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EXACT RECTIFICATION OF A GENERAL CURVE OF NYSTEDT AND OTHERS, USING HYPERGEOMETRIC FUNCTION APPROACH

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Abstract

In this paper, we established some formulas for exact rectification of a curve of P. Fermat, H. van Heuraet, W. Neile and P. Nysted by means of Mellin-Barnes type contour integral representation of generalized hypergeometric function, analytic continuation formula of Gauss function. Some examples are also discussed for verification of our derived formulas of rectification.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

The following was said by the Greek philosopher Aristotle (384 – 322 B.C.) in relation to comparisons of motions along straight lines and along circles:

If the motions are similar, we run into the aforementioned problem, where a straight line becomes equal to a circle. However, these cannot be compared. (Heath[9][p.141]).

The Greek mathematician, astronomer, physicist, engineer: Archimedes (287-212 B.C.) rectification of the circle was based on a study spiral with some exceptions (Richesnon[24]).

The following was mentioned by French philosopher, scientist and mathematician: René Descartes (1596-1650) in his work *La Géométrie* from 1637:

No conclusion based on such ratios can be recognized as rigorous and correct because the ratios between straight and curved lines are unknown and, in my opinion, cannot be determined by human intellect. (Smith-Latham[25, p.91]).

The above statement was disproved by the following mathematicians:

French lawyer and mathematician: P. Fermat (1601-1665), Dutch mathematician: H. van Heuraet (1634-1660) and English mathematician: W. Neile (1637-1670) were able to find the length of cubical parabola using the method of exhaustion, the polygonal line-method, etc., independently (Traub[27]). After that Isaac Newton (1642-1726) and G. W. Leibniz (1646-1716) fully developed calculus, including integral-based arc length formulas (Edwards[4, p.217 and 242])

In Cartesian form, the length of the arc of the curve $y = f(x)$ in the interval $x_1 < x < x_2$ is given by

$$s = \int_{x_1}^{x_2} \sqrt{\left\{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}}\, \mathrm{d}x, \tag{1.1}$$

where y and $\frac{\mathrm{d}y}{\mathrm{d}x}$ are continuous and single valued functions in the interval $x_1 < x < x_2$ and the integrand $\sqrt{\{1 + (\frac{\mathrm{d}y}{\mathrm{d}x})^2\}}$ is always positive in this interval.

P. Nystedt[15] considered rectification of the general curve of the type

$$f(x) = b^{\frac{1}{n}}\, x^{\frac{n+1}{n}} \tag{1.2}$$

To simplify our computations, we will choose a specific value for the positive real constant b and positive integers n (see Traub[27]) i.e. $b^{\frac{1}{n}} = \frac{n}{n+1}$.

He gave an integral formula for rectification who presented the arc length as a limit of polygonal sums where the subdivision of the intervals is uniform, which is as follows:

Proposition[15, p.315]
If n is a positive integer and $0 \leq x_1 < x_2$, then the function

$$f(x) = \frac{n}{n+1}\, x^{\frac{n+1}{n}} \tag{1.3}$$

is rectifiable over $[x_1, x_2]$ with length

$$L_1 = \frac{n}{2^{n+1}} \int_{t_1}^{t_2} (t^2 - 1)^{n-1}\, (t^2 + 1)^2\, t^{-n-1}\, \mathrm{d}t, \tag{1.4}$$

where t_1 and t_2 are strictly positive numbers given by $t_1 = x_1^{\frac{1}{n}} + \sqrt{1 + x_1^{\frac{2}{n}}}$ and $t_2 = x_2^{\frac{1}{n}} + \sqrt{1 + x_2^{\frac{2}{n}}}$.

(i) We think that above formula (1.4) is not beneficial to students/researchers for calculation purpose.

(ii) For calculation of the length of the arc of the curve $f(x) = \frac{x^3}{3}$ over the interval $[0, 1]$, the corresponding integral for rectification will be

$$L_2 = \int_0^1 \sqrt{(1 + x^4)}\, \mathrm{d}x \tag{1.5}$$

There is no standard formula for (1.5), available in graduate level text-books on integral calculus. Such integrals are known as elliptic integrals (see Hancock[7]) and is impossible to calculate exact length in terms of elementary functions.

To overcome above difficulties (i)-(ii) in rectification, our aim is to obtain some formulas for exact rectification of the general algebraic curve in the form: $y = f(x) = \frac{m}{m+1}\, x^{\frac{m+1}{m}}$, where m is any positive real number, using the approach of hypergeometric functions.

In the development of our work, we shall apply the following definitions, preliminaries, etc.

Pochhammer's Symbol
The Pochhammer's symbol or generalized factorial function is defined by

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1); & \text{if } k = 1,2,3,\ldots;\ \lambda \in \mathbb{C} \\ 1 & ; \text{ if } k = 0;\ \lambda \in \mathbb{C}\backslash\mathbb{Z}_0^- \\ k! & ; \text{ if } \lambda = 1;\ k = 1,2,3,\ldots \end{cases} \tag{1.6}$$

where the notation Γ stands for Gamma function.

Legendre's duplication formula
Srivastava-Manocha[26, p.23 (Equation 25)]

$$\sqrt{\pi}\,\Gamma(2\lambda) = 2^{2\lambda-1}\,\Gamma(\lambda)\,\Gamma\Big(\lambda+\frac{1}{2}\Big);\ \ 2\lambda \neq 0,\ -1,\ -2,\ \ldots \tag{1.7}$$

Some identities

$$\sin^{-1}(i\theta) = i\sinh^{-1}(\theta) \tag{1.8}$$

$$\sinh^{-1}(x) = \ln\left\{x + \sqrt{(1+x^2)}\right\} \tag{1.9}$$

$$(1-z)^{-a} = {}_1F_0\left[\begin{matrix} a & ; & \\ & & z \\ \text{---} & ; & \end{matrix}\right];\ a \in \mathbb{C}\backslash\mathbb{Z}_0^- \tag{1.10}$$

For the power series definition of generalized hypergeometric functions ${}_pF_q$, their convergence conditions and related results the interested readers/researchers are advised to refer the beautiful monographs: Abramowitz *et al.*[1], Erdélyi *et al.*[5], Hansen[8], Lebedev[11], Magnus *et al.*[12], Olmstead[16], Prudnikov *et al.*[17], Rainville[22] and Srivastava-Manocha[26].

Mellin-Barnes type contour integral representation
Rainville[22, pp.100-101]

$$\frac{1}{2\pi\omega}\int_{-\omega\infty}^{+\omega\infty}\frac{\Gamma(-\zeta)\Gamma(\zeta+a_1)\Gamma(\zeta+a_2)\cdots\Gamma(\zeta+a_{q+1})}{\Gamma(\zeta+b_1)\Gamma(\zeta+b_2)\cdots\Gamma(\zeta+b_q)}\,(-z)^{\zeta}\,\mathrm{d}\zeta$$

$$= \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{q+1})}{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_q)}\ {}_{q+1}F_q\left[\begin{matrix} a_1,\ a_2,\ \ldots,\ a_{q+1} & ; & \\ & & z \\ b_1,\ b_2,\ \ldots,\ b_q & ; & \end{matrix}\right];\ z \neq 0 \tag{1.11}$$

where $|\arg(-z)| < \pi$; $a_1, a_2, \ldots, a_{q+1}, b_1, b_2, \ldots, b_q \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\omega = \sqrt{-1}$.

Analytic continuation formula
Abramowitz *et al.*[1, p.559(15.3.7)]; Erdélyi *et al.*[5, p.108(2)]; Magnus *et al.*[12, p.48]; Lebedev[11, p.249(9.5.9)]; Prudnikov *et al.*[17, p.454 (Entry 6)]

For $|z| > 1$,

$$
{}_2F_1\left[\begin{matrix} a, b & ; \\ & & z \\ c & ; \end{matrix}\right] = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1\left[\begin{matrix} a,\ 1-c+a & ; \\ & & \frac{1}{z} \\ 1-b+a & ; \end{matrix}\right] + \\
+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1\left[\begin{matrix} b,\ 1-c+b & ; \\ & & \frac{1}{z} \\ 1-a+b & ; \end{matrix}\right]; z \neq 0 \quad (1.12)
$$

where $|\arg(-z)| < \pi$, $|\arg(1-z)| < \pi$ and $b - a \neq \pm m; m = 0, 1, 2, \ldots.$

$$
{}_2F_1\left[\begin{matrix} a, b & ; \\ & & 0 \\ c & ; \end{matrix}\right] = {}_2F_1\left[\begin{matrix} 0, b & ; \\ & & z \\ c & ; \end{matrix}\right] = 1;\ c \in \mathbb{C}\backslash\mathbb{Z}_0^- \quad (1.13)
$$

Some reduction formulas of Gauss function
Prudnikov *et al.*[17, p.468(Entry 3)]

$$
{}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{1}{2} & ; \\ & & z \\ \frac{3}{2} & ; \end{matrix}\right] = \frac{1}{2}\left\{\frac{\sin^{-1}\sqrt{z}}{\sqrt{z}} + \sqrt{(1-z)}\right\} \quad (1.14)
$$

Prudnikov *et al.*[17, p.469 (Entry 11)]

$$
{}_2F_1\left[\begin{matrix} -\frac{1}{2}, 1 & ; \\ & & z \\ 2 & ; \end{matrix}\right] = \frac{2}{3\,z}\left\{1 - (1-z)^{\frac{3}{2}}\right\} \quad (1.15)
$$

Prudnikov *et al.*[17, p.469(Entry 19)]

$$
{}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{3}{2} & ; \\ & & z \\ \frac{5}{2} & ; \end{matrix}\right] = \frac{3}{8\,z}\left\{\frac{\sin^{-1}\sqrt{z}}{\sqrt{z}} - (1-2z)\sqrt{(1-z)}\right\} \quad (1.16)
$$

Prudnikov *et al.*[17, p.469(Entry 27)]

$$
{}_2F_1\left[\begin{matrix} -\frac{1}{2}, 2 & ; \\ & & z \\ 3 & ; \end{matrix}\right] = \frac{4}{15\,z^2}\left\{2 - (2+3z)(1-z)^{\frac{3}{2}}\right\} \quad (1.17)
$$

Prudnikov *et al.*[17, p.470(Entry 35)]

$$ {}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{5}{2} & ; & \\ & & z \\ \frac{7}{2} & ; & \end{matrix}\right] = \frac{5}{48\,z^2}\left\{\frac{3\sin^{-1}\sqrt{z}}{\sqrt{z}} - \frac{(3-z-10z^2+8z^3)}{\sqrt{(1-z)}}\right\} \tag{1.18} $$

Prudnikov *et al.*[17, p.470(Entry 43)]

$$ {}_2F_1\left[\begin{matrix} -\frac{1}{2}, 3 & ; & \\ & & z \\ 4 & ; & \end{matrix}\right] = \frac{2}{35\,z^3}\left\{8-(8+12z+15z^2)\,(1-z)^{\frac{3}{2}}\right\} \tag{1.19} $$

A summation theorem of Choi-Rathie-Malani
[3, p.1523-1524 (Equation 2.3)]; see also [23, p.828(3)]

$$ {}_2F_1\left[\begin{matrix} a, b & ; & \\ & & -1 \\ 1+a-b+p & ; & \end{matrix}\right] = \frac{\Gamma(1+a-b+p)}{2\,\Gamma(a)\,(1-b)_p}\sum_{r=0}^{p}\left\{\frac{p!\,(-1)^r\,\Gamma(\frac{r+a}{2})}{r!\,(p-r)!\,\Gamma(\frac{r+a}{2}+1-b)}\right\} \tag{1.20} $$

$$ \left(\Re(b) < \frac{p+2}{2};\ a,\ 1-b,\ 1+a-b+p \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ p \in \mathbb{N}_0\right) $$

During our investigation, we have also studied the articles: Gil[6], Khan[10], Maksimovi *et al.*[13], Nystedt[14], Qureshi *et al.*([18];[19];[20];[21]).

This paper is organized as follows:
In section 2, we derived three formulas for rectification of the general curve $y = \frac{m}{(m+1)}\,x^{\frac{(m+1)}{m}}$ for a particular interval, using Mellin-Barnes type contour integral of generalized hypergeometric function and analytic continuation formula of Gauss function.

To verify our three cases, in section 3, we gave some examples of rectification for particular values of x_1, x_2 and m, using some results recorded by Prudnikov *et al.*[17] and Choi-Rathie-Malani[3].

2. ARC LENGTH OF GENERAL ALGEBRAIC CURVE

Theorem 2.1: For algebraic curve $y = \frac{m}{(m+1)}\,x^{\frac{(m+1)}{m}}$ (where m is any positive real number), the lengths of the arc "s" in the interval $0 \leq x_1 < x < x_2$ are given in terms of Gauss functions:
Case-I: When $0 \leq x_1^{\frac{2}{m}} \leq 1$ and $0 < x_2^{\frac{2}{m}} \leq 1$, then

$$ s = x_2\ {}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{m}{2} & ; & \\ & & -x_2^{\frac{2}{m}} \\ \frac{m+2}{2} & ; & \end{matrix}\right] - x_1\ {}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{m}{2} & ; & \\ & & -x_1^{\frac{2}{m}} \\ \frac{m+2}{2} & ; & \end{matrix}\right]. \tag{2.1} $$

Case-II: When $0 \leq x_1^{\frac{2}{m}} \leq 1$; $x_2^{\frac{2}{m}} > 1$ and $\frac{(-m\pm 1)}{2} \neq 0, -1, -2, \ldots$, then

$$s = \frac{m\, x_2^{\frac{1}{m}+1}}{(m+1)}\, {}_2F_1\left[\begin{array}{ll} -\frac{1}{2}, \frac{-m-1}{2} & ; \\ & \quad - x_2^{-\frac{2}{m}} \\ \frac{1-m}{2} & ; \end{array}\right] - x_1\, {}_2F_1\left[\begin{array}{ll} -\frac{1}{2}, \frac{m}{2} & ; \\ & \quad - x_1^{\frac{2}{m}} \\ \frac{m+2}{2} & ; \end{array}\right] - \frac{\Gamma(\frac{m+2}{2})\,\Gamma(\frac{-m-1}{2})}{2\sqrt{\pi}}. \tag{2.2}$$

Case-III: When $x_1^{\frac{2}{m}} > 1$; $x_2^{\frac{2}{m}} > 1$ and $\frac{-m+1}{2} \neq 0, -1, -2, \ldots$, then

$$s = \frac{m\, x_2^{\frac{1}{m}+1}}{(m+1)}\, {}_2F_1\left[\begin{array}{ll} -\frac{1}{2}, \frac{-m-1}{2} & ; \\ & \quad - x_2^{-\frac{2}{m}} \\ \frac{1-m}{2} & ; \end{array}\right] - \frac{m\, x_1^{\frac{1}{m}+1}}{(m+1)}\, {}_2F_1\left[\begin{array}{ll} -\frac{1}{2}, \frac{-m-1}{2} & ; \\ & \quad - x_1^{-\frac{2}{m}} \\ \frac{1-m}{2} & ; \end{array}\right]. \tag{2.3}$$

Note: Here sum of denominator parameters − sum of numerator paramter $= \frac{3}{2}$ in each Gauss function which is greater than zero as well as −1, therefore each series is absolutely convergent when $x^{\frac{2}{m}} = 1$ and $x^{-\frac{2}{m}} = -1$.

Derivation:

Consider the curve:

$$y = \frac{m}{(m+1)}\, x^{\frac{(m+1)}{m}}; \quad 0 \leq x_1 < x < x_2. \tag{2.4}$$

Differentiate (2.4) with respect to x, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^{\frac{1}{m}}. \tag{2.5}$$

Therefore, length of the arc is given below by means of (1.1) and (1.11)

$$s = \int_{x_1}^{x_2} \sqrt{(1 + x^{\frac{2}{m}})}\,\mathrm{d}x \tag{2.6}$$

$$= \int_{x_1}^{x_2} {}_1F_0\left[\begin{array}{ll} -\frac{1}{2} & ; \\ & \quad - x^{\frac{2}{m}} \\ \text{---} & ; \end{array}\right] \mathrm{d}x. \tag{2.7}$$

When $\frac{2}{m} = 3$ or 4 then corresponding integral (2.6) is called elliptic integral. When $\frac{2}{m} \in \{5, 6, \ldots\}$ then corresponding integral (2.6) is called hyper elliptic integral (see [2]).

Remark 1: For the general positive interval $0 \leq x_1 < x < x_2$ and all positive real values of m, the value of $x^{\frac{2}{m}}$ is always positive which may be less than or (equal to) or greater than unity. We have checked it graphically using Mathematica Software.

Therefore, we shall represent Binomial theorem ${}_1F_0[-\frac{1}{2}; \text{---}; -x^{\frac{2}{m}}]$ by means of contour integral (in place of power series expansion) due to ambiguity in the values of $x^{\frac{2}{m}}$.

Now using contour integral representation of ${}_1F_0$ (by putting $q = 0$ in (1.11)), we get

$$s = \int_{x_1}^{x_2} \left\{ \frac{1}{2\pi\omega\,\Gamma(-\frac{1}{2})} \int_{-\omega\infty}^{+\omega\infty} \Gamma(-\zeta)\Gamma(-\frac{1}{2}+\zeta)\,(x)^{\frac{2\zeta}{m}}\,\mathrm{d}\zeta \right\} \mathrm{d}x;\; x \neq 0 \tag{2.8}$$

Now change the order of integration in above double integral because all four limits in (2.8) are constants, therefore we have

$$s = -\frac{1}{4\pi\omega\,\sqrt{\pi}} \int_{-\omega\infty}^{+\omega\infty} \Gamma(-\zeta)\Gamma(-\frac{1}{2}+\zeta) \left\{ \int_{x_1}^{x_2} x^{\frac{2\zeta}{m}}\,\mathrm{d}x \right\} \mathrm{d}\zeta \tag{2.9}$$

$$= -\frac{m\,x_2}{4\pi\omega\,\sqrt{\pi}} \int_{-\omega\infty}^{+\omega\infty} \Gamma(-\zeta)\Gamma(-\frac{1}{2}+\zeta) \left(\frac{x_2^{\frac{2\zeta}{m}}}{2\zeta+m} \right) \mathrm{d}\zeta + $$

$$+ \frac{m\,x_1}{4\pi\omega\,\sqrt{\pi}} \int_{-\omega\infty}^{+\omega\infty} \Gamma(-\zeta)\Gamma(-\frac{1}{2}+\zeta) \left(\frac{x_1^{\frac{2\zeta}{m}}}{2\zeta+m} \right) \mathrm{d}\zeta. \tag{2.10}$$

$$= -\frac{m\,x_2}{4\pi\omega\,\sqrt{\pi}} \int_{-\omega\infty}^{+\omega\infty} \Gamma(-\zeta)\Gamma(-\frac{1}{2}+\zeta)\, \frac{\Gamma(2\zeta+m)\,x_2^{\frac{2\zeta}{m}}}{\Gamma(2\zeta+m+1)}\,\mathrm{d}\zeta +$$

$$+ \frac{m\,x_1}{4\pi\omega\,\sqrt{\pi}} \int_{-\omega\infty}^{+\omega\infty} \Gamma(-\zeta)\Gamma(-\frac{1}{2}+\zeta)\, \frac{\Gamma(2\zeta+m)\,x_1^{\frac{2\zeta}{m}}}{\Gamma(2\zeta+m+1)}\,\mathrm{d}\zeta. \tag{2.11}$$

Now applying Legendre's duplication formula(1.7) for $\Gamma(2\zeta + m)$ and $\Gamma(2\zeta + m + 1)$, after simplification we get

$$s = -\frac{m\,x_2}{8\pi\omega\,\sqrt{\pi}} \int_{-\omega\infty}^{+\omega\infty} \frac{\Gamma(-\zeta)\Gamma(-\frac{1}{2}+\zeta)\Gamma(\frac{m}{2}+\zeta)\,(x_2^{\frac{2}{m}})^{\zeta}}{\Gamma(\frac{m+2}{2}+\zeta)}\,\mathrm{d}\zeta +$$

$$+ \frac{m\,x_1}{8\pi\omega\,\sqrt{\pi}} \int_{-\omega\infty}^{+\omega\infty} \frac{\Gamma(-\zeta)\Gamma(-\frac{1}{2}+\zeta)\Gamma(\frac{m}{2}+\zeta)\,(x_1^{\frac{2}{m}})^{\zeta}}{\Gamma(\frac{m+2}{2}+\zeta)}\,\mathrm{d}\zeta. \tag{2.12}$$

Now expressing above contour integrals of (2.12) in the form of Gauss function ${}_2F_1$ (put $q = 1$ in (1.11)), we obtain

$$s = x_2\;{}_2F_1\left[\begin{matrix} -\frac{1}{2},\, \frac{m}{2} & ; & \\ & & -\,x_2^{\frac{2}{m}} \\ \frac{m+2}{2} & ; & \end{matrix}\right] - x_1\;{}_2F_1\left[\begin{matrix} -\frac{1}{2},\, \frac{m}{2} & ; & \\ & & -\,x_1^{\frac{2}{m}} \\ \frac{m+2}{2} & ; & \end{matrix}\right], \tag{2.13}$$

$$\left(|\arg(x_1^{\frac{2}{m}})| < \pi \text{ and } |\arg(x_2^{\frac{2}{m}})| < \pi \right)$$

The contour integral representation (2.12) or (2.13) is the general formula for rectification under the conditions cited above. For calculation and verification purposes, we shall discuss following possible three cases.

When $0 \leq x_1^{\frac{2}{m}} \leq 1$ and $0 < x_2^{\frac{2}{m}} \leq 1$. In this case, power series representation of two Gauss functions obtained by (2.12) or (2.13) are always convergent, therefore case-I arises.

When $0 \leq x_1^{\frac{2}{m}} \leq 1$ and $x_2^{\frac{2}{m}} > 1$ then apply analytic continuation formula (1.12) in first Gauss function of R.H.S. of (2.13), we obtain case-II.

When $x_1^{\frac{2}{m}} > 1$ and $x_2^{\frac{2}{m}} > 1$ then apply analytic continuation formula (1.12) for both Gauss functions of (2.13), we obtain case-III.

3. SOME NUMERICAL EXAMPLES

Example 1:

The rectification of semi cubical parabola $f(x) = \frac{2}{3} x^{\frac{3}{2}}$, can be solved by the substitution $m = 2$ in general curve of theorem 2.1 with interval [3, 8]. Here $x_1^{\frac{2}{m}} > 1$ and $x_2^{\frac{2}{m}} > 1$, therefore, we shall apply case-III for rectification. Hence length of the arc of $y = \frac{2}{3} x^{\frac{3}{2}}$ in the interval [3, 8] is given by

$$s_1 = \frac{16\sqrt{8}}{3}\, {}_1F_0\left[\begin{array}{ccc} -\frac{3}{2} & ; & \\ & & -\frac{1}{8} \\ \text{—} & ; & \end{array}\right] - 2\sqrt{3}\, {}_1F_0\left[\begin{array}{ccc} -\frac{3}{2} & ; & \\ & & -\frac{1}{3} \\ \text{—} & ; & \end{array}\right] \tag{3.1}$$

$$s_1 = \frac{32\sqrt{2}}{3}\left(1 + \frac{1}{8}\right)^{\frac{3}{2}} - 2\sqrt{3}\left(1 + \frac{1}{3}\right)^{\frac{3}{2}} = \frac{38}{3} \text{ unit.} \tag{3.2}$$

Example 2:

The rectification of semi cubical parabola $f(x) = \frac{2}{3} x^{\frac{3}{2}}$, can be solved by the substitution $m = 2$ in general curve with interval [0, 3]. Here $x_1^{\frac{2}{m}} = 0$ and $x_2^{\frac{2}{m}} > 1$, therefore, we shall apply case-II for rectification. Hence length of the arc of the curve $y = \frac{2}{3} x^{\frac{3}{2}}$ in the interval [0, 3] is given by

$$s_2 = \frac{6\sqrt{3}}{3}\, {}_1F_0\left[\begin{array}{ccc} -\frac{3}{2} & ; & \\ & & -\frac{1}{3} \\ \text{—} & ; & \end{array}\right] - \frac{\Gamma(2)\,\Gamma(-\frac{3}{2})}{2\sqrt{\pi}} \tag{3.3}$$

$$s_2 = 2\sqrt{3}\left(1 + \frac{1}{3}\right)^{\frac{3}{2}} - \frac{2}{3} = \frac{14}{3} \text{ unit.} \tag{3.4}$$

Example 3:

The rectification of $f(x) = \frac{(2k+1)}{(2k+3)} x^{\frac{(2k+3)}{(2k+1)}}$ can be solved by the substitution $m = \frac{2k+1}{2}$; $k \in \mathbb{Z}_0^+$ in general curve with interval [0, 1]. Here $x_1^{\frac{4}{2k+1}} = 0$ and $x_2^{\frac{4}{2k+1}} = 1$, therefore, we shall apply case-I for rectification. Hence length of the arc in the interval [0, 1] is given by

$$s_3 = {}_2F_1\left[\begin{array}{ccc} -\frac{1}{2}, \frac{2k+1}{4} & ; & \\ & & -1 \\ \frac{2k+5}{4} & ; & \end{array}\right] \tag{3.5}$$

Now use a summation theorem (1.20) of Choi-Rathi-Malani (with $a = -\frac{1}{2}$, $b = \frac{2k+1}{4}$ and $p = k + 1$) in (3.5) and apply some properties of Gamma functions, we get

$$s_3 = -\frac{(2k+1)\,\Gamma(\frac{2k+1}{4})}{4\,\sqrt{\pi}\,(3-2k)\,(\frac{7-2k}{4})_k} \sum_{r=0}^{k+1} \left\{ \frac{(k+1)!\,(-1)^r\,\Gamma(\frac{2r-1}{4})}{r!\,(k-r+1)!\,\Gamma(\frac{r-k+1}{2})} \right\} \text{unit.} \tag{3.6}$$

Example 4:
For rectification of cubical parabola $f(x) = \frac{x^3}{3}$ substitute $m = \frac{1}{2}$ in general curve with interval [0, 1]. Here $x_1^{\frac{2}{m}} = 0$ and $x_2^{\frac{2}{m}} = 1$, therefore, we shall apply case-I for rectification. Hence length of the arc in the interval [0, 1] is given by

$$s_4 = {}_2F_1 \left[\begin{array}{lll} -\frac{1}{2}, \frac{1}{4} & ; & \\ & & -1 \\ \frac{5}{4} & ; & \end{array} \right] \tag{3.7}$$

Now use a result (1.20) of Choi-Rathi-Malani (with $a = -\frac{1}{2}$, $b = \frac{1}{4}$ and $p = 1$) in (3.7) or $k = 0$ in (3.6) and apply some properties of Gamma functions, we get the exact length of the curve discussed by Nystedt and others during seventeenth century.

$$s_4 = -\frac{\Gamma(\frac{5}{4})}{3\,\sqrt{\pi}} \left\{ \frac{\Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{2})} - \Gamma\Big(\frac{1}{4}\Big) \right\} = \frac{\sqrt{2}}{3} + \frac{\{\Gamma(\frac{1}{4})\}^2}{12\,\sqrt{\pi}} \text{ unit.} \tag{3.8}$$

Example 5:
The rectification of $f(x) = \frac{3}{5}\,x^{\frac{5}{3}}$ can be solved by the substitution $m = \frac{3}{2}$ in general curve with interval [0, 1]. Here $x_1^{\frac{2}{m}} = 0$ and $x_2^{\frac{2}{m}} = 1$, therefore, we shall apply case-I for rectification. Hence length of the arc in the interval [0, 1] is given by

$$s_5 = {}_2F_1 \left[\begin{array}{lll} -\frac{1}{2}, \frac{3}{4} & ; & \\ & & -1 \\ \frac{7}{4} & ; & \end{array} \right] \tag{3.9}$$

Now using a result (1.20) of Choi-Rathi-Malani (with $a = -\frac{1}{2}$, $b = \frac{3}{4}$ and $p = 2$) in (3.9) or $k = 1$ in (3.6), we get

$$s_5 = -\frac{3\,\Gamma(\frac{3}{4})}{5\,\sqrt{\pi}} \left\{ \Gamma\Big(\frac{3}{4}\Big) - \frac{2\,\Gamma(\frac{1}{4})}{\sqrt{\pi}} \right\} = \frac{6\,\sqrt{2}}{5} - \frac{3\{\Gamma(\frac{3}{4})\}^2}{5\,\sqrt{\pi}} \text{ unit.} \tag{3.10}$$

Example 6:
The rectification of parabola $f(x) = \frac{x^2}{2}$ can be solved by the substitution $m = 1$ in general curve with interval [0, 1]. Here $x_1^{\frac{2}{m}} = 0$ and $x_2^{\frac{2}{m}} = 1$, therefore, we shall apply case-I for rectification. Hence length of the arc in the interval [0, 1] is given by

$$s_6 = {}_2F_1 \left[\begin{array}{lll} -\frac{1}{2}, \frac{1}{2} & ; & \\ & & -1 \\ \frac{3}{2} & ; & \end{array} \right]. \tag{3.11}$$

Now using a result (1.14) of Prudnikov *et al.* with $z = -1$. After that apply some properties of Gamma functions, we get

$$s_6 = \frac{1}{2}\left(\frac{\sin^{-1} i}{i} + \sqrt{2}\right) = \frac{1}{2}\{\sinh^{-1}(1) + \sqrt{2}\} = \frac{1}{2}\{\ln(1 + \sqrt{2}) + \sqrt{2}\}\,\text{unit}. \tag{3.12}$$

Example 7:
The rectification of semi cubical parabola $f(x) = \frac{2}{3}x^{\frac{3}{2}}$ can be solved by the substitution $m = 2$ in general curve with interval [0, 1]. Here $x_1^{\frac{2}{m}} = 0$ and $x_2^{\frac{2}{m}} = 1$, therefore, we shall apply case-I for rectification. Hence length of the arc in the interval [0, 1] is given by

$$s_7 = {}_2F_1\left[\begin{matrix} -\frac{1}{2}, 1 & ; & \\ & & -1 \\ 2 & ; & \end{matrix}\right]. \tag{3.13}$$

Now using a result (1.15) of Prudnikov *et al.* with $z = -1$. After that apply some properties of Gamma functions, we get

$$s_7 = \frac{-2}{3}\left\{1 - (1+1)^{\frac{3}{2}}\right\} = \frac{2}{3}(2\sqrt{2} - 1)\,\text{unit}. \tag{3.14}$$

Example 8:
The rectification of $f(x) = \frac{3}{4}x^{\frac{4}{3}}$ can be solved by the substitution $m = 3$ in general curve with interval [0, 1]. Here $x_1^{\frac{2}{m}} = 0$ and $x_2^{\frac{2}{m}} = 1$, therefore, we shall apply case-I for rectification. Hence length of the arc in the interval [0, 1] is given by

$$s_8 = {}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{3}{2} & ; & \\ & & -1 \\ \frac{5}{2} & ; & \end{matrix}\right]. \tag{3.15}$$

Now using a result (1.16) of Prudnikov *et al.* with $z = -1$. After that apply some properties of Gamma functions, we get

$$s_8 = \frac{9\sqrt{2}}{8} - \frac{3}{8}\ln(1 + \sqrt{2})\,\text{unit}. \tag{3.16}$$

Example 9:
The rectification of $f(x) = \frac{4}{5}x^{\frac{5}{4}}$ can be solved by the substitution $m = 4$ in general curve with interval [0, 1]. Here $x_1^{\frac{2}{m}} = 0$ and $x_2^{\frac{2}{m}} = 1$, therefore, we shall apply case-I for rectification. Hence length of the arc in the interval [0, 1] is given by

$$s_9 = {}_2F_1\left[\begin{matrix} -\frac{1}{2}, 2 & ; & \\ & & -1 \\ 3 & ; & \end{matrix}\right]. \tag{3.17}$$

Now using a result (1.17) of Prudnikov *et al.* with $z = -1$, after simplification we get

$$s_9 = \frac{8}{15}(1 + \sqrt{2})\,\text{unit}. \tag{3.18}$$

Example 10:
The rectification of $f(x) = \frac{5}{6} x^{\frac{6}{5}}$ can be solved by the substitution $m = 5$ in general curve with interval [0, 1]. Here $x_1^{\frac{2}{m}} = 0$ and $x_2^{\frac{2}{m}} = 1$, therefore, we shall apply case-I for rectification. Hence length of the arc in the interval [0, 1] is given by

$$s_{10} = {}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{5}{2} & ; & \\ & & -1 \\ \frac{7}{2} & ; & \end{matrix}\right]. \tag{3.19}$$

Now using a result (1.18) of Prudnikov *et al.* with $z = -1$, after simplification we get

$$s_{10} = \frac{35}{24\sqrt{2}} + \frac{15}{48} \ln(1 + \sqrt{2}) \text{ unit.} \tag{3.20}$$

Example 11:
The rectification of $f(x) = \frac{6}{7} x^{\frac{7}{6}}$ can be solved by the substitution $m = 6$ in general curve with interval [0, 1]. Here $x_1^{\frac{2}{m}} = 0$ and $x_2^{\frac{2}{m}} = 1$, therefore, we shall apply case-I for rectification. Hence length of the arc in the interval [0, 1] is given by

$$s_{11} = {}_2F_1\left[\begin{matrix} -\frac{1}{2}, 3 & ; & \\ & & -1 \\ 4 & ; & \end{matrix}\right]. \tag{3.21}$$

Now using a result (1.19) of Prudnikov *et al.* with $z = -1$, after simplification we get

$$s_{11} = \frac{4}{35}(11\sqrt{2} - 4) \text{ unit.} \tag{3.22}$$

Remark 2: Taking suitable values of m, x_1 and x_2 in integral (2.6) according to examples, we have verified the answers of examples 1-11 numerically.

CONCLUSION

We conclude our present investigations by observing that several hypergeometric forms of rectification of transcendental curves or other curves (in suitable intervals) can be obtained in an analogous manner by using the theory of Mellin-Barnes contour integral and analytic continuation formula. It may be useful for further analysis in the field of mathematical physics, astronomy and other branches of science and engineering.

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