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STUDY OF η-EINSTEIN SOLITONS IN LORENTZIAN β-KENMOTSU MANIFOLD

SHYAM KISHOR and ANJUM ARA ✉

Abstract

This paper is dedicated to the study of η-Einstein solitons in Lorentzian β-Kenmotsu Manifolds, focusing on specific conditions. We will begin by presenting the fundamental concepts of Lorentzian β-Kenmotsu Manifolds in a clear and concise manner. Subsequently, our research will delve into the study of conformal η-Einstein soliton on n-dimensional Lorentzian β-Kenmotsu Manifold with h Parallel. We will also explore n-dimensional Lorentzian β-Kenmotsu Manifold admitting conformal η-Einstein soliton. Additionally, we will study conformal η-Einstein solitons on n-dimensional Lorentzian β-Kenmotsu Manifold satisfying R(ξ, X).S = 0 . After that, we investigate einstein semi-symmetric n-dimensional Lorentzian β-Kenmotsu Manifold admitting conformal η-Einstein solitons. Lastly, we consider conformal η-Einstein solitons on n-dimensional Lorentzian β-Kenmotsu Manifold with torse-forming vector field.

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1. Introduction

In 1982, Ricci solitons were introduced by R.S. Hamilton as natural generalizations of Einstein metrics. The Ricci fow on a smooth manifold M with Riemannian metric g(t) is given by

$$\frac{\partial}{\partial t} g(t) = -2Ric, \tag{1.1}$$

where, 'Ric' is the Ricci tensor of the metric $g(t)$. A Ricci soliton is a solution of Ricci flow [7] , defined on a pseudo-Riemannian manifold (M, g) by

$$\frac{1}{2} L_V g + Ric = \lambda g, \tag{1.2}$$

where, L_V denotes the Lie-derivative with respect to $V \in \chi(M)$, 'Ric' is the Ricci tensor of g and λ is a constant. The Ricci soliton is shrinking, steady, and expanding depending on $\lambda < 0, \lambda = 0, \lambda > 0$ respectively. Otherwise, it will be called indefnite. Cho and Kimura [10], generalized the notion of Ricci soliton, by introducing the notion

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of η-Ricci soliton. Later Calin and Crasmareanu [11] studies it on Hopf hypersurfaces in complex space forms. An η-Ricci soliton equation is given by:

$$L_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.3}$$

where, S is the Ricci tensor and λ and μ are constants.
The η-Einstein soliton [12] on a Riemannian manifold (M, g) is given by,

$$L_\xi g + 2S + (2\lambda - r)g + 2\mu\eta \otimes \eta = 0, \tag{1.4}$$

where, r is the scalar curvature of the metric g and λ and μ are constants. For $\mu = 0$, the data (g, ξ, λ) is called Einstein soliton[13]. On an n-dimensional manifold conformal Einstein soliton [14]is given by,

$$L_V g + 2S + (2\lambda - r + (p + \frac{2}{n}))g = 0, \tag{1.5}$$

where, λ is real constant, p is a scalar non-dynamical field. Moreover, an n-dimensional Riemannian manifold (M, g) is said to admit conformal η-Einstein soliton if the following equation is satisfied,

$$L_\xi g + 2S + (2\lambda - r + (p + \frac{2}{n})g + 2\mu\eta \otimes \eta = 0. \tag{1.6}$$

Note that, the conformal η-Einstein soliton becomes the Einstein soliton (g, ξ, λ).
Ricci solitons and Einstein solitons are considered by many authors in different contexts for instant: on Kahler manifolds [1], on contact and Lorentzian manifolds[2], on K-contact manifolds[3] etc. In 2017, Yaning Wang[4] proved that if cosymplectic manifold M_3 admits a Ricci soliton, then either M_3 is locally flat or the potential vector field is an infnitesimal contact transformation. Also in [5, 17], some authors have provided insight on Lorentzian β-Kenmotsu Manifold. Dey et al[6]. also have set up some new results on conformal η-Einstein soliton. Very recently, $*$-Ricci soliton and Yamabe soliton and their generalizations and related research have been studied by many authors.
The paper is organized as follows:
After preliminaries in Section 2, we study conformal η-Einstein soliton on n-dimensional Lorentzian β-Kenmotsu Manifold with h Parallel. Section 3 is devoted to n-dimensional Lorentzian β-Kenmotsu Manifold admitting conformal η-Einstein soliton. In the next section, we study conformal η-Einstein solitons on n-dimensional Lorentzian β-Kenmotsu Manifold satisfying R(ξ, X).S = 0. In Section 5, we consider einstein semi-symmetric n-dimensional Lorentzian β-Kenmotsu Manifold admitting conformal η-Einstein solitons. Section 6 deals with the study of conformal η-Einstein solitons on n-dimensional Lorentzian β-Kenmotsu Manifold with torse-forming vector field.

2. Preliminaries

A differentiable manifold of dimension n is called Lorentzian β-Kenmotsu manifold if it admits a (1, 1)-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy[15]

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi\xi = 0, \tag{2.2}$$

$$\eta(\phi X) = 0, \tag{2.3}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.4}$$

$$g(X, \xi) = \eta(X), \tag{2.5}$$

$$g(\phi(X), \phi(Y)) = g(X, Y) + \eta(X)\eta(Y), \tag{2.6}$$

for all $X, Y \in T(M)$.
Also, an Lorentzian β-Kenmotsu manifold M is satisfying

$$\nabla_X \xi = -\beta[X + \eta(X)\xi], \tag{2.7}$$

$$\nabla_X \eta(Y) = \beta[g(X, Y) - \eta(X)\eta(Y)], \tag{2.8}$$

$$(\nabla_X \phi)Y = \beta[g(\phi X, Y) + \eta(Y)\phi X], \tag{2.9}$$

where '∇' denotes the operator of covariant differentiation with respect to the Lorentzian metric g.
Further, on an Lorentzian β-Kenmotsu manifold M the following relations hold

$$\eta(R(X, Y)Z) = \beta^2[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \tag{2.10}$$

$$R(\xi, X)Y = \beta^2(\eta(Y)X - g(X, Y)\xi), \tag{2.11}$$

$$R(X, Y)\xi = \beta^2(\eta(X)Y - \eta(Y)X), \tag{2.12}$$

$$S(X, \xi) = -(n-1)\beta^2\eta(X), \tag{2.13}$$

$$Q\xi = -(n-1)\beta^2\xi, \tag{2.14}$$

$$S(\xi,\xi) = (n-1)\beta^2, \tag{2.15}$$

$$g(\xi,\xi) = \eta(\xi) = -1, \tag{2.16}$$

Definition 2.1. A Lorentzian β-Kenmotsu manifold M is said to be a generalized η-Einstein manifold if its Ricci tensor S is of the form [16]

S(X,Y)= a g(X, Y)+b $\eta(X)\eta(Y)$,

where, a and b are smooth functions on M.
In particular, if $b = 0$, then an η-Einstein manifold is an Einstein manifold.

3. Conformal η-Einstein Soliton on n-dimensional Lorentzian β-Kenmotsu Manifold with h Parallel

In this section, we study Conformal η-Einstein Soliton on n-dimensional Lorentzian β-Kenmotsu Manifold with symmetric $(0,2)$-tensor field h that is parallel($\nabla h = 0$).

Theorem 3.1. *If an n-dimensional Lorentzian β-Kenmotsu manifold assumes that a symmetric* $(0,2)$*-tensor field* $h = L_\xi g + 2S + 2\mu\eta\otimes\eta$ *is parallel with respect to the Levi-Civita connection associated with g,then* (g,ξ) *yields a conformal η-Einstein soliton.*

Proof. Given that, symmetric $(0,2)$-tensor field $h = L_\xi g + 2S + 2\mu\eta\otimes\eta$ is parallel with respect to the Levi-Civita connection ∇. i.e. $\nabla h = 0$.
Applying the Ricci commutation identity, we get

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0.$$

From the above relation, we obtain

$$h(R(X,Y)Z,W) + h(Z,R(X,Y)W) = 0$$

Putting $Z = W = \xi$ in the above equation and using (2.12), we get

$$\eta(X)h(Y,\xi) - \eta(Y)h(X,\xi) = 0.$$

Again, by putting $X = \xi$ and using equation (2.1)-(2.4), the above equation reduces to

$$h(Y,\xi) = -\eta(Y)h(\xi,\xi). \tag{3.1}$$

for all vector fields Y on M. Differentiating the above equation covariantly with respect to X,

$$\nabla_X(h(Y,\xi)) = -\nabla_X(\eta(Y)h(\xi,\xi)).$$

Now, expanding the above equation by using (3.1),(2.7),(2.8) and the property $\nabla h = 0$, we obtain

$$h(X,Y) = -h(\xi,\xi)g(X,Y), \tag{3.2}$$

for all vector fields X, Y on M. Let us take

$$h = L_\xi g + 2S + 2\mu\eta \otimes \eta. \tag{3.3}$$

Now, Since

$$(L_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \tag{3.4}$$

for all vector fields X, Y on M.
Using (2.7) in above relation, we get

$$(L_\xi g)(X,Y) = -2\beta g(\phi X, \phi Y) \tag{3.5}$$

Also,

$$S(X,Y) = [\beta - \lambda + \frac{r}{2} - (\frac{p}{2} + \frac{1}{n})]g(X,Y) + (\beta - \mu)\eta(X)\eta(Y) \tag{3.6}$$

which is later proved in section 4 of this paper and given by equation (4.3)
Then, by using equation (3.5), (27) in (3.3), we obtain

$$h(\xi,\xi) = 2\lambda + (p + \frac{2}{n}) - r. \tag{3.7}$$

Again, by using (3.3) and (3.7), equation (3.2) becomes

$$(L_\xi g)(X,Y) + 2S(X,Y) + [2\lambda + (p + \frac{2}{n}) - r]g(X,Y) + 2\mu\eta(X)\eta(Y) = 0,$$

which is the conformal η-Einstein soliton. Hence, we complete the proof. □

Corollary 3.2. *Let $(M, g, F, \eta, \xi, \alpha, \beta)$ be an n-dimensional Lorentzian β-Kenmotsu manifold with α and β constants ($\beta \neq 0$). If the symmetric $(0,2)$ tensor field h satisfies the condition*

$$(L_\xi)g(X,Y) + 2S(X,Y) + 2\mu\eta(X)\eta(Y) = 0,$$

and is parallel with respect to the Levi-Civita connection associated to g, then (g, ξ, μ) becomes an η-Ricci soliton.

Next, we obtain some results on n-dimensional Lorentzian β-manifold satisfying a conformal η-Einstein soliton when the manifold is Ricci-symmetric has η-recurrent Ricci curvature tensor.

Theorem 3.3. *Let (M, g) be an n-dimensional Lorentzian-β-Kenmotsu manifold with α and β constants ($\beta \neq 0$) satisfying a conformal η-Einstein soliton.*

1. *If the manifold (M, g) is Ricci symmetric (i.e., ∇S = 0) then $\mu = \beta$.*

2. *If the Ricci tensor is η-recurrent (i.e., ∇S= η ⊗S) then* $\mu = \lambda - \frac{r}{2} + (\frac{p}{2} + \frac{1}{n})$.

Proof. From the equation (1.6), we get

$$2S(X,Y) = -g(\nabla_X\xi, Y) - g(X, \nabla_Y\xi) - [2\lambda - r + (p + \frac{2}{n})]g(X,Y) - 2\mu\eta(X)\eta(Y). \quad (3.8)$$

Now, we use the equation (2.7) into the identity (3.8) to yield

$$S(X,Y) = [\frac{r}{2} - \lambda + \beta - (\frac{p}{2} + \frac{1}{n})]g(X,Y) + (\beta - \mu)\eta(X)\eta(Y), \quad (3.9)$$

and

$$S(X,\xi) = [\frac{r}{2} - \lambda - (\frac{p}{2} + \frac{1}{n}) + \mu]\eta(X). \quad (3.10)$$

Also employing the identity (2.13) to (3.10), we obtain

$$-(n-1)\beta^2 = [\frac{r}{2} - \lambda - (\frac{p}{2} + \frac{1}{n}) + \mu]. \quad (3.11)$$

The Ricci operator Q is defned by $g(QX, Y) = S(X, Y)$. Then, we get

$$Q(X) = [\beta - \mu - (n-1)\beta^2]X + (\beta - \mu)\eta(X)\xi. \quad (3.12)$$

1. We consider that the manifold (M, g) is Ricci symmetric i.e.,

$$\nabla S = 0. \quad (3.13)$$

Now, we have

$$\nabla_X S(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(\nabla_X Z, Y). \quad (3.14)$$

Using the equation (3.9) and (3.13), we obtain

$$(\beta - \mu)[\beta g(X,Z)\eta(Y) + \eta(Z)g(X,Y) - 2\beta\eta(X)\eta(Y)\eta(Z)] = 0. \quad (3.15)$$

By putting $Y = Z = \xi$, the above equation becomes $\mu = \beta$.

2. We assume that the manifold (M, g) is η-recurrent, i.e.,

$$\nabla S = \eta \otimes S. \quad (3.16)$$

Now, we have

$$(\nabla_X S)(Y,Z) = \eta(X)S(Y,Z), \quad (3.17)$$

for all vector felds X, Y, Z. Using the equation (3.9) and (3.17), we obtain

$$\mu = \lambda - \frac{r}{2} + (\frac{p}{2} + \frac{1}{n}).$$

Hence, we complete the proof. □

4. An n-dimensional Lorentzian β-Kenmotsu Manifold Acceding Conformal η-Einstein Soliton

Here, we study an n-dimensional Lorentzian β-Kenmotsu Manifold Acceding Conformal η-Einstein Soliton.

Theorem 4.1. *If an n-dimensional Lorentzian β-Kenmotsu manifold (M, g) admits a conformal η-Einstein soliton (g, ξ, λ, μ), then the manifold (M, g) becomes an η-Einstein manifold of constant scalar curvature* $r = \frac{2n}{(n-2)}(\frac{p}{2} + \frac{1}{n}) + \frac{2n}{(n-2)}\lambda - \frac{2n}{(n-2)}\beta - \frac{2}{(n-2)}(\beta - \mu)$.
where, $\lambda = -(\frac{p}{2} + \frac{1}{n}) - \frac{(n-1)}{2}\mu + \frac{(n+1)}{2}\beta - \frac{(n-1)(n-2)}{2}\beta^2$.
Furthermore, the soliton is shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ respectively.

Proof. Let us consider an n-dimensional Lorentzian β-Kenmotsu manifold (M, g) admitting a conformal η-Einstein soliton (g, ξ, λ, μ). Then, from the equation (1.6), we can write

$$(L_\xi g)(X, Y) + 2S(X, Y) + [2\lambda + (p + \frac{2}{n}) - r]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \qquad (4.1)$$

for all $X, Y \in T(M)$.
Using $(L_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)$ and equation (2.7) we get,

$$(L_\xi g)(X, Y) = -2\beta[g(X, Y) + \eta(X)\eta(Y)]. \qquad (4.2)$$

Using equations (4.1) and (4.2), we achieve

$$S(X, Y) = [\frac{r}{2} - \lambda + \beta - (\frac{p}{2} + \frac{1}{n})]g(X, Y) + (\beta - \mu)\eta(X)\eta(Y). \qquad (4.3)$$

Consequently, (M, g) is an η-Einstein manifold. Also, we plug $Y = \xi$ into (4.3) to find

$$S(X, \xi) = [\frac{r}{2} - \lambda - (\frac{p}{2} + \frac{1}{n}) + \mu]\eta(X). \qquad (4.4)$$

Comparing the above equation (4.4) with the identity (2.13), we obtain

$$-(n-1)\beta^2 = \mu - \lambda + \frac{r}{2} - (\frac{p}{2} + \frac{1}{n}). \qquad (4.5)$$

Taking an orthonormal basis $(e_1, e_2, e_3, ..., e_n)$ of (M, g) and then setting $X = Y = e_i$ in the equation (4.3) and summation over i we obtain

$$r = \frac{2n}{(n-2)}(\frac{p}{2} + \frac{1}{n}) + \frac{2n}{(n-2)}\lambda - \frac{2n}{(n-2)}\beta - \frac{2}{(n-2)}(\beta - \mu), \qquad (4.6)$$

Finally, by combining equations (4.5) and (4.6), we arrive at

$$\lambda = -(\frac{p}{2} + \frac{1}{n}) - \frac{(n-1)}{2}\mu + \frac{(n+1)}{2}\beta - \frac{(n-1)(n-2)}{2}\beta^2. \qquad (4.7)$$

Hence, we complete the proof. □

5. Conformal η-Einstein Solitons on n-dimensional Lorentzian β-Kenmotsu Manifold Satisfying $R(\xi, X).S = 0$

In this section, we study Conformal η-Einstein Solitons on n-dimensional Lorentzian β-Kenmotsu Manifold Satisfying $R(\xi, X).S = 0$.

THEOREM 5.1. *Let (M, g) be an n-dimensional Lorentzian β-Kenmotsu manifold admitting a conformal η-Einstein soliton (g, ξ, λ, μ). If the manifold satisfes the curvature condition $R(\xi, X).S = 0$, then the manifold becomes an Einstein manifold of constant scalar curvature $r = \frac{2n}{(n-2)}[(\frac{p}{2} + \frac{1}{n}) + \lambda - \beta$.*

PROOF. - Given that an n-dimensional Lorentzian β-Kenmotsu manifold admits a conformal η-Einstein soliton (g, ξ, λ, μ) and the manifold satisfies the curvature condition $R(\xi, X).S = 0$,
Then, we can write

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0. \tag{5.1}$$

Now, using the equation (4.3) into (5.1), we get

$$[\frac{r}{2} - \lambda + \beta - (\frac{p}{2} + \frac{1}{n})][g(R(\xi, X)Y, Z) + g(Y, R(\xi, X)Z)] + (\beta - \mu)[\eta(R(\xi, X)Y)\eta(Z) \tag{5.2}$$
$$+\eta(R(\xi, X)Z)\eta(Y)]$$

Using (2.11) in the previous equation, we obtain

$$(\beta - \mu)[2\beta^2\eta(X)\eta(Y)\eta(Z) + \beta^2(g(X, Y)\eta(Z) + g(X, Z)\eta(Y))] = 0. \tag{5.3}$$

By taking $Z = \xi$ in equation (5.3) and using (2.1), we get

$$(\beta - \mu)\beta^2 g(\phi X, \phi Y) = 0, \tag{5.4}$$

for all $X, Y \in T(M)$.
As $g(\phi X, \phi Y) \neq 0$ and for non-trivial case $\alpha^2 \neq \beta^2$, we can conclude from the equation (5.4) that $\mu = \beta$. Thus, using equation (4.3) we have

$$S(X, Y) = [\frac{r}{2} - \lambda + \beta - (\frac{p}{2} + \frac{1}{n})]g(X, Y), \tag{5.5}$$

for all $X, Y \in T(M)$.
On contracting equation (5.5), we have

$$r = \frac{2n}{(n-2)}[(\frac{p}{2} + \frac{1}{n}) + \lambda - \beta]. \tag{5.6}$$

Hence proved. □

6. An Einstein Semi-Symmetric *n*-dimensional Lorentzian β-Kenmotsu Manifold Acceding Conformal η-Einstein Solitons

Definition 6.1. An n-dimensional Lorentzian β-Kenmotsu Manifold (M, g) is called Einstein semi-symmetric [9] if $R.E = 0$, where E is the Einstein tensor given by

$$E(X, Y) = S(X, Y) - \frac{r}{3}g(X, Y), \tag{6.1}$$

for all vector fields $X, Y \in T(M)$ and r is the scalar curvature of the manifold.

Let us consider that an n-dimensional Lorentzian β-kenmotsu manifold is Einstein semi-symmetric i.e. the manifold satisfes the curvature condition $R.E = 0$. Then, for all the vector fields $X, Y, Z, W \in TM$, we can write

$$E(R(X, Y)Z, W) + E(Z, R(X, Y)W) = 0. \tag{6.2}$$

In view of equation (6.1), the equation (6.2) becomes

$$S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = \frac{r}{3}[g(R(X, Y)Z, W) + g(Z, R(X, Y)W)]. \tag{6.3}$$

Replacing X = Z = ξ in the above equation (6.3) and then using (2.11), we arrive at

$$-\beta^2 S(Y, W) = \beta^2[\eta(Y)S(\xi, W) - \eta(W)S(\xi, Y) + g(Y, W)S(\xi, \xi)]. \tag{6.4}$$

Thus, considering (2.13), the preceding equation (6.4) ultimately yields

$$S(Y, W) = -(n-1)\beta^2 g(Y, W), \tag{6.5}$$

for all $Y, W \in T(M)$. This implies that the manifold is an η-Einstein manifold.
Hence, we have the following:

Lemma 6.2. *An Einstein semi-symmetric n-dimensional Lorentzian β-Kenmotsu manifold is an η-Einstein manifold.*

Now, let us assume that the Einstein semi-symmetric n-dimensional Lorentzian β-Kenmotsu manifold (M, g) admits a conformal η-Einstein soliton (g, ξ, λ, μ). Then equation (4.3) holds and combining (4.3) with the above equation (6.5), we get

$$r = 2(\frac{p}{2} + \frac{1}{n}) + 2\lambda - 2\mu - 2(n-1)\beta^2. \tag{6.6}$$

Again, by recalling the equation (4.6) in the above (6.6), we achieve

$$\lambda = -(\frac{p}{2} + \frac{1}{n}) - \frac{(n-1)}{2}\mu + \frac{(n+1)}{2(n-2)}\beta - \frac{(n-1)(n-2)}{2}\beta^2. \tag{6.7}$$

Therefore, we can state the following:

THEOREM 6.3. *Let (M, g) be an n-dimensional Lorentzian β-kenmotsu manifold admitting a conformal η-Einstein soliton (g, ξ, λ, μ). If the manifold is Einstein semi-symmetric, then the manifold becomes an Einstein manifold of constant scalar curvature $r = 2(\frac{p}{2}+\frac{1}{n})+2\lambda-2\mu-2(n-1)\beta^2$, where $\lambda = -(\frac{p}{2}+\frac{1}{n})-\frac{(n-1)}{2}\mu+\frac{(n+1)}{2(n-2)}\beta-\frac{(n-1)(n-2)}{2}\beta^2$ and the soliton is shrinking, steady or expanding as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.*

7. Conformal η-Einstein Solitons on n-dimensional Lorentzian β-Kenmotsu Manifold with Torse-Forming Vector Field

Here, we study the nature of conformal η-Einstein solitons on n-dimensional Lorentzian β-Kenmotsu Manifold with torse-forming vector field.

DEFINITION 7.1. A vector field $\mathcal{V}$ on n-dimensional Lorentzian β-Kenmotsu Manifold is a torse-forming vector field [8] if

$$\nabla_{\mathcal{V}} = fX + \gamma(X)\mathcal{V}, \tag{7.1}$$

where, f is a smooth function and γ is a 1-form.

Suppose (g, ξ, λ, μ) be a conformal η-Einstein soliton on n-dimensional Lorentzian β-Kenmotsu Manifold (M, g) and assume that the Reeb vector field ξ of the manifold is a torse-forming vector feld. Then, with ξ being a torse-forming vector field, by defnition (7.1), we have

$$\nabla_X\xi = fX + \gamma(X)\xi, \tag{7.2}$$

for all $X \in T(M)$.
Taking the inner product in equation (2.7) with ξ, we can write

$$g(\nabla_X\xi, \xi) = 0. \tag{7.3}$$

Taking the inner product in equation (7.2) with ξ, we obtain

$$g(\nabla_X\xi, \xi) = f\eta(X) + \gamma(X). \tag{7.4}$$

We combine (7.3) and (7.4) to get, $\gamma = -f\eta$.
Thus, from equation (7.2) it implies that, for torse forming vector field ξ in n-dimensional Lorentzian β-Kenmotsu Manifold, we have

$$\nabla_X\xi = f(X - \eta(X)\xi). \tag{7.5}$$

Now, from the formula of Lie diferentiation and using equation (7.5) yields

$$\begin{aligned}(L_\xi g)(X, Y) &= g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi),\\ &= 2f[g(X, Y) - \eta(X)\eta(Y)],\end{aligned} \tag{7.6}$$

Since (g, ξ, λ, μ) is a conformal η-Einstein soliton, the equation (1.6) holds. So, in view of equation (7.6), the equation (1.6) reduces to

$$S(X,Y) = g(X,Y)[\frac{r}{2} - (\frac{p}{2} + \frac{1}{n}) - \lambda - f] + (f - \mu)\eta(X)\eta(Y). \tag{7.7}$$

Thus, the manifold is an η-Einstein manifold. On letting $Y = \xi$ in equation (7.7) we get

$$S(X,\xi) = [\frac{r}{2} - (\frac{p}{2} + \frac{1}{n}) - \lambda - 2f + \mu]\eta(X). \tag{7.8}$$

Combining equation (7.8) with the equation (2.13) implies

$$(n-1)\beta^2 = \lambda - \frac{r}{2} + (\frac{p}{2} + \frac{1}{n}) + 2f - \mu. \tag{7.9}$$

On taking trace in equation (7.7) we obtain

$$r = \frac{2n}{(n-2)}\lambda + \frac{2n}{(n-2)}(\frac{p}{2} + \frac{1}{n}) + \frac{2(n-1)}{(n-2)}f + \frac{2}{(n-2)}\mu. \tag{7.10}$$

Using equation (7.10) in equation (7.9), we get $\lambda = -\frac{(n-1)(n-2)}{2}\beta^2 - (\frac{p}{2} + \frac{1}{n} + \frac{(n-3)}{2}f - \frac{(n-1)}{2}\mu$, and we have the following

Theorem 7.2. *Let (M, g) be a conformal η-Einstein soliton on an n-dimensional Lorentzian β-kenmotsu manifold (M, g) with torse-forming vector feld ξ, then the manifold becomes an η -Einstein manifold and the soliton is shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, where*
$\lambda = -\frac{(n-1)(n-2)}{2}\beta^2 - (\frac{p}{2} + \frac{1}{n} + \frac{(n-3)}{2}f - \frac{(n-1)}{2}\mu.$

Conclusion: In conclusion, this paper demonstrated the existence of conformal η-Einstein solitons in n-dimensional Lorentzian β-Kenmotsu manifolds under various conditions, including h-parallel and torse-forming vector fields. The curvature condition $R(\xi, X).S = 0$ was found to significantly influence soliton properties, and Einstein semi-symmetric manifolds were shown to admit such solitons. These results enhance our understanding of the interplay between solitons and the geometry of Lorentzian β-Kenmotsu manifolds, with potential applications in further geometric and physical studies.

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Shyam Kishor, Department of Mathematics & Astronomy, University of Lucknow, Lucknow, India
e-mail: skishormath@gmail.com

Anjum Ara, Department of Mathematics & Astronomy, University of Lucknow, Lucknow, India
e-mail: anjumara891997@gmail.com