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HYPONORMALITY OF TOEPLITZ OPERATORS WITH CIRCULANT-TYPE SYMMETRIC TRIGONOMETRIC POLYNOMIAL SYMBOLS

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Abstract

This paper presents some necessary and sufficient conditions for the hyponormality of Toeplitz operators on Hardy space $H^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle in the complex plane, in terms of circulant-type trigonometric polynomial symbols under certain conditions concerning the coefficients of the polynomial.

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1. Introduction

Let H be a complex Hilbert space and $\mathcal{B}(H)$ be the collection of all bounded linear operators on H. Then a bounded linear operator $T \in \mathcal{B}(H)$ is said to be hyponormal if its self commutator $[T^*, T] := T^*T - TT^*$ is positive semi-definite. Let $\mathbb{T}$ denote the unit circle in the complex plane $\mathbb{C}$. For any $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator on the Hardy space $H^2(\mathbb{T})$ with the symbol φ, denoted by T_φ, is given by $T_\varphi f = P(\varphi \cdot f)$, where f belongs to $H^2(\mathbb{T})$ and P denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. The objective of this article is to study the hyponormality of Toeplitz operators with circulant-type trigonometric polynomials that has been remained unobserved in the previous literature.

2. Preliminaries

In 1988, Cowen[1] dealt with the problems of determining which symbols induce hyponormal Toeplitz operators and gave a very elegant characterisation. However, here in our work we will employ the variant of Cowen's theorem proposed by Nakazi and Takahashi in the paper [7] which is stated below.

Theorem 2.1. *Suppose that $\varphi \in L^\infty(\mathbb{T})$ is arbitrary and write $\mathcal{E}(\varphi) = \{k \in H^\infty(\mathbb{T}) : \|k\|_\infty \leq 1$ and $\varphi - k\bar{\varphi} \in H^\infty(\mathbb{T})\}$. Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is non empty.*

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Later on, for arbitrary trigonometric polynomials, Zhu in his paper [9], employed variant of the Cowen's Theorem very exclusively to show the hyponormality of Toeplitz operators with trigonometric polynomial symbols via classical Schur's algorithm in function theory and by denoting the Schur's functions as a sequence $\{\Phi_n\}$, he obtained an abstract characterisation (Theorem 2.2) for the hyponormality of T_φ and also listed first three Schur's function Φ_0, Φ_1 and Φ_2. Furthermore, by applying this characterisation, he gave a set of necessary and sufficient conditions explicitly for the hyponormality of T_φ, where $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$, whenever $N \leq 3$.

THEOREM 2.2. *[9] If $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$, where $a_N \neq 0$ and if*

$$\begin{pmatrix} \bar{c}_0 \\ \bar{c}_1 \\ \vdots \\ \bar{c}_{N-1} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{N-1} & a_N \\ a_2 & a_3 & \cdots & a_N & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_N & 0 & \cdots & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \bar{a}_{-1} \\ \bar{a}_{-2} \\ \vdots \\ \bar{a}_{-N} \end{pmatrix}, \tag{2.1}$$

then T_φ is hyponormal if and only if $|\Phi_n(c_0, ..., c_n)| \leq 1$ for each $n = 0, 1, ..., N-1$.

In this theorem, it is shown that if $k(z) = \sum_{j=0}^{\infty} c_j z^j$ satisfies $\varphi - k\bar{\varphi} \in H^\infty(\mathbb{T})$ then the hyponormality of T_φ is independent of c_j's for $j \geq N$. In our article, we employ this idea of Zhu exclusively to study the hyponormality of Toeplitz operators under certain conditions called circulant-type trigonometric polynomials.
However, Zhu's theorem was reformulated and further simplified by Kim and Lee [6] in order to give a tractable necessary and sufficient conditions for the hyponormality of T_φ, in the case, where the coefficients of φ satisfy a symmetry condition. If $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ then, it is said to be a symmetric-type trigonometric polynomial if the coefficients of φ satisfy the following condition:

$$\overline{a_N} \begin{pmatrix} a_{-1} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \overline{a_1} \\ \vdots \\ \overline{a_N} \end{pmatrix} \tag{2.2}$$

THEOREM 2.3. *[6] If $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where $m \leq N$ and $a_N \neq 0$, and $\{\Phi_n\}$ is the sequence of Schur's functions, then T_φ is hyponormal if and only if $|\Phi_n(c_0, \ldots, c_n)| \leq 1$ for each $n = 0, 1, \ldots, N-1$, where c_n's are given by the following recurrence relation:*

$$\begin{cases} c_0 = c_1 = \ldots = c_{N-m-1} = 0 \\ c_{N-m} = \frac{a_{-m}}{\overline{a_N}} \\ c_n = (\overline{a_N})^{-1} \left(a_{-N+n} - \sum_{j=N-m}^{n-1} c_j \overline{a_{N-n+j}} \right) \quad for \\ \quad n = N-m+1, \ldots, N-1. \end{cases} \tag{2.3}$$

In [4], Schur's function Φ_3 was evaluated explicitly and then by using this result the hyponormality of T_φ was studied under certain conditions about the coefficients of φ. In ([5], [2]), the symmetric-type trigonometric polynomials were considered under various conditions and then hyponormality was discussed. In our paper, we employ the following lemma exclusively to determine the hyponormality of T_φ.

LEMMA 2.4. *[5] Suppose that $k(z) = \sum_{j=0}^{\infty} c_j z^j$ is in the closed unit ball of $H^{\infty}(\mathbb{T})$ and that $\{\Phi_n\}$ is the sequence of Schur's functions associated with $\{c_n\}$. If $c_1 = c_2 = c_3 = ... = c_{n-1} = 0$ and $c_n \neq 0$, then $\Phi_0 = c_0$ and $\Phi_1 = \Phi_2 = \Phi_3 ... = \Phi_{n-1} = 0$; $\Phi_n = \frac{c_n}{1-|c_0|^2}$; $\Phi_{n+1} = \frac{c_{n+1}}{(1-|c_0|^2)(1-|\Phi_n|^2)}$; $\Phi_{n+2} = \frac{(1-|\Phi_n|^2)c_{n+2}c_n + |\Phi_n|^2 c_{n+1}^2}{c_n(1-|c_0|^2)(1-|\Phi_n|^2)^2(1-|\Phi_{n+1}|^2)}$.*

Circulant-type Symbols:[3] Let $a_0, a_1, a_2, ..., a_N$ be fixed complex numbers and let $\alpha \in \mathbb{C}$ be such that $|\alpha| = 1$. Then a trigonometric polynomial $\varphi(e^{i\theta}) = \sum_{-m}^{N} b_n e^{in\theta}$, $(m \leq N)$, is said to be a circulant-type trigonometric polynomial, if the coefficients b_n satisfy the following combinatorial constraints:

$$\begin{cases} b_n & = a_n \qquad (0 \leq n \leq N-m) \\ & = a_n + \alpha a_{n+1} \qquad (N-m+1 \leq n \leq N-1); \\ & = a_N \qquad (n = N) \\ b_{-n} & = e^{i\omega}(a_{N-n+1} + \bar{\alpha} a_{N-n}) \qquad (1 \leq n \leq m-1) \\ & = e^{i\omega} a_{N-m+1} \qquad (n = m) \end{cases} \tag{2.4}$$

Remark: In the above expression, if $\alpha = 0$ and if $a_1 = a_2 = ... = a_{N-m} = 0$ when $m < N$, then φ reduces to a **circulant trigonometric polynomial with argument** ω. In **Theorem 2.1** of the paper [8], the second author considered an extremal case of a circulant and symmetric-type trigonometric polynomial symbol and gave a set of necessary and sufficient conditions to determine the hyponormality of T_φ. The theorem can be restated below:

THEOREM 2.5. *If $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ (with $a_N \neq 0$) is a circulant polynomial with argument $2n\pi$ for some integer n, which satisfies*

$$\overline{a_N} \begin{pmatrix} a_{-4} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \overline{a_4} \\ \vdots \\ \overline{a_N} \end{pmatrix} \tag{2.5}$$

and $\beta = \frac{a_1\overline{a_2} - a_2\overline{a_3}}{|a_1|^2 - |a_3|^2}$, then T_φ is hyponormal if and only if

(i) $|a_3| \leq |a_1| \leq |a_N|$

(ii) $|\beta| \leq 1$

(iii) $\left|1 - \beta\left(\frac{a_2}{a_1}\right) + \beta^2\left(\frac{a_3}{a_1}\right)\right| \leq 1 - |\beta|^2$

In **Theorem 3.1** of Section 3 of our article, we deal with the hyponormality of T_φ by considering circulant-type trigonometric polynomial symbols given by (2.4)where the coefficients of φ satisfy a set of different symmetric conditions.

3. Main Results

This section gives some necessary and sufficient conditions for the hyponormality of the Toeplitz operator T_φ with circulant-type trigonometric polynomial symbols concerning the coefficients of the symbol φ.

Theorem 3.1. *Let $a_0, a_1, a_2, ..., a_N$ be fixed complex numbers and let $\alpha \in \mathbb{C}$ be such that $|\alpha| = 1$. Let $\varphi(e^{i\theta}) = \sum_{n=-N}^{N} b_n e^{in\theta}$ be a circulant-type trigonometric polynomial with the following combinatorial constraints:*

$$\begin{cases} b_n & = a_n \quad (n = 0) \\ & = a_n + \alpha a_{n+1} \quad 1 \le n \le N-1) \\ & = a_N \quad (n = N) \\ b_{-n} & = e^{i\omega}(a_{N-n+1} + \bar{\alpha} a_{N-n}) \quad (1 \le n \le N-1) \\ & = e^{i\omega} a_1 \quad (n = N) \end{cases} \tag{3.1}$$

Let $A = |a_N|^2 - |a_1|^2$, $D_1 = det\begin{pmatrix} \overline{a_N} & a_1 \\ a_2 + \alpha a_3 & a_{N-1} + \bar{\alpha} a_{N-2} \end{pmatrix}$ and

$D_2 = det\begin{pmatrix} \overline{a_N} & a_1 \\ a_3 + \alpha a_4 & a_{N-2} + \bar{\alpha} a_{N-3} \end{pmatrix}$.

(I). *Suppose that the coefficients of φ satisfy the following condition:*

$\overline{b_N}\begin{pmatrix} b_{-2} \\ \vdots \\ b_{-N} \end{pmatrix} = b_{-N}\begin{pmatrix} \overline{b_2} \\ \vdots \\ \overline{b_N} \end{pmatrix}$. *Then T_φ is hyponormal if and only if*

1. $|a_1| \le |a_N|$;
2. $\left|1 + \bar{\alpha}\frac{a_{N-1}\overline{a_N} - a_1\overline{a_2}}{A}\right| \le 1$.

(II). *Suppose that the coefficients of φ satisfy the following condition:*

$\overline{b_N}\begin{pmatrix} b_{-1} \\ b_{-3} \\ \vdots \\ b_{-N} \end{pmatrix} = b_{-N}\begin{pmatrix} \overline{b_1} \\ \overline{b_3} \\ \vdots \\ \overline{b_N} \end{pmatrix}$. *Then T_φ is hyponormal if and only if*

1. $|a_1| \le |a_N|$;
2. $|D_1| \le A$;
3. $\left|D_1\left(\overline{\frac{a_{N-1}}{a_N} + \alpha}\right)\right| \le \frac{A^2 - |D_1|^2}{A}$.

***(III).*:** *Suppose that the coefficients of φ satisfy the following condition:*

$\overline{b_N}\begin{pmatrix} b_{-3} \\ \vdots \\ b_{-N} \end{pmatrix} = b_{-N}\begin{pmatrix} \overline{b_3} \\ \vdots \\ \overline{b_N} \end{pmatrix}$. *Then T_φ is hyponormal if and only if*

1. $|a_1| \le |a_N|$;
2. $|D_1| \le A$;
3. $\left|A + \bar{\alpha}(\bar{a}_N a_{N-1} - \bar{a}_2 a_1) - D_1\left(\overline{\frac{a_{N-1}}{a_N} + \alpha}\right)\right| \le \frac{A^2 - |D_1|^2}{A}$

***(IV).*:** *Suppose that the coefficents of φ satisfy the following condition:*

$\overline{b_N}\begin{pmatrix} b_{-1} \\ b_{-2} \\ b_{-4} \\ \vdots \\ b_{-N} \end{pmatrix} = b_{-N}\begin{pmatrix} \overline{b_1} \\ \overline{b_2} \\ \overline{b_4} \\ \vdots \\ \overline{b_N} \end{pmatrix}$ *Then, T_φ is hyponormal if and only if*

1. $|a_1| \leq |a_N|$;
2. $|D_2| \leq A$;
3. $\left|\frac{a_{N-1}}{a_N} + \alpha\right| \leq \left|\frac{A}{D_2} - \frac{\overline{D_2}}{A}\right|$;
4. $\left|D_2[A^2(\bar{a}_{N-1}^2 - \overline{a_N a_{N-2}} + (\overline{\alpha a_N})^2 + \overline{\alpha a_{N-1} a_N}) + |D_2|^2(\overline{\alpha a_{N-1} a_N + a_N a_{N-2}})]\right| \leq \left|A\left(|a_N|^2|A - \frac{|D_2|^2}{A}|^2 - |D_2|^2|a_{N-1} + \alpha a_N|^2\right)\right|$.

***(V).**: Suppose that the coefficients of φ satisfy the following condtion:*
$\overline{b_N}\begin{pmatrix} b_{-4} \\ \vdots \\ b_{-N} \end{pmatrix} = b_{-N}\begin{pmatrix} \overline{b_4} \\ \vdots \\ \overline{b_N} \end{pmatrix}$. *Then T_φ is hyponormal if and only if*
1. $|a_1| \leq |a_N|$;
2. $|D_2| \leq A$;
3. $\left|D_1 - D_2\overline{(\frac{a_{N-1}}{a_N} + \alpha)}\right| \leq A - \frac{|D_2|^2}{A}$;
4. $A|\overline{a_N}^2 A + \overline{\alpha a_N}^2(a_{N-1}\bar{a}_N - a_1\bar{a}_2) - D_2\bar{a}_N(\overline{a_{N-2} + \alpha a_{N-1}}) - D_1\bar{a}_N(\overline{a_{N-1} + \alpha a_N}) + D_2(\overline{a_{N-1} + \alpha a_N})^2 + \left(\frac{\overline{D_2}}{A^2 - |D_2|^2}\right)(\bar{a}_N D_1 - D_2(\overline{a_{N-1} + \alpha a_N}))^2 \,| \leq ||a_N|^2(A^2 - |D_2|^2) - \left(\frac{A^2}{A^2 - |D_2|^2}\right)$ $|\bar{a}_N D_1 - D_2(\overline{a_{N-1} + \alpha a_N})|^2|$.

4. Proof of Theorem 3.1

Proof. Under the combinatorial constraints, the structure of φ will be
$\varphi(z) = e^{i\omega} a_1 z^{-N} + e^{i\omega} \sum_{i=1}^{N-1}(a_{i+1} + \bar{\alpha} a_i) z^{-N+i} + a_0 + \sum_{i=1}^{N-1}(a_i + \alpha a_{i+1}) z^i + a_N z^N$
In view of the variant of Cowen's Theorem, to show the hyponormality of T_φ, our objective will be to determine a unique analytic polynomial $k(z) = \sum_{j=0}^{\infty} c_j z^j$ in the closed unit ball of $H^\infty(\mathbb{T})$ such that $\varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})$, which is equivalent to saying that T_φ will be hyponormal if there exists a unique analytic polynomial $k(z) = \sum_{j=0}^{\infty} c_j z^j$ in the closed unit ball of $H^\infty(\mathbb{T})$ satisfying the condition that

$$k \sum_{n=1}^{N} \overline{b_n} z^{-n} - \sum_{n=1}^{N} b_{-n} z^{-n} \in H^\infty(\mathbb{T}) \tag{4.1}$$

By a straightforward calculation and then by equating the co-analytic coefficients to zero, the equation (4.1) ultimately reduces to the recurrence relation given by the equation (2.3). Thus, the Theorem 2.3 can be applied to determine the hyponormality of T_φ via Zhu's Theorem.

(I). We observe that φ together with the combinatorial constraints will automatically satisfy the following symmetric condition:

$$\overline{a_N}\begin{pmatrix} a_{N-1} + \bar{\alpha} a_{N-2} \\ a_{N-2} + \bar{\alpha} a_{N-3} \\ \vdots \\ a_2 + \bar{\alpha} a_1 \\ a_1 \end{pmatrix} = a_1 \begin{pmatrix} \overline{a_2 + \alpha a_3} \\ \overline{a_3 + \alpha a_4} \\ \vdots \\ \overline{a_{N-1} + \alpha a_N} \\ \overline{a_N} \end{pmatrix} \tag{4.2}$$

Now, by applying the equations (2.3) and (4.2), c_i's can be computed as follows:
$c_0 = \frac{b_{-N}}{\overline{b_N}} = \frac{e^{i\omega}a_1}{\overline{a_N}}$;
$c_1 = (\overline{b_N})^{-1}\{b_{-N+1} - c_0\overline{b_{N-1}}\} = (\overline{a_N})^{-2}e^{i\omega}\{\overline{a_N}(a_2 + \bar{\alpha}a_1) - a_1(\overline{a_{N-1}} + \overline{\alpha a_N})\} = 0$;
$c_2 = \ldots = c_{N-2} = 0$;
$c_{N-1} = (\overline{b_N})^{-1}(b_{-1} - \sum_{j=0}^{N-2} c_j\overline{b_{j+1}})$
$= (\overline{b_N})^{-1}(b_{-1} - c_0\overline{b_1} - \ldots - c_{N-2}\overline{b_{N-1}})$
$= (\overline{a_N})^{-1}\{e^{i\omega}(a_N + \bar{\alpha}a_{N-1}) - \frac{e^{i\omega}a_1}{\overline{a_N}}(\overline{a_1 + \alpha a_2})\}$
$= (\overline{a_N})^{-2}e^{i\omega}\{\overline{a_N}(a_N + \bar{\alpha}a_{N-1}) - a_1(\overline{a_1 + \alpha a_2})\}$
$= (\overline{a_N})^{-2}e^{i\omega}\{A + \bar{\alpha}(a_{N-1}\bar{a}_N - a_1\bar{a}_2)\}$

Now, if $\hat{k}(0), \hat{k}(1), \ldots\ldots, \hat{k}(N-1)$ are the Fourier coefficients of k, then $\hat{k}(n) = c_n$ for $n = 1, 2, .., N-1$, and also, as the hyponormality of T_φ is independent of c_j's for $j \geq N$, $k(z) = c_0 + c_{N-1}z^{N-1}$ is the required unique analytic polynomial of degree less than N in $H^\infty(\mathbb{T})$. Now, if $\{\Phi_n\}$ is the sequence of Scur's function associated with c_n's, then by Lemma 2.4, Φ_n's can be computed as follows:

$\Phi_0 = c_0 = \frac{e^{i\omega}a_1}{\overline{a_N}}; \Phi_1 = \ldots = \Phi_{N-3} = \Phi_{N-2} = 0;$
$\Phi_{N-1} = \frac{c_{N-1}}{(1-|c_0|^2)} = (\overline{a_N})^{-2}e^{i\omega}\{A + \bar{\alpha}(a_{N-1}\bar{a}_N - a_1\bar{a}_2)\}.\frac{|a_N|^2}{A}.$

Now, by Theorem 2.3 the result follows.
Then by applying the same technique as that of the proof (I), the other results (II) through (V) can be computed easily □

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