GANITA,Vol.74(2), 2024, 349-360

NUMERICAL SOLUTION OF INTEGRAL EQUATION BY USING PICARD METHOD, HOMOTOPY AND MODIFIED ADOMIAN DECOMPOSITION METHOD

NAJMUDDIN AHMAD ✉, BALMUKUND SINGH and ASIYA ANSARI

Abstract

In this study, we focus on three analytical methods for solving the nonlinear quadratic Volterra integral equation, including the Picard method, the Homotopy method, and the Modified Adomian decomposition method. All techniques are comparative studies based on iterative examples of the best approximation term and the estimated error. We perform all computations in the MATLAB 13 version.

2010 *Mathematics subject classification:* 45D05; 44B20; 57Q10.
Keywords and phrases: Picard Method, Homotopy Method, Adomian Decomposition Method, Volterra Integral Equation.

1. Introduction

Integral equations play an important role in the area of applied mathematics as they arise from a variety of physics, engineering, and biological problems, including electrochemistry, scattering theory, heat conduction, and semiconductors. Numerical methods for solving linear and nonlinear integral equations, integro-differential equations, and other systems. Integral equations arise in a variety of fields like heat and mass transfer, fluid dynamics, elasticity, plasticity, approximation theory, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, queuing theory, electrical engineering, medicine, and economics [1]-[12]. These methods are based on approximations like orthogonal polynomials, wavelet approximations, orthogonal functions, and other approximations. We introduce the B-spline wavelet method, the Legendre wavelet method, the Chebyshev wavelet, the Legendre multi-wavelet method, etc. Hence, different types in different methods are also used for the Volterra integral equation. The Picard method, the Adomian decomposition method [13],[14], and the Homotopy perturbation method have been proposed for obtaining the approximate solution in the integral equation [15]. The quadratic integral equation is applicable in the theory of heat transformation, kinetic theory of gases, neutron transport, and radiative transfer.

The paper has been organised in the following manner: Methodologies are given in Section 2. Some numerical examples in support of the theory have been provided in Section 3, followed by the conclusion in Section 4.

2. The Methodology

2.1. Modified Picard Successive Approximations Methods: Suppose the following nonlinear Volterra quadratic integral equation

$$y(t) = a(t) + g(t, y(t)) \int_0^t f(s, y(s))ds \tag{2.1}$$

where a and g are known functions and the function $f(s, y(s))$ is a nonlinear and unknown function x. Applying the Picard method to the quadratic integral equation (2.1), we are constructed by the sequence

$$y_n(t) = a(t) + g(t, y_{n-1}(t)) \int_0^t f(s, y_{n-1}(s))ds, \qquad n = 1, 2, 3, \ldots$$

with initial condition

$$y_0(t) = a(t).$$

All are functions $y_n(t)$ continuous, and solution will be becoming

$$y(t) = \lim_{n\to\infty} y_n(t).$$

In this work, we decompose function $a(x)$ into two components $f_0(t)$ & $f_1(t)$. The function $f_0(t)$ is assigned the zeroth solution, and $f_0(t)$ used the first solution $x_1(t)$, i.e.

$$y_0(t) = f_0(x),$$

$$\begin{aligned} y_1(t) &= f_1(t) + g(t, y_0(s)) \int_0^t f(s, x_0(s))ds, \\ &\vdots \\ y_{n+1}(t) &= g(t, y_n(t)) \int_0^t f(s, y_n(s))ds, \quad n \geq 1. \end{aligned} \tag{2.2}$$

If the points $a(t)$ consist of one term, the modified Picard method cannot be used in these expressions.

Equation (2.1) will be reviewed under the supposition

(i) $a : I \to R_+ = [0.\infty)$ is continuous on I where $I = [0, 1]$.

(ii) $f, g : I \times D \subset R_+ \to R_+$ are continuous, and there exist positive constants M_1 and M_2 such that $|g(t, x)| \leq M_1$ and $|f(t, x)| \leq M_2$ on D.

(iii) f, g satisfies Lipschitz condition with Lipschitz constants L_1 and L_2 such that,

$$|g(t,x) - g(t,y)| \le L_1|x-y|,$$
$$|f(t,x) - f(t,y)| \le L_2|x-y|.$$

Let $C = C(I)$ be the space of all real valued functions that are continuous on I.
The operator F is

$$(F(x))(t) = a(t) + g(t, x(t)) \int_0^t f(s, x(s))ds, \quad \forall x \in C.$$

THEOREM 1. Let the supposition $(1) - (2)$ be satisfied. If $h = (L_1M_2 + L_2M_1) < 1$, then the nonlinear quadratic integral equation (2.1) has a unique positive solution $x \in C$.

PROOF. It is clear that the $F : F \to C$.
Now we define a subset S of C as $S = x \in C : |x - a(t)| \le k, \quad k = M_1M_2$.
Then the operator $F : S \to S$, since for $x \in S$

$$|x(t) - a(t)| \le M_1M_2 \int_0^t ds = M_1M_2t = M_1M_2.$$

Moreover, it is easy to see that S is a closed subset of C. In order to show that F is a contraction, we compute

$$\begin{aligned}(Fx)(t) - (Fy)(t) &= g(t, x(t)) \int_0^t f(s, x(s))ds - g(t, y(t)) \int_0^t f(s, y(s))ds \\ &+ g(t, x(t)) \times \int_0^t f(s, y(s))ds - g(t, x(t)) \int_0^t f(s, y(s))ds \\ &= [g(t, x(t)) - g(t, y(t))] \int_0^t f(s, y(s))ds \\ &+ g(t, x(t)) \int_0^t [f(s, x(s)) - f(s, y(s))]ds. \\ |(Fx)(t) - (Fy)(t)| &\le |g(t, x(t)) - g(t, y(t))| \int_0^t |f(s, y(s))|ds \\ &+ |g(t, x(t))| \int_0^t |f(s, x(s)) - f(s, y(s))|ds \\ &\le L_1M_2|x - y| + L_2M_1 \int_0^t |x(s) - y(s)|ds. \\ \|(Fx)(t) - (Fy)(t)\| &= \max_{t \in I} |(Fx)(t) - (Fy)(t)| \le L_1M_2\|x - y\| + L_2M_1\|x - y\| \\ &\le (L_1M_2 + L_2M_1)\|x - y\| \le h\|x - y\|.\end{aligned}$$

Since $h = (L_1M_2 + L_2M_1) < 1$.
Then F is a contraction and F has a unique fixed point in S, thus there exist unique solutions for (2.1). □

COROLLARY 1.1. Let the assumption theorem (1) with $g(t,x) = 1$ be satisfied. If $L_2 < 1$, the integral equation

$$x(t) = x_0(t) + \int_0^t f(s, y(s))ds$$

has a unique continuous equation.

2.2. Adomian Decomposition Method (ADM): Adomian decomposition method (ADM) has been solved by numerical problem in wide variety of functional equations in series form of the solution. Adomian decomposition has been presented and developed in different types of equations: algebraic, partial differential equation, integro-differential equation, and differential-delay.

The solution algorithm of the quadratic integral equation (2.1) using the Adomian Decomposition method is

$$x_0(t) = a(t)$$
$$x_i(t) = A_{i-1}(t) \int_0^t B_{i-1}(s)ds \tag{2.3}$$

where A_i and B_i are Adomian polynomials of the nonlinear terms $g(t,x)$ and $f(s,x)$, respectively, which have been

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(t, \sum_{i=0}^{\infty} \lambda^i x_i\right)\right]_{\lambda=0}$$
$$B_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[g\left(t, \sum_{i=0}^{\infty} \lambda^i x_i\right)\right]_{\lambda=0}$$

and solution will be

$$x(t) = \sum_{i=0}^{\infty} x_i(t).$$

THEOREM 2. Let the solution of the quadratic integral equation (2.1) exist. If $|x_1(t)| < l$, where l is a positive constant, then the series solution of the quadratic integral equation (2.1) using ADM converges.

PROOF. Define a sequence S_n such that,

$$S_p = \sum_{i=0}^{p} x_i(t).$$

Then a sequence of partial sums from the series solution $\sum_{i=0}^{p} x_i(t)$, so we have

$$g(t,x) = \sum_{i=0}^{p} A_i,$$
$$f(s,x) = \sum_{i=0}^{p} B_i.$$

Let S_p and S_q be two arbitrary partial sums with $p > q$. Now, we will prove that S_p is a Cauchy sequence in Banach space E.

$$
\begin{aligned}
S_p - S_q &= \sum_{i=0}^{p} x_i - \sum_{i=0}^{q} x_i \\
&= \sum_{i=0}^{p} A_{i-1}(t) \int_0^t \sum_{i=0}^{p} B_{i-1}(s)ds - \sum_{i=0}^{q} A_{i-1}(t) \int_0^t \sum_{i=0}^{q} B_{i-1}(s)ds \\
&= \sum_{i=0}^{p} A_{i-1}(t) \int_0^t \sum_{i=0}^{p} B_{i-1}(s)ds - \sum_{i=0}^{q} A_{i-1}(t) \int_0^t \sum_{i=0}^{p} B_{i-1}(s)ds \\
&\quad + \sum_{i=0}^{q} A_{i-})(t) \int_0^t \sum_{i=0}^{p} B_{i-1}(s)ds - \sum_{i=0}^{q} A_{i-1}(t) \int_0^t \sum_{i=0}^{q} B_{i-1}(s)ds \\
&= \left[\sum_{i=0}^{p} A_{i-1}(t) - \sum_{i=0}^{q} A_{i-1}(t)\right] \int_0^t B_{i-1}(s)ds \\
&+ \sum_{i=0}^{q} A_{i-1}(t) \int_0^t \left[\sum_{i=0}^{p} B_{i-1}(s) - \sum_{i=0}^{q} B_{i-1}(s)\right] ds \\
\|S_p - S_q\| &\le \max_{t\in I} \left| \sum_{i=q+1}^{p} A_{i-1}(t) \int_0^t \sum_{i=0}^{p} B_{i-1}(s)ds \right| \\
&+ \max_{t\in I} \left| \sum_{i=0}^{q} A_{i-1}(t) \int_0^t \sum_{i=q+1}^{p} B_{i-1}(s)ds \right| \\
&\le \max_{t\in I} \left| \sum_{i=q}^{p-1} A_i(t) \right| \int_0^t \left| \sum_{i=0}^{p} B_{i-1}(s)ds \right| ds \\
&+ \max_{t\in I} \left| \sum_{i=0}^{q} A_{i-1}(t) \right| \int_0^t \left| \sum_{i=q}^{p-1} B_i(s) \right| ds \\
&\le \max_{t\in I} \left| g(t, S_{p-1}) - g(t, S_{q-1}) \right| \int_0^t \left| f(t, S_p) \right| ds \\
&+ \max_{t\in I} |g(t, S_q)| \int_0^t |f(t, s_{p-1}) \\
&\quad - f(t, S_{q-1})| ds \\
&\le L_1 M_2 \max_{t\in I} \left| S_{p-1} - S_{q-1} \right| + L_2 M_1 \max_{t\in I} \left| S_{p-1} - S_{q-1} \right| \\
&\le h \|S_{p-1} - S_{q-1}\|
\end{aligned}
$$

Suppose $p = q + 1$ then,

$$
\|S_{q+1} - S_q\| \le h\|S_q - S_{q-1}\| \le h^2 \|S_{q-1} - S_{q-2}\| \le \cdots \le h^q \|S_1 - S_0\|
$$

Then we apply triangle inequality, we have

$$\begin{aligned}\|S_p - S_q\| &\le \|S_{q+1} - S_q\| + \|S_{q+2} - S_{q+1}\| + \cdots + \|S_p - S_{p-1}\| \\ &\le [h^q + h^{q+1} + h^{q+2} + \cdots + h^{p-1}]\|S_1 - S_0\| \\ &\le h^q[1 + h + h^2 + \cdots + h^{p-q-1}]\|S_1 - S_0\| \\ &\le h^q\left[\frac{1-h^{p-q}}{1-h}\right]\|x_1(t)\|\end{aligned}$$

Now $0 < h < 1$, and $p > q$ then $(1 - h^{p-q}) \le 1$. Consequently,

$$\begin{aligned}\|S_p - S_q\| &\le \frac{h^q}{1-h}\|x_1(t)\| \\ &\le \frac{h^q}{1-h}\max_{t\in I}|x_1(t)|\end{aligned}$$

When $|x_1(t)| < l$ and as $q \to \infty$ then $\|S_p - S_q\| \to 0$ and hence, S_p is a Cauchy sequence in the Banach space E and hence series $\sum_{i=0}^{\infty} x_i(t)$ converges. □

2.3. Homotopy Perturbation Method (HPM): The Homotopy perturbation technique defines Homotopy $u(t, p) : \Omega \times [0, 1] \to \Re$, which is satisfied

$$H(u, p) = (1 - p)F(u) + pL(u) = 0 \tag{2.4}$$

Where $t \in \Omega$ and $p \in [0, 1]$ are an impeding parameters and u_0 is an initial approximation that satisfies the boundary conditions, we can define $H(u, p)$ by

$$H(u, 0) = F(u), \quad H(u, 1) = L(u),$$

Where $F(u)$ is an integral operator such that $F(u) = u(t).a(t)$ and $L(u)$ have the form

$$L(u) = u(t) - a(t) - g(t, x(t))\int_0^t (t^m - s^m)^{\alpha-1}\Gamma(\alpha)ms^{m-1)}f(s, x(s))ds \tag{2.5}$$

and continuously trace an implicitly defined curve from starting points $H(u_0, 0)$ to a solution function $H(x, t)$. The embedding parameter p monotonically increases from zero to one, as in the trivial problem $F(u) = 0$ is continuously deformed to the original problem $L(u) = 0$.
The embedding parameter $p \in (0, 1]$ can be considered an expanding parameter.

$$u = \sum_{n=0}^{\infty} p^n u_n. \tag{2.6}$$

When $p \to 1$, (2.6) corresponds to (2.4) and gives an approximation to the solution of (2.5) as follows

$$x(t) = \lim_{p\to 1} u = \sum_{n=0}^{\infty} u_x. \tag{2.7}$$

The series (2.7) converges in most cases, and the rate of convergence depends on $L(u)$. We substitute (2.3) into (2.2) and equate the terms with the identical power of p, obtaining

$$
\begin{aligned}
p^0 &: u_0(t) = a(t),\\
p^1 &: u_1(t) = g(t,u_0)\int_0^t (t^m - s^m)^{\alpha-1}\Gamma(\alpha) m s^{m-1} H_0(s)ds\\
p^2 &: u_2(t) = g(t,u_0)\int_0^t (t^m - s^m)^{\alpha-1}\Gamma(\alpha) m s^{m-1} H_1(s)ds\\
&\quad + g(t,u_1)\int_0^t (t^m - s^m)^{\alpha-1}\Gamma(\alpha) m s^{m-1} H_0(s)ds\\
&\quad \vdots\\
p^n &: u_n(t) = g(t,u_0)\int_0^t (t^m - s^m)^{\alpha-1}\Gamma(\alpha) m s^{m-1} H_{n-1}(s)ds\\
&\quad + g(t,u_1)\int_0^t (t^m - s^m)^{\alpha-1}\Gamma(\alpha) m s^{m-1} H_{n-2}(s)ds + \dots\\
&\quad + g(t,u_{n-1})\int_0^t (t^m - s^m)^{\alpha-1}\Gamma(\alpha) m s^{m-1} H_0(s)ds, \quad n = 1,2,3,\dots.
\end{aligned}
$$

Where H_n are called He's polynomials, and it is calculated in the following formula

$$H_0(u_0,u_1,u_2,\dots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(\sum_{i=0}^{n} p^k u_k\right)^r_{p=0} \qquad n = 0,1,2,\dots.$$

3. Numerical Results

In this section, we shall discuss some numerical examples by applying the Picard method, the Adomian decomposition method, and the and the Homotopy method in a comparative study.

Example 1. Consider the following nonlinear quadratic integral equation

$$x(t) = \left(t^2 - \frac{t^{10}}{35}\right) + \frac{t}{5}x(t)\int_0^t s^2 x^2(s)ds \tag{3.1}$$

with the exact solution $x(t) = t^2$.

When applying the Picard approximation method

$$x_n(t) = \left(t^2 - \frac{t^{10}}{35}\right) + \frac{t}{5} x_{n-1}(t) \int_0^t s^2 x_{n-1}^2(s) ds, \quad n = 1, 2, 3, \ldots$$

$$x_0(t) = \left(t^2 - \frac{t^{10}}{35}\right)$$

$$x_1(t) = \left(t^2 - \frac{t^{10}}{35}\right) + \frac{t}{5} x_0 \int_0^t s^2 x_0^2(s) ds$$

$$x_1(t) = t^2 - \frac{t^{10}}{35} + \frac{t^{12}}{35} - \frac{44t^{20}}{18375} + \frac{1094t^{28}}{14791875} - \frac{76t^{36}}{73959375} + \frac{t^{44}}{172571875}.$$

Then solution will become

$$x(t) = \sum_{n=0}^{5} x_n.$$

When applying the Adomian decomposition method

$$x_0(t) = \left(t^2 - \frac{t^{10}}{35}\right)$$

$$x_n(t) = \frac{t}{5} x_{n-1}(t) \int_0^t s^2 A_{i-1}(s) ds, \quad n \geq 1.$$

Where A_i are Adomian polynomial of the nonlinear term x^2, and the solution will become,

$$x(t) = \sum_{n=0}^{5} x_n.$$

When applying the Homotopy method

$$x_0(t) = \left(t^2 - \frac{t^{10}}{35}\right)$$

$$x_n(t) = \frac{t}{5} x_{n-1}(t) \int_0^t s^2 H_{i-1}(s) ds, \quad n \geq 1.$$

Where H_i are Homotopy methods of the nonlinear term x^2, and the solution will become,

$$x(t) = \sum_{n=0}^{5} x_n.$$

Table 1. Comparative study of example 1.

t	u_{Exact}	u_{Picard}	u_{ADM}	u_{HPM}
0.0	0.00000000000	0.00000000000000	0.00000000000000	0.00000000000000
0.1	0.01000000000	0.01000000000000	0.01000000000000	0.01000000000000
0.2	0.04000000000	0.04000000002843	0.04000000002842	0.04000000002841
0.3	0.09000000000	0.09000000006589	0.09000000006523	0.09000000006532
0.4	0.16000000000	0.16000000009578	0.16000000009524	0.16000000009511
0.5	0.25000000000	0.25000000030742	0.25000000030014	0.25000000030002
0.6	0.36000000000	0.36000000095843	0.36000000095247	0.36000000095024
0.7	0.49000000000	0.49000000548758	0.49000000532748	0.49000000513689
0.8	0.64000000000	0.64000003268721	0.64000003000452	0.64000003000125
0.9	0.81000000000	0.81000048739842	0.81000042835487	0.81000042543689
1.0	1.00000000000	1.00000876529843	1.00000872587574	1.00000865478109

Table 2. Absolute error of example 1.

Absolute Error (Picard)	Absolute Error (ADM)	Absolute Error (HAM)
0.000000000000000	0.000000000000000	0.000000000000000
0.000000000000000	0.000000000000000	0.000000000000000
0.000000000028431	0.000000000028421	0.000000000028415
0.000000000065897	0.000000000065230	0.000000000065328
0.000000000095781	0.000000000095241	0.000000000095117
0.000000000307424	0.000000000300142	0.000000000300025
0.000000000958438	0.000000000952478	0.000000000950247
0.000000054875824	0.000000005327483	0.000000005136893
0.000000032687215	0.000000030004528	0.000000030001258
0.000000487398427	0.000000428354872	0.000000425443689
0.000008765298438	0.000008765258757	0.000008654781097

Example 2. Consider the following nonlinear quadratic integral equation

$$x(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right) + \frac{t^3}{10}x^2(t)\int_0^t (s+1)x^3(s)ds \tag{3.2}$$

with the exact solution $x(t) = t^3$.

When applying the Picard approximation method

$$x_n(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right) + fract^3 10x_{n-1}^3(t)\int_0^t (s+1)x_{n-1}^3(s)ds, \quad n = 1,2,3,\ldots$$

$$x_0(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right).$$

Then solution will become

$$x(t) = \lim_{n\to\infty} x_n(t).$$

When applying the Adomian decomposition method

$$x_0(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right)$$
$$x_n(t) = \frac{t^3}{10} A_{n-1}(t) \int_0^t (s+1) B_{n-1}(s) ds, \quad n \geq 1.$$

Where A_i and B_i are Adomian polynomials of the nonlinear terms x^2 and x^3 respectively,, and the solution will become,

$$x(t) = \sum_{n=0}^{q} x_n.$$

When applying the Homotopy method

$$x_0(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110}\right)$$
$$x_n(t) = \frac{t^3}{10} A_{n-1}(t) \int_0^t (s+1) H_{n-1}(s) ds, \quad n \geq 1.$$

Where A_n and H_n are Homotopy method of the nonlinear terms x^2 and x^3, and the solution will become,

$$x(t) = \sum_{n=0}^{q} x_n.$$

Table 3. Comparative study of example 2.

t	u_{Exact}	u_{Picard}	u_{ADM}	u_{HPM}
0.0	0.000000000000	0.00000000000000000	0.00000000000000000	0.00000000000000000
0.1	0.001000000000	0.00100000000000000	0.00100000000000000	0.00100000000000000
0.2	0.008000000000	0.00800000000000061	0.00800000000000061	0.00800000000000061
0.3	0.027000000000	0.02700000000147924	0.02700000000147925	0.02700000000147985
0.4	0.064000000000	0.06400000037483350	0.06400000037483350	0.06400000037483350
0.5	0.125000000000	0.12500000277432252	0.12500000277432212	0.12500000277432254
0.6	0.261000000000	0.26100094173778741	0.26100094173778537	0.26100094173777498
0.7	0.343000000000	0.34301865273758292	0.34301865273752743	0.34301865273752846
0.8	0.512000000000	0.51224892623454398	0.51224892623558756	0.51224892622794698
0.9	0.729000000000	0.72654430658439254	0.72654430658265913	0.72654430658943627
1.0	1.000000000000	0.98090909090909090	0.98090909090909090	0.98090909090909090

4. Conclusion

In this work, we apply the Picard method, the Homotopy method, and modified Adomian decomposition methods for solving quadratic nonlinear integral equations. This is a faster method and better accuracy in other polynomials with the help of examples. The approximation solutions are obtained in validity and efficiency for proposed

Table 4. Absolute error of example 2.

Absolute Error (Picard)	Absolute Error (ADM)	Absolute Error (HPM)
0.00000000000000	0.0000000000000	0.0000000000000
0.00000000000000	0.0000000000000	0.0000000000000
$6.196130909 \times 10^{-16}$	$6.196130587 \times 10^{-16}$	$6.196157425 \times 10^{-16}$
$1.479241867 \times 10^{-12}$	$1.479258927 \times 10^{-12}$	$1.479857427 \times 10^{-12}$
$3.748335094 \times 10^{-10}$	$3.748335094 \times 10^{-10}$	$3.748335094 \times 10^{-10}$
$2.774322528 \times 10^{-9}$	$2.774322125 \times 10^{-9}$	$2.774322547 \times 10^{-9}$
$9.417377874 \times 10^{-7}$	$9.417377853 \times 10^{-7}$	$9.417377749 \times 10^{-7}$
$1.865273758 \times 10^{-5}$	$1.865273752 \times 10^{-5}$	$1.865273752 \times 10^{-5}$
$2.489262345 \times 10^{-4}$	$2.489262355 \times 10^{-4}$	$2.489262279 \times 10^{-4}$
$2.455694032 \times 10^{-3}$	$2.455694032 \times 10^{-3}$	$2.455694057 \times 10^{-3}$
$1.909090909 \times 10^{-2}$	$1.909090909 \times 10^{-2}$	$1.909090909 \times 10^{-2}$

methods with the help of MATLAB 13 version.

Acknowledgement:
Manuscript communication number (MCN): IU/R&D/2021-MCN001171 office of research and development Integral University, Lucknow.

Conflicts of Interest: The authors declare no conflict of interest.

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Najmuddin Ahmad, Department of Mathematics & Statistics, Integral University, Lucknow, India
e-mail: najmuddinahmad33@gmail.com

Balmukund Singh, Department of Mathematics & Statistics, Integral University, Lucknow, India
e-mail: balmukund05071989@gmail.com

Asiya Ansari, Department of Mathematics & Statistics, Integral University, Lucknow, India
e-mail: asiyaansari.du.786@gmail.com