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GENERALIZATION OF THE ENESTRÖM-KAKEYA THEOREM THROUGH POLYNOMIAL COEFFICIENTS RELAXATION

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Abstract

This study introduces novel Eneström-Kakeya Theorem generalizations, selectively relaxing polynomial coefficients. Analyzing these relationships enriches classical theorem understanding, revealing fresh insights and mathematical opportunities within the Eneström-Kakeya context.

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1. Introduction and Presentation of Findings

In the following section, we examine the dependence of polynomial zero bounds on various coefficient types and how these bounds evolve as the coefficient types change. Pioneers, such as Gauss and Cauchy [3], have laid the foundation for exploring the bounds of polynomial zeros, sparking a rich tradition of research on this subject [9]. Although many studies have contributed to enhancing our understanding of zero bounds, they have frequently focused on restricted coefficients, whether real or complex numbers, as integral components of these bounds [1–3, 11, 12]. In this study, we embark on a comprehensive analysis within the complex plane to ascertain the precise regions encompassing all zeros of polynomials characterized by these restricted coefficients. Our analysis also identified the parameters that can be fine-tuned to match the desired precision and intensities.

Cauchy's classical theorem provides a straightforward bound for the moduli of zeros of a polynomial based on its coefficients. It states that:

Theorem 1.1. *[5] For a polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n, all its zeros reside within the circle $|z| \leq 1 + M$, where $M = \max_{0\leq j\leq n-1} \left|\frac{a_j}{a_n}\right|$.*

The computational simplicity of Theorem 1.1 distinguishes it from other bounds. Cauchy-type polynomials have been a focal point of extensive research for over a century. This research has branched out in numerous directions and has resulted in several publications [8–10]. Mathematical concepts related to polynomials and the spatial relationships of their zeros remain an active field of research. Each year, numerous research papers appear in various journals, employing diverse approaches to serve various purposes. In this study, we investigated polynomial regions containing zeros and those free from zeros subject to certain coefficient constraints.

The following Eneström-Kakeya Theorem [9] has significantly advanced our understanding of the modulus of zeros in polynomials as an enhancement of the Cauchy bound, particularly when working with restricted coefficients.

Theorem 1.2. *[9] If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n that satisfies the condition* $a_n \geq a_{n-1} \geq \ldots \geq a_0 > 0$*, then all the zeros of* $P(z)$ *lie within the region* $|z| \leq 1$.

Numerous extensions and generalizations of the Eneström-Kakeya Theorem have been documented in the literature [[1]-[4], [6]]. Joyal et al. [8] expanded upon the Eneström-Kakeya Theorem and included polynomials with coefficients that follow a monotonic pattern, without the restriction of non-negativity. They obtained the following outcome:

Theorem 1.3. *[8] For a polynomial* $P(z) = \sum_{j=0}^{n} a_j z^j$ *of degree* n*, where the coefficients satisfy the condition*

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0,$$

all the zeros are contained within the disk

$$|z| \leq \frac{|a_n| - a_0 + |a_0|}{|a_n|}.$$

Aziz and Zargar [4] relaxed Theorem 1.2's assumptions in various ways and presented extensions and generalizations of this result as follows:

Theorem 1.4. *[4] If a polynomial* $P(z) = \sum_{j=0}^{n} a_j z^j$ *of degree* n *satisfies the following conditions for some* $k \geq 1$ *and* $0 < \rho \leq 1$*:*

$$ka_n \geq a_{n-1} \geq \ldots \geq a_1 \geq \rho a_0 > 0,$$

then all the zeros of $P(z)$ *are contained within the disk*

$$|z + k - 1| \leq k + \frac{2a_0}{a_n}(1 - \rho).$$

Shah et al. recently introduced an expanded version of Theorem 1.3, detailed in [Theorem 1, [14]], where they eased the monotonicity condition for specific coefficients. Additionally, within the same work, they derived a subsequent finding regarding regions devoid of polynomial zeros.

In a recent publication, Zargar et al. [15] extended the results presented by Shah et al. in [14].

In this investigation, the focus is on the polynomials $P(z) = 30z^6 + 32z^5 + 24z^4 + 25z^3 + 26z^2 + 15z + 20$ and $Q(z) = 14z^6 + 12z^5 + 8z^4 + 15z^3 + 10z^2 + 8z + 10$. It is noted that neither the findings in Theorem 1 of [14] nor Theorem 2.1 of [15] offer a definitive region encompassing all roots of these polynomials. Moreover, neither Theorem 2 in [14] nor Theorem 2.6 in [15] establishes a unique zero-free region for these polynomials.

For the case of $p = 6$ and $q = 4$, the coefficients of $Q(z)$ adhere to the monotonicity conditions outlined in [Theorem 1, 2 [14]], indicating that all zeros lie within the region $|z| \leq 3.43$, while the region with no zeros extends to $|z| < 0.19$.

Again, when the polynomial $Q(z)$ satisfies Theorem 1 and Theorem 2 in [14] for $p = 3$ and $q = 1$, its zeros are confined to the region $|z| \leq 1.07$, with none within $|z| < 0.28$.

The polynomial $P(z)$ satisfies both Theorem 2.1 and Theorem 2.6 conditions as outlined in [15] for two parameter sets: $p = 6$, $q = 3$, $k = \frac{32}{30}$, $\sigma = \frac{24}{25}$, as well as for $p = 4$, $q = 2$, $k = \frac{25}{24}$, $\sigma = \frac{25}{26}$. In the former scenario, all roots of $P(z)$ are confined within the disk $|z| \leq 2.6$, with none falling inside the region $|z| < 0.23$. In the latter situation, all roots of $P(z)$ are situated within the disk $|z| \leq 1.6$, and no roots are present within the region $|z| < 0.34$.

These results highlight the necessity of further investigation of the studies conducted by Shah et al.(2021) and Zargar et al.(2023).

2. Main Results

The primary objective of the study is to expand upon existing findings by relaxing initial assumptions. We extend Theorem 1.3 and generalize Theorem 1.4 to enhance understanding of this subject. Additionally, our goal is to establish a definite zero-free region, similar to the one found in [14] and [15].

It is observed that neither Theorem 1.3 nor Theorem 1.4 can be applied to the polynomial $30z^6 + 32z^5 + 24z^4 + 25z^3 + 26z^2 + 15z + 20$ to determine a definitive region containing all of its zeros. Motivated by this observation, the primary finding of this study is presented as follows:

Theorem 2.1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that, for a positive integer $k \geq 1$ and for all non-negative integers $\sigma_0, \sigma_1, ..., \sigma_{p-1}, \sigma_p \leq 1$, the following inequalities hold:*

$$ka_n \geq a_{n-1} \geq \ldots \geq \sigma_p a_p \geq \sigma_{p-1} a_{p-1} \geq \ldots \geq \sigma_0 a_0, \ 1 \leq p < n.$$

Then all the zeros of $P(z)$ are contained within the disk

$$|z| \leq \frac{(k-1)|a_n| + ka_n - \sigma_0 a_0 + (2-\sigma_0)|a_0| + 2\sum_{j=1}^{p}(1-\sigma_j)|a_j|}{|a_n|}.$$

REMARK 2.2. 1. *If all the coefficients are non-negative, and $k = \sigma_0 = ... = \sigma_p = 1$, Theorem 2.1 simplifies to the Enström-Kakeya theorem.*

2. *When $k = \sigma_0 = ... = \sigma_p = 1$, Theorem 2.1 reduces to Theorem 1.3.*
3. *Since the circle $|z + k - 1| \leq k + \frac{2a_0}{a_n}(1 - \rho)$ is enclosed by the circle $|z| \leq (2k - 1) + \frac{2a_0}{a_n}(1 - \rho)$, it logically follows that all roots of the polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$, meeting the conditions stated in Theorem 1.4, are located within the circle $|z| \leq (2k - 1) + \frac{2a_0}{a_n}(1 - \rho)$. In this scenario, if all coefficients are positive and $\sigma_1 = \sigma_2 = \ldots = \sigma_p = 1$, Theorem 2.1 reduces to Theorem 1.4.*

EXAMPLE 2.3. *Let us consider the polynomial $P(z) = 30z^6 + 32z^5 + 24z^4 + 25z^3 + 26z^2 + 15z + 20$ to demonstrate the applicability of Theorem 2.1, while neither Theorem 1.3 nor Theorem 1.4 is applicable.*
Here, with $n = 6$, $a_n = 30$, $p = 3$, $a_p = 25$, $a_0 = 20$, taking $k = \frac{32}{30}$, and $\sigma_3 = \frac{24}{25}$, $\sigma_2 = \frac{24}{26}$, $\sigma_1 = 1$, and $\sigma_0 = \frac{15}{20}$, it follows from Theorem 2.1 that all the zeros of $P(z)$ lie within

$$\begin{aligned}|z| &\leq \frac{(k-1)|a_n| + ka_n - \sigma_0 a_0 + (2 - \sigma_0)|a_0| + 2\sum_{j=1}^{p}(1 - \sigma_j)|a_j|}{|a_n|} \\ &= \frac{2 + 32 - 15 + 25 + 2(0 + 2 + 1)}{30} \\ &= 1.67.\end{aligned}$$

On the other hand, by Cauchy's Theorem, all the zeros of $P(z)$ are confined within

$$|z| \leq 2.07.$$

Hence, Theorem 2.1 provides a sharper bound than Cauchy's theorem.

The following result is introduced to provide a definitive zero-free region similar to the ones described in Theorem 2 in [14] and Theorem 2.6 in [15].

THEOREM 2.4. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $P(0) \neq 0$. For a positive integer $k \geq 1$ and for all non-negative integers $\sigma_0, \sigma_1, ..., \sigma_{p-1}, \sigma_p \leq 1$, if the following inequalities hold:*

$$ka_n \geq a_{n-1} \geq \ldots \geq \sigma_p a_p \geq \sigma_{p-1} a_{p-1} \geq \ldots \geq \sigma_0 a_0, \text{ where } 1 \leq p < n,$$

then there are no zeros of $P(z)$ contained within the disk

$$|z| < \frac{|a_0|}{2\sum_{j=1}^{p}(1 - \sigma_j)|a_j| + (1 - \sigma_0)|a_0| + ka_n - \sigma_0 a_0 + k|a_n|}.$$

EXAMPLE 2.5. *To justify the applicability of Theorem 2.4, let's examine the polynomial $P(z)$ mentioned in Example 2.3. According to Theorem 2.4, there are no zeros of the polynomial $P(z)$ within the defined region $|z| < 0.33$. However, Theorem 2.6 in [15] provides two zero-free regions: one with $|z| < 0.23$ and the other with $|z| < 0.34$.*

Remark 2.6. 1. *The parameters $k \geq 1$ and $0 \leq \sigma_i \leq 1$ are essential for generalizing existing theorems while ensuring valid zero-containing and zero-free regions. Altering these constraints could lead to inaccurate or less meaningful bounds.*

2. *An important observation arises when comparing the numerator of the bound in Theorem 2.1 and the denominator of the bound in Theorem 2.4. Both of these bounds are sharper when the smallest value of k is chosen, along with the largest values of all $\sigma_0, \sigma_1, \ldots, \sigma_{p-1}, \sigma_p$, among the values that satisfy the conditions of these theorems.*

3. Proof of Theorem 2.1

Considering $Q(z) = (1 - z)P(z)$, we get that

$$\begin{aligned}
|Q(z)| &= \Bigg| -a_n z^{n+1} + (a_n - a_{n-1})z^n + \sum_{j=p+2}^{n-1} (a_j - a_{j-1})z^j + (a_{p+1} - a_p)z^{p+1} + \\
&\quad \sum_{j=1}^{p}(a_j - a_{j-1})z^j + a_0 \Bigg| \\
&\geq |a_n||z|^{n+1} - \Bigg| (a_n - a_{n-1})z^n + \sum_{j=p+2}^{n-1} (a_j - a_{j-1})z^j + (a_{p+1} - a_p)z^{p+1} + \\
&\quad \sum_{j=1}^{p}(a_j - a_{j-1})z^j + a_0 \Bigg| .
\end{aligned} \tag{3.1}$$

Now, when $|z| > 1$ i.e., $\frac{1}{|z|} < 1$, and if $k \geq 1$ and $0 \leq \sigma_0,\ \sigma_1, ..., \sigma_{p-1},\ \sigma_p \leq 1$, it follows that

$$\begin{aligned}
&\left| (a_n - a_{n-1})z^n + \sum_{j=p+2}^{n-1} (a_j - a_{j-1})z^j + (a_{p+1} - a_p)z^{p+1} + \sum_{j=1}^{p}(a_j - a_{j-1})z^j + a_0 \right| \\
&\leq |z|^n \left[|a_n - a_{n-1}| + \sum_{j=p+2}^{n-1} \frac{|a_j - a_{j-1}|}{|z|^{n-j}} + \frac{|a_{p+1} - a_p|}{|z|^{n-p-1}} + \sum_{j=1}^{p} \frac{|a_j - a_{j-1}|}{|z|^{n-j}} + \frac{|a_0|}{|z|^n} \right] \\
&< |z|^n \left[|a_n - a_{n-1}| + \sum_{j=p+2}^{n-1} |a_j - a_{j-1}| + |a_{p+1} - a_p| + \sum_{j=1}^{p} |a_j - a_{j-1}| + |a_0| \right]
\end{aligned}$$

$$= |z|^n \Bigg[|(1-k)a_n + (ka_n - a_{n-1})| + \sum_{j=p+2}^{n-1} |a_j - a_{j-1}| + |(\sigma_p - 1)a_p + (a_{p+1} - \sigma_p a_p)| +$$

$$\sum_{j=1}^{p} |(1-\sigma_j)a_j + (\sigma_j a_j - \sigma_{j-1}a_{j-1}) + (\sigma_{j-1} - 1)a_{j-1}| + |a_0| \Bigg]$$

$$\le |z|^n \Bigg[(k-1)|a_n| + (ka_n - a_{n-1}) + \sum_{j=p+2}^{n-1} (a_j - a_{j-1}) + (1-\sigma_p)|a_p| + (a_{p+1} - \sigma_p a_p) +$$

$$\sum_{j=1}^{p} (\sigma_j a_j - \sigma_{j-1}a_{j-1}) + \sum_{j=1}^{p}(1-\sigma_j)|a_j| + \sum_{j=1}^{p}(1-\sigma_{j-1})|a_{j-1}| + |a_0| \Bigg]$$

$$\le |z|^n \Big[(k-1)|a_n| + (ka_n - a_{n-1}) + (a_{n-1} - a_{p+1}) + (1-\sigma_p)|a_p| + (a_{p+1} - \sigma_p a_p) +$$

$$(\sigma_p a_p - \sigma_0 a_0) + (1-\sigma_p)|a_p| + 2\sum_{j=1}^{p-1}(1-\sigma_j)|a_j| + (1-\sigma_0)|a_0| + |a_0| \Bigg]$$

$$= |z|^n \Bigg[(k-1)|a_n| + ka_n - \sigma_0 a_0 + 2\sum_{j=1}^{p}(1-\sigma_j)|a_j| + (1-\sigma_0)|a_0| + |a_0| \Bigg].$$

Hence, for $|z| > 1$, $k \ge 1$ and $0 \le \sigma_0,\ \sigma_1, ..., \sigma_{p-1},\ \sigma_p \le 1$, we deduce from (3.1) that

$$|Q(z)| \ge |z|^n \Bigg[|a_n||z| - \Bigg((k-1)|a_n| + ka_n - \sigma_0 a_0 + 2\sum_{j=1}^{p}(1-\sigma_j)|a_j| +$$

$$(1-\sigma_0)|a_0| + |a_0| \Big) \Big] > 0$$

$$\text{if } |z| > \frac{(k-1)|a_n| + ka_n - \sigma_0 a_0 + 2\sum_{j=1}^{p}(1-\sigma_j)|a_j| + (1-\sigma_0)|a_0| + |a_0|}{|a_n|}.$$

Therefore, outside the disk $|z| \le 1$, no zeros of $Q(z)$ lie in

$$|z| > \frac{(k-1)|a_n| + ka_n - \sigma_0 a_0 + 2\sum_{j=1}^{p}(1-\sigma_j)|a_j| + (1-\sigma_0)|a_0| + |a_0|}{|a_n|}.$$

Consequently, all the zeros of $Q(z)$ and hence $P(z)$ are contained within the region

$$|z| \le \frac{(k-1)|a_n| + ka_n - \sigma_0 a_0 + 2\sum_{j=1}^{p}(1-\sigma_j)|a_j| + (1-\sigma_0)|a_0| + |a_0|}{|a_n|}.$$

This completes the proof of Theorem 2.1. □

4. Proof of Theorem 2.4

Let us consider

$$S(z) = z^n P\left(\frac{1}{z}\right)$$

$$T(z) = (z-1)S(z).$$

Then,

$$|T(z)| = \left| a_0 z^{n+1} + \sum_{j=0}^{p-1}(a_{j+1}-a_j)z^{n-j} + (a_{p+1}-a_p)z^{n-p} + \right.$$

$$\left. \sum_{j=p+1}^{n-2}(a_{j+1}-a_j)z^{n-j} + (a_n - a_{n-1})z - a_n \right|$$

$$\geq |a_0||z|^{n+1} - \left| \sum_{j=0}^{p-1}(a_{j+1}-a_j)z^{n-j} + (a_{p+1}-a_p)z^{n-p} + \right.$$

$$\left. \sum_{j=p+1}^{n-2}(a_{j+1}-a_j)z^{n-j} + (a_n - a_{n-1})z - a_n \right|. \tag{4.1}$$

Now, for $|z| > 1$ i.e., $\frac{1}{|z|} < 1$, and $k \geq 1$, $0 \leq \sigma_0,\ \sigma_1, ..., \sigma_{p-1},\ \sigma_p \leq 1$, we get that

$$\left| \sum_{j=0}^{p-1}(a_{j+1}-a_j)z^{n-j} + (a_{p+1}-a_p)z^{n-p} + \sum_{j=p+1}^{n-2}(a_{j+1}-a_j)z^{n-j} + (a_n - a_{n-1})z - a_n \right|$$

$$\leq |z|^n \left[\sum_{j=0}^{p-1} \frac{|a_{j+1}-a_j|}{|z|^j} + \frac{|a_{p+1}-a_p|}{|z|^p} + \sum_{j=p+1}^{n-2} \frac{|a_{j+1}-a_j|}{|z|^j} + \frac{|a_n - a_{n-1}|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \right]$$

$$\leq |z|^n \left[\sum_{j=0}^{p-1} |a_{j+1}-a_j| + |a_{p+1}-a_p| + \sum_{j=p+1}^{n-2} |a_{j+1}-a_j| + |a_n - a_{n-1}| + |a_n| \right]$$

$$= |z|^n \left[\sum_{j=0}^{p-1} |(1-\sigma_{j+1})a_{j+1} + \sigma_{j+1}a_{j+1} - \sigma_j a_j + (\sigma_j - 1)a_j| + |a_{p+1} - \sigma_p a_p + \right.$$

$$\left. (\sigma_p - 1)a_p| + \sum_{j=p+1}^{n-2} |a_{j+1}-a_j| + |(1-k)a_n + ka_n - a_{n-1}| + |a_n| \right]$$

$$\leq |z|^n \left[\sum_{j=0}^{p-1}(1-\sigma_{j+1})|a_{j+1}| + \sum_{j=0}^{p-1}(1-\sigma_j)|a_j| + \sum_{j=0}^{p-1}(\sigma_{j+1}a_{j+1} - \sigma_j a_j) + \right.$$

$$\left. (a_{p+1} - \sigma_p a_p) + (1-\sigma_p)|a_p| + \sum_{j=p+1}^{n-2}(a_{j+1}-a_j) + (k-1)|a_n| + (ka_n - a_{n-1}) + |a_n| \right]$$

$$= |z|^n \left[(1-\sigma_p)|a_p| + 2\sum_{j=1}^{p-1}(1-\sigma_j)|a_j| + (1-\sigma_0)|a_0| + (\sigma_p a_p - \sigma_0 a_0) + \right.$$

$$\left. (a_{p+1} - \sigma_p a_p) + (1-\sigma_p)|a_p| + (a_{n-1} - a_{p+1}) + (k-1)|a_n| + (ka_n - a_{n-1}) + |a_n| \right]$$

$$= |z|^n \left[2\sum_{j=1}^{p}(1-\sigma_j)|a_j| + (1-\sigma_0)|a_0| + ka_n - \sigma_0 a_0 + k|a_n| \right].$$

Therefore, when $|z| > 1$, $k \geq 1$, and $0 \leq \sigma_0, \sigma_1, ..., \sigma_{p-1}, \sigma_p \leq 1$, we obtain from (4.1) that

$$|T(z)| \geq |z|^n \left[|a_0||z| - \left(2\sum_{j=1}^{p}(1-\sigma_j)|a_j| + (1-\sigma_0)|a_0| + ka_n - \sigma_0 a_0 + k|a_n| \right) \right] > 0$$

$$\text{if } |z| > \frac{2\sum_{j=1}^{p}(1-\sigma_j)|a_j| + (1-\sigma_0)|a_0| + ka_n - \sigma_0 a_0 + k|a_n|}{|a_0|}.$$

Hence, all the zeros of $T(z)$ and, consequently, of $S(z)$ lie within the disk

$$|z| \leq \frac{2\sum_{j=1}^{p}(1-\sigma_j)|a_j| + (1-\sigma_0)|a_0| + ka_n - \sigma_0 a_0 + k|a_n|}{|a_0|}.$$

Since all the zeros of $P(z)$ are the reciprocals of the zeros of $S(z)$, it follows that $P(z)$ has no zeros within the region

$$|z| < \frac{|a_0|}{2\sum_{j=1}^{p}(1-\sigma_j)|a_j| + (1-\sigma_0)|a_0| + ka_n - \sigma_0 a_0 + k|a_n|}.$$

Thus Theorem 2.4 is proved. □

5. Conclusion

The research has extended the Enström-Kakeya theorem through innovative methods that relax polynomial coefficients. This approach provides valuable insights into the distribution of zeros across a broader spectrum of polynomials, offering new avenues for understanding and characterizing polynomial behavior.

6. Future Course of Work

The findings presented in this paper can be extended to polynomials involving several variables, potentially opening a new area of research.

Declaration of Conflicting Interest. The author asserts the absence of any conflicts of interest.

Data Availability. This article does not involve data sharing since no datasets were created or analyzed in the course of the current study.

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