{a}, -\sqrt{a}, aq/d, aq/e, q^{1-n}/a, aq^{n+1}; q)_m} \left(\frac{q^2}{cde}\right)^m \times \left\{\frac{(aq^{m+1}/c, ae, bq^{m+1}/c, be; q)_\infty}{(abeq^m/c; q)_\infty}\right\}$$
$$= \left(\frac{1}{e}\right)^n \frac{(-aq, ae; q)_n}{(aq/e, a; q)_n} \sum_{m=0}^{n} \frac{(aq/cd, e, a^2q^n, q^{-n}; q)_m}{(q, aq/c, aq/d; q)_m} q^m \times \left\{\frac{(aeq^m, aq/c, beq^m/c, bq/c; q)_\infty}{(abeq^m/c; q)_\infty}\right\}.$$

Now, multiplying $(abe/c; q)_\infty$ both sides, we obtain (3.4).

Proof of (3.5): After rewriting (1.4), and then applying operator $T(eD_q)$, we get

$$\sum_{m=0}^{n} \frac{(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_m}{(q, \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, k/\sqrt{a}, -k/\sqrt{a}, k\sqrt{q/a}, -k\sqrt{q/a}; q)_m} \times \frac{(k^2q^n/a, q^{-n}; q)_m}{(aq^{1-n}/k, kq^{n+1}; q)_m} q^m T(eD_q)\left\{\frac{1}{(d, kdq^m/a; q)_\infty}\right\}$$
$$= \frac{(kq, k^2/a^2, -k/\sqrt{a}; q)_n}{(k/a, k^2/a, -kq/\sqrt{a}; q)_n} \sum_{m=0}^{n} \frac{(a, q\sqrt{a}, b, c, q^{-n}; q)_m}{(\sqrt{a}, aq/b, aq/c, aq/d, a^2q^{1-n}/k^2, q; q)_m} q^m \times T(eD_q)\left\{\frac{1}{(kd/a, dq^m; q)_\infty}\right\}.$$

Now using (2.4), we have

$$\sum_{m=0}^{n} \frac{(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_m}{(q, \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, k/\sqrt{a}, -k/\sqrt{a}, k\sqrt{q/a}, -k\sqrt{q/a}; q)_m} \times \frac{(k^2q^n/a, q^{-n}; q)_m}{(aq^{1-n}/k, kq^{n+1}; q)_m} q^m \left\{\frac{(kdeq^m/a; q)_\infty}{(d, kdq^m/a, e, keq^m/a; q)_\infty}\right\}$$

$$= \frac{(kq, k^2/a^2, -k/\sqrt{a}; q)_n}{(k/a, k^2/a, -kq/\sqrt{a}; q)_n} \sum_{m=0}^{n} \frac{(a, q\sqrt{a}, b, c, q^{-n}; q)_m}{(\sqrt{a}, aq/b, aq/c, aq/d, a^2q^{1-n}/k^2, q; q)_m} q^m \times \left\{ \frac{(kdeq^m/a; q)_\infty}{(kd/a, dq^m, ke/a, eq^m; q)_\infty} \right\}.$$

We obtain (3.5) on multiplying both sides by $1/(kde/a; q)_\infty$.

Proof of (3.6): Putting $b = q\sqrt{a}/c$ in (1.4) and then applying operator $E(e\theta)$, we have

$$\sum_{m=0}^{n} \frac{(k, q\sqrt{k}, -q\sqrt{k}, kq/c\sqrt{a}, kc/a, kd/a, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_m}{(q, \sqrt{k}, -\sqrt{k}, aq/c, aq/d, k/\sqrt{a}, -k/\sqrt{a}, k\sqrt{q/a}, -k\sqrt{q/a}; q)_m} \times \frac{(k^2q^n/a, q^{-n}; q)_m}{(aq^{1-n}/k, kq^{n+1}; q)_m} q^m E(e\theta) \left\{ (cq^m\sqrt{a}, \sqrt{a}; q)_\infty \right\}$$
$$= \frac{(kq, k^2/a^2, -k/\sqrt{a}; q)_n}{(k/a, k^2/a, -kq/\sqrt{a}; q)_n} \sum_{m=0}^{n} \frac{a, q\sqrt{a}, q\sqrt{a}/c, c, d, q^{-n}; q)_m}{(q, c\sqrt{a}, aq/c, aq/d, a^2q^{1-n}/k^2); q)_m} \times q^m E(e\theta) \left\{ (q^m\sqrt{a}, c\sqrt{a}; q)_\infty \right\},$$

where $k = a\sqrt{a}/d$.

Next, using (2.6), we get

$$\sum_{m=0}^{n} \frac{(k, q\sqrt{k}, -q\sqrt{k}, kq/c\sqrt{a}, kc/a, kd/a, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq}; q)_m}{(q, \sqrt{k}, -\sqrt{k}, aq/c, aq/d, k/\sqrt{a}, -k/\sqrt{a}, k\sqrt{q/a}, -k\sqrt{q/a}; q)_m} \times \frac{(k^2q^n/a, q^{-n}; q)_m}{(aq^{1-n}/k, kq^{n+1}; q)_m} q^m \left\{ \frac{(cq^m\sqrt{a}, \sqrt{a}, cq^m\sqrt{e}, \sqrt{e}; q)_\infty}{(cq^m\sqrt{ae}/q; q)_\infty} \right\}$$
$$= \frac{(kq, k^2/a^2, -k/\sqrt{a}; q)_n}{(k/a, k^2/a, -kq/\sqrt{a}; q)_n} \sum_{m=0}^{n} \frac{a, q\sqrt{a}, q\sqrt{a}/c, c, d, q^{-n}; q)_m}{(q, c\sqrt{a}, aq/c, aq/d, a^2q^{1-n}/k^2); q)_m} q^m \times \left\{ \frac{(q^m\sqrt{a}, c\sqrt{a}, q^m\sqrt{e}, c\sqrt{e}; q)_\infty}{(cq^m\sqrt{ae}/q; q)_\infty} \right\},$$

where $k = a\sqrt{a}/d$.

After multiplying $(c\sqrt{ae}/q; q)_\infty$ both sides, we get (3.6).

Proof of (3.7): Taking $a = \sqrt{b}$ in (1.5), and using operator $T(dD_q)$, we have

$$\sum_{m=0}^{n} \frac{(q\sqrt{b}, c, q^{-n}; q)_m}{(\sqrt{b}, bq/c, c^2q^{1-n}, q; q)_m} q^m T(dD_q) \left\{ \frac{1}{(b/c^2, bq^m; q)_\infty} \right\}$$
$$= \frac{(1/c, -q\sqrt{b}/c; q)_n}{(bq/c, 1/c^2, -\sqrt{b}/c; q)_n} T(dD_q) \left\{ \frac{1}{(b, bq^n/c^2; q)_\infty} \right\}.$$

Now, applying (2.4), we get

$$\sum_{m=0}^{n} \frac{(q\sqrt{b}, c, q^{-n}; q)_m}{(\sqrt{b}, bq/c, c^2q^{1-n}, q; q)_m} q^m \left\{ \frac{(bdq^m/c^2; q)_\infty}{(b/c^2, bq^m, d/c^2, dq^m; q)_\infty} \right\}$$
$$= \frac{(1/c, -q\sqrt{b}/c; q)_n}{(bq/c, 1/c^2, -\sqrt{b}/c; q)_n} \left\{ \frac{(bdq^m/c^2; q)_\infty}{(b, bq^n/c^2, d, dq^n/c^2; q)_\infty} \right\}.$$

Multiplying $1/(bd/c^2; q)_\infty$ both sides, we get (3.7).

Proof of (3.8): Taking $c = -c$ in (1.5) and then using operator $E(d\theta)$, we get

$$\sum_{m=0}^{n} \frac{(a^2, aq, -c, q^{-n}; q)_m}{(q, -a^2q/c, c^2q^{1-n}; q)_m} q^m E(d\theta) \{(aq^m, a/c; q)_\infty\}$$
$$= \frac{(a^2/c^2, -1/c, aq/c; q)_n}{(-a^2q/c, 1/c^2; q)_n} E(d\theta) \{(aq^n/c, a; q)_\infty\}.$$

After, using (2.6), we have

$$\sum_{m=0}^{n} \frac{(a^2, aq, -c, q^{-n}; q)_m}{(q, -a^2q/c, c^2q^{1-n}; q)_m} q^m \left\{ \frac{(aq^m, a/c, dq^m, d/c; q)_\infty}{(adq^m/cq; q)_\infty} \right\}$$
$$= \frac{(a^2/c^2, -1/c, aq/c; q)_n}{(-a^2q/c, 1/c^2; q)_n} \left\{ \frac{(aq^n/c, a, dq^n/c, d; q)_\infty}{(adq^n/cq; q)_\infty} \right\}.$$

Multiplying $(ad/cq; q)_\infty$ both sides, we obtain (3.8).

4. Special Cases

As $d \longrightarrow \infty$ in (3.1), we get

$$_4\phi_3\left(a, 1/a, c, q^{-n}; e, g, h; q, q/ae\right)$$
$$= \frac{(g/c, eg; q)_n}{(g, eg/c; q)_n} {}_4\phi_3\left(e/a, ae, c, q^{-n}; e, cq^{1-n}/g, cq^{1-n}/h; q, q/a\right). \quad (4.1)$$

Again, if $d \longrightarrow \infty$ in (3.2), we get

$$_4\phi_3\left(a, b, c, q^{-n}; e, ecq, h; q, e\right)$$
$$= \frac{(eq, e^2cq/ab; q)_n}{(ecq, e^2q/ab; q)_n} {}_4\phi_3\left(e/a, e/b, c, q^{-n}; e, q^{-n}/e, cq^{1-n}/h; q, ecq\right). \quad (4.2)$$

If we take $b \longrightarrow \infty$ in (3.3), we get

$$_8\phi_7\Big(a, q\sqrt{a}, -q\sqrt{a}, c, d, e, f, q^{-n}; \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f, aq^{n+1}; q, aq^2/ef\Big)$$
$$= \frac{(-aq, aq/ef; q)_n}{(aq/e, aq/f; q)_n} {}_4\phi_3\left(aq/cd, e, f, q^{-n}; aq/c, aq/d, efq^{-n}/a; q, q/a\right). \quad (4.3)$$

For $c = 1, d = q$ in (4.3), we get

$$ {}_3\phi_2\left(e, f, q^{-n}; aq, efq^{-n}/a; q, q/a\right) = \frac{(aq/e, aq/f; q)_n}{(-aq, aq/ef; q)_n}. \tag{4.4}$$

For $b \longrightarrow \infty$ in (3.4), we get

$$ {}_8\phi_7\Big(a, q\sqrt{a}, -q\sqrt{a}, c, d, e, a^2q^n, q^{-n};\ \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, q^{1-n}/a, aq^{n+1}; q, aq/cd\Big) = \left(\frac{1}{e}\right)^n \frac{(-aq, ae; q)_n}{(aq/e, a; q)_n}\, {}_4\phi_3\Big(aq/cd, e, a^2q^n, q^{-n}; ae, aq/c, aq/d; q, aq/c\Big). \tag{4.5}$$

As $d \longrightarrow \infty$ in (4.5), we get

$$ {}_7\phi_6\Big(a, q\sqrt{a}, -q\sqrt{a}, c, e, a^2q^n, q^{-n};\ \sqrt{a}, -\sqrt{a}, aq/c, aq/e, q^{1-n}/a, aq^{n+1}; q, aq/c\Big) = \left(\frac{1}{e}\right)^n \frac{(-aq, ae; q)_n}{(aq/e, a; q)_n}\, {}_3\phi_2\Big(e, a^2q^n, q^{-n}; ae, aq/c; q, aq/c\Big). \tag{4.6}$$

For $c = 1$ in (4.6), we get

$$ {}_3\phi_2\Big(e, a^2q^n, q^{-n}; ae, aq; q, aq\Big) = (e)^n \frac{(aq/e, a; q)_n}{(-aq, ae; q)_n}. \tag{4.7}$$

Taking $e \longrightarrow \infty$ in (3.5), we get

$$ {}_{12}\phi_{11}\Big(k, q\sqrt{k}, -q\sqrt{k}, kb/a, kc/a, kd/a, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, k^2q^n/a, q^{-n};\ \sqrt{k}, -\sqrt{k}, aq/b, aq/c, aq/d, k/\sqrt{a}, -k/\sqrt{a}, k\sqrt{q/a}, -k\sqrt{q/a}, aq^{1-n}/k, kq^{n+1}; q, q/d\Big) = \frac{(kq, k^2/a^2, -k/\sqrt{a}; q)_n}{(k/a, k^2/a, -kq/\sqrt{a}; q)_n} \times {}_6\phi_5\Big(a, q\sqrt{a}, b, c, d, q^{-n};\ \sqrt{a}, aq/b, aq/c, aq/d, a^2q^{1-n}/k^2; q, aq/kd\Big), \tag{4.8}$$

where $k = a^2q/bcd$.

If we take $d = 1, b = \sqrt{a}$ and $c = -\sqrt{a}$ in (4.8), we get

$$ {}_8\phi_7\Big(k, q\sqrt{k}, -q\sqrt{k}, k/a, \sqrt{aq}, -\sqrt{aq}, k^2q^n/a, q^{-n};\ \sqrt{k}, -\sqrt{k}, aq, k\sqrt{q/a}, -k\sqrt{q/a}, aq^{1-n}/k, kq^{n+1}; q, q\Big) = \frac{(kq, k^2/a^2, -k/\sqrt{a}; q)_n}{(k/a, k^2/a, -kq/\sqrt{a}; q)_m}, \tag{4.9}$$

where $k = -aq$.

Taking $e \longrightarrow \infty$ in (3.6), we get

$$
\begin{aligned}
&{}_{12}\phi_{11}\Big(k, q\sqrt{k}, -q\sqrt{k}, kc/a, kd/a, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, kq/c\sqrt{a},\\
&\qquad k^2q^n/a, q^{-n};\ \sqrt{k}, -\sqrt{k}, aq/c, aq/d, k/\sqrt{a}, -k/\sqrt{a}, k\sqrt{q/a},\\
&\qquad\qquad -k\sqrt{q/a}, c\sqrt{a}, aq^{1-n}/k, kq^{n+1}; q, \sqrt{a}\Big)\\
&\qquad = \frac{(kq, k^2/a^2, -k/\sqrt{a}; q)_n}{(k/a, k^2/a, -kq/\sqrt{a}; q)_n}\times\\
&{}_6\phi_5\Big(a, q\sqrt{a}, q\sqrt{a}/c, c, d, q^{-n}; c\sqrt{a}, \sqrt{a}, aq/c, aq/d, a^2q^{1-n}/k^2; q, c\sqrt{a}\Big), \qquad (4.10)
\end{aligned}
$$

where $k = a\sqrt{a}/d$.

On taking $c = q\sqrt{a}, d = q$ in (4.10), we get

$$
\begin{aligned}
&{}_{12}\phi_{11}\Big(k, q\sqrt{k}, -q\sqrt{k}, kq/\sqrt{a}, k/a, kq/a, q\sqrt{a}, -q\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, k^2q^n/a,\\
&\qquad q^{-n};\ \sqrt{k}, -\sqrt{k}, \sqrt{a}, a, aq, k/\sqrt{a}, -k/\sqrt{a}, k\sqrt{q/a}, -k\sqrt{q/a}, aq^{1-n}/k,\\
&\qquad\qquad aq^{1-n}/k, kq^{n+1}; q, \sqrt{a}\Big)\\
&\qquad = \frac{(kq, k^2/a^2, -k/\sqrt{a}; q)_n}{(k/a, k^2/a, -kq/\sqrt{a}; q)_n}, \qquad (4.11)
\end{aligned}
$$

where $k = a\sqrt{a}/q$.

For $d \longrightarrow \infty$ in (3.7), we get

$$
\begin{aligned}
&{}_4\phi_3\Big(b, q\sqrt{b}, c, q^{-n};\ \sqrt{b}, bq/c, c^2q^{1-n}; q, c^2q/b\Big)\\
&\qquad = \left(\frac{1}{b}\right)^n \frac{(b/c^2, 1/c, -q\sqrt{b}/c; q)_n}{(bq/c, 1/c^2, -\sqrt{b}/c; q)_n}. \qquad (4.12)
\end{aligned}
$$

As $d \longrightarrow \infty$ in (3.8), we get

$$
\begin{aligned}
&{}_4\phi_3\Big(a^2, aq, -c, q^{-n}; a, -a^2q/c, c^2q^{1-n}; q, a/c\Big)\\
&\qquad = \left(\frac{a}{q}\right)^n \frac{(a^2/c^2, -1/c, aq/c; q)_n}{(-a^2q/c, 1/c^2, a/c; q)_n}. \qquad (4.13)
\end{aligned}
$$

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S. Ahmad Ali, Department of Mathematics & Computer Science, Babu Banarasi Das University, Lucknow-226028 (India)
e-mail: ali.bbdu@gmail.com

Saloni Kushvaha, Department of Mathematics & Computer Science, Babu Banarasi Das University, Lucknow-226028 (India)
e-mail: salonikushvaha478@gmail.com