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ON THE CONVERGENCE OF S-ITERATION PROCESS (SIP) OF INEXACT NEWTON METHOD (INM)

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Abstract

In the present article, we introduce a SIP of inexact Newton-type algorithm to approximate the solution of operator equations in Banach space and discussed its semilocal convergence analysis (SCA) under weak and γ-Lipschitz condition (LC). In the special cases of the main result, we obtained some well established results and discussed their convergence analysis under weak conditions.

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1. Introduction

In the whole paper, the following notations have been used: D is a nonempty open convex subset of a Banach space X and Y be another Banach space, $\mathcal{F}'_u$ is the Fréchet derivative (FD) of nonlinear operator $\mathcal{F} : D \to Y$ at point $u \in D$. For given $u \in D$, a real number $r > 0$, $B_r[u] = \{v \in D : \|v - u\| \leq r\}$ and $B_r(u) = \{v \in D : \|v - u\| < r\}$ will designate the closed and open bolls respectively and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Most of the problems of applied mathematics, physical sciences, economics and engineering such as differential and integral equations, system of linear and nonlinear equations can be written in the following form:

$$\text{to find } u \in D \subseteq X \text{ such that } \mathcal{F}(u) = 0. \tag{1.1}$$

To find the exact solution of operator equation (1.1) is a challenging problem. The Newton's method (NM) is a powerful tool for finding approximate solutions of such operator equations. With initial approximation $u_0 \in D$, the NM is defined by

$$u_{n+1} = u_n - \mathcal{F}'^{-1}_{u_n}\mathcal{F}(u_n) \quad \text{for all } n \in \mathbb{N}_0. \tag{1.2}$$

In 1948, Kantorovich [1] first discussed the SCA of NM (1.2) with a convergence criteria that the involved operator $\mathcal{F}$ has bounded second FD $\mathcal{F}''$ or the FD $\mathcal{F}'$ satisfies the LC (see[2]). After Kantorovich, various generalization of NM (1.2) has

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been studied by several researchers [9–20] which either improve the Kantorovich convergence criteria or has better performance than Newton-Kantorovich method (1.2). In 1986, an important result was presented by Smale in the 20^{th} international conference of Mathematicians [3], where the concept of approximating the solution of an operator equation was proposed and the convergence criteria depending only on initial point was provided which really improve the Kantorovich's convergence criteria. Wang et al. [4–7] introduced the γ-condition which improve and generalized the Smale's main result. On the survey of literature, one can easily observe that there are two drawbacks in the NM, first is to find FD of involved operator at each iteration and second is to calculate exact solution of the following Newton's equation:

$$\mathcal{F}'_{u_n}(u_{n+1} - u_n) = -\mathcal{F}(u_n) \quad \text{for all } n \in \mathbb{N}_0$$

at each iteration. To overcome the first drawback of the NM, many Newton-type iterative methods have been proposed in literature (see [2, 12–14]) and to overcome the second drawback of the NM, Dembo et al. [21] in 1982 introduced the following INM:

$$\begin{cases} u_{n+1} = u_n + v_n; \\ \mathcal{F}'_{u_n}(v_n) = -\mathcal{F}(u_n) + r_n \quad \text{for all } n \in \mathbb{N}_0, \end{cases} \tag{1.3}$$

where $\mathcal{F} : D \to Y$ is a continuously nonlinear Fréchet differentiable operator (NFDO) and $\{r_n\}$ is a sequence in Y which depends on $\{u_n\}$. Dembo et al. [21] discussed the local convergence analysis of (1.3) under the condition that $\{r_n\}$ satisfy the following residual condition:

$$\|r_n\| \leq \kappa_n \|\mathcal{F}(u_n)\| \text{ for all } n \in \mathbb{N}_0, \tag{1.4}$$

where $0 \leq \kappa_n < 1$ for all $n \in \mathbb{N}_0$ is a forcing sequence. In place of assumption (1.4), Ypma [22] and Guo [23] used the conditions $\|\mathcal{F}'^{-1}_{u_n}(r_n)\| \leq \kappa_n \|\mathcal{F}'^{-1}_{u_n}\mathcal{F}(u_n)\|$ and $\|\mathcal{F}'^{-1}_{u_0}(r_n)\| \leq \kappa_n \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_n)\|$ for all $n \in \mathbb{N}_0$ respectively, to studied the local and SCA of (1.3). In 2009, Shen and Li [24], used the condition $\|\mathcal{F}'^{-1}_{u_0}(r_n)\| \leq \kappa_n \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_n)\|^{\rho+1}$ for all $n \in \mathbb{N}_0$, where $0 \leq \rho \leq 1$ and established a Kantorovich-type convergence criterion for the INM (1.3). With $\rho = 1$ and γ-condition, Shen and Li [25] established the Smale α-theory for the convergence analysis of (1.3). Recently, V. K. Singh [26, 27] introduced an inexact Newton-type algorithm for solving generalized operator equations involving non Fréchet differentiable operator and studied its SCA under weak and γ-LC.

In 2007, Agarwal et al. [28] presented the S-iteration process in Banach space for finding approximate fixed point of nonlinear operators. It is well known that for finding the approximate fixed point the SIP is faster than the Mann and Picard iteration processes (see [29, 31, 32]) for contraction operators. The SIP is also independent of Ishikawa [33] and Mann [34] iteration processes. In 2011, Sahu [30] introduced the normal SIP in Banach space and discussed its applications in various areas.

To find an approximate solution of real equation $f(t) = 0$, the following SIP of Newton-type method was presented by Sahu [35]:

$$\begin{cases} v_{n+1} = \omega_n - \frac{f(\omega_n)}{f'(\omega_n)}, \\ \omega_n = (1-\delta_n)v_n + \delta_n \mu_n, \\ \mu_n = v_n - \frac{f(v_n)}{f'(v_n)} \quad \text{for all } n \in \mathbb{N}_0, \end{cases}$$

where $\{\delta_n\}$ is a sequence in $(0, 1)$. In 2012, Sahu et al. [16] discussed the semi local convergence of normal SIP for modified NM and in 2017, Sahu et al. [36] studied the semilocal convergence of SIP of Newton-Kantorovich type method in the Banach space setting.

In the present article, motivated by the work of Sahu [30, 35], Dembo et al. [21], Shen and Li [25] and V. K. Singh [26, 27], we introduce a new SIP of inexact Newton-type to approximate the solution of nonlinear operator equation (1.1):

Algorithm 1.1. *Let D be nonempty open convex subset of a Banach space X, Y be another Banach space and let $u_0 \in D$. Suppose $\mathcal{F} : D \to Y$ is a NFDO such that $\mathcal{F}'^{-1}_{u_0}$ exists. For initial guess $u_0 \in D$, the SIP of inexact Newton-type method is defined as follows:*

$$\begin{cases} \mathcal{F}'_{w_n}(u_{n+1} - w_n) = -\mathcal{F}(w_n) + r_n, \\ w_n = (1-\delta)u_n + \delta v_n, \\ \mathcal{F}'_{u_n}(v_n - u_n) = -\mathcal{F}(u_n) + r'_n \quad \textit{for all } n \in \mathbb{N}_0, \end{cases} \tag{1.5}$$

where $\delta \in (0, 1)$ and $\{r_n\}$, $\{r'_n\}$ are two sequences in Y depending on $\{u_n\}$.

The purpose of this article is to study the SCA of Algorithm 1.1 under the following assumptions:

$$\|\mathcal{F}'^{-1}_{u_0} r_n\| \le \kappa'_n \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_n)\|^2; \tag{1.6}$$

$$\|\mathcal{F}'^{-1}_{u_0} r'_n\| \le \kappa''_n \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_n)\|^2 \quad \text{for all } n \in \mathbb{N}_0, \tag{1.7}$$

where $\kappa''_n = \delta\kappa'_n$, $0 \le \kappa'_n < 1$, $\kappa = \sup_{n\in\mathbb{N}}\{\kappa'_n\}$ and $\mathcal{F}'^{-1}_{u_0}\mathcal{F}'$ satisfies the following weak LC: (see[27, 38, 39]):

$$\|\mathcal{F}'^{-1}_{u_0}(\mathcal{F}'_u - \mathcal{F}'_v)\| \le \int_{\rho(u)}^{\rho(u,v)} L(t)dt \tag{1.8}$$

for all $u \in B_r(u_0), v \in B_{r-\rho(u)}[u_0]$, where $\rho(u) = \|u - u_0\|, \rho(u, v) = \rho(u) + \|v - u\| \le r$ and $L : [0, r] \to \mathbb{R}$ be a non-negative, non-decreasing integrable function. The Kantorovich's and Smale's type convergence conditions are the special cases of our convergence criteria. More precisely, when $\kappa_n = 0$ for all $n \in \mathbb{N}_0$ our Algorithm 1.1 reduces to SIP of Newton-Kantorovich method [36]. Our results extend and improve some established results in the context of SIP of INM in the Banach space setting.

The remaining part of the article is arranged as follows: In Section 2, we provide some basic results. In Section 3, we discussed the SCA of proposed algorithm under the weak LC and some related corollaries are also discussed. In Section 4, we studied the SCA of same algorithm under the γ-Lipschitz condition.

2. Preliminaries

In the present section, we provide some basic definitions and results which are very useful in the proof of our results.

LEMMA 2.1. *Let σ, r, a and τ be positive real numbers with $\sigma \geq 0$. Let $L : [0, r] \to \mathbb{R}$ be a positive non decreasing function and $f : [0, r] \to \mathbb{R}$ be a function defined by*

$$f(\nu) = \tau - \nu + \sigma\nu^2 + a \int_0^{\nu} L(\theta)(\nu - \theta)d\theta. \tag{2.1}$$

Define two real numbers ω and b_ω by

$$\omega = \sup\left\{\nu \in (0, r) : a \int_0^{\nu} L(\theta)d\theta + 2\sigma\nu \leq 1\right\},$$

$$b_\omega = \omega - \sigma\omega^2 - a \int_0^{\omega} L(\theta)(\omega - \theta)d\theta. \tag{2.2}$$

If $\tau \leq b_\omega$, then f is strictly decreasing on $[0, \omega]$ and it has a unique zero ν^* in $[0, \omega]$ with $\tau < \nu^*$.

Proof: One can easily observe that $f'(\nu) = -1 + 2\sigma\nu + a\int_0^{\nu} L(\theta)d\theta$, $f''(\nu) = 2\sigma + aL(\nu) \geq 0$ for all $\nu \in [0, r]$. Which implies that f' is increasing on $[0, r]$ and $f'(0) < 0$ and $f'(\omega) \leq 0$ hence f is decreasing on $[0, \omega]$. From (2.1) and (2.2), we have $f(0) = \tau > 0$ and $f(\omega) \leq 0$. Which shows that $f(\nu)$ has a unique zero ν^* in $[0, \omega]$ and

$$f(\tau) = \sigma\tau^2 + a \int_0^{\tau} L(\theta)(\tau - \theta)d\theta > 0,$$

gives $\tau < \nu^*$. □

LEMMA 2.2. *Let $L_0 : [0, r] \to \mathbb{R}$ be a non-negative, non-decreasing function and let all the assumptions of Lemma 2.1 hold. Define $g : [0, r] \to \mathbb{R}$ by*

$$g(\nu) = \tau - \nu + a \int_0^{\nu} L_0(\theta)(\nu - \theta)d\theta. \tag{2.3}$$

If $L_0(\nu) \leq L(\nu)$ for all $\nu \in [0, r]$ and ν^* is the positive solution of $f(\nu) = 0$ in $[0, s]$. Then the sequence $\{\nu_n\}$ defined by

$$\begin{cases} \nu_0 = 0, \\ \mu_n = \nu_n - \frac{f(\nu_n)}{g'(\nu_n)}, \\ \omega_n = (1 - \delta)\nu_n + \delta\mu_n, \\ \nu_{n+1} = \omega_n - \frac{f(\omega_n)}{g'(\omega_n)} \text{ for all } n \in \mathbb{N}_0, \end{cases} \tag{2.4}$$

where $\delta \in (0, 1)$ is increasing and converges to ν^*. Moreover, the following assertions hold:

(a) $\nu_n \leq \omega_n \leq \nu_{n+1} < \nu^*$,
(b) $a \int_0^{\nu_n - \omega_{n-1}} (\nu_n - \omega_{n-1} - \nu)L(\omega_{n-1} + \nu)d\nu + \sigma(\nu_n - \omega_{n-1})^2 \leq f(\nu_n)$,

(c) $a\int_0^{\omega_n-\nu_n}(\omega_n-\nu_n-\nu)L(\nu_n+\nu)d\nu + a(1-\delta)f(\nu_n)+\delta^2\sigma(\mu_n-\nu_n)^2 \le f(\omega_n)$ *for each* $n\in\mathbb{N}_0$.

Proof. We shall prove part (a) by mathematical induction method. Note that $0=\nu_0<\mu_0=\tau$, $\omega_0=\delta\mu_0$ and

$$\nu_1=\omega_0-\frac{f(\omega_0)}{g'(\omega_0)}\ge\omega_0.$$

Let $k>1$ be an integer and suppose that the following hold for each $n=1,2,3,\cdots,k$

$$\nu_{k-1}\le\omega_{k-1}<\nu_k<\nu^*. \tag{2.5}$$

Since $g''(\nu)=aL_0(\nu)>0$ for all $\nu\in[0,r]$, this shows that $-g'$ is strictly decreasing on $[0,\omega]$ and for all $\nu\in[0,\omega]$, we obtained

$$-g'(\nu)>-g'(\nu^*)\ge-g'(\omega)=1-a\int_0^{\omega}L_0(\theta)d\theta>1-a\int_0^{\omega}L(\theta)d\theta>0.$$

Note that $f(\nu_k)>0$. It follows that

$$\omega_k=\nu_k-\delta\frac{f(\nu_k)}{g'(\nu_k)}\ge\nu_k \text{ and } \nu_{k+1}=\omega_k-\frac{f(\omega_k)}{g'(\omega_k)}\ge\omega_k.$$

Thus (2.5) hold for $n=k+1$. Therefore, by the method of mathematical induction, (2.5) is true for each $n\in\mathbb{N}$.

Define $\mathcal{N},\mathcal{S}:[0,\nu^*]\to[0,\nu^*]$ by

$$\mathcal{N}(\nu)=\nu-\delta\frac{f(\nu)}{g'(\nu)} \quad\text{and}\quad \mathcal{S}(\mathcal{N}(\nu))=\mathcal{N}(\nu)-\frac{f(\mathcal{N}(\nu))}{g'(\mathcal{N}(\nu))}.$$

Since $g'(\nu)<0$ on $[0,\nu^*]$ and $f(\nu^*)=0$, therefore $\lim_{\nu\to\nu^*}\frac{f(\nu)}{g'(\nu)}=0$, $\mathcal{N}(\nu^*)=\nu^*$ and $\mathcal{S}(\mathcal{N}(\nu^*))=\nu^*$. Thus the functions $\mathcal{N}(\nu)$ and $\mathcal{S}(\nu)$ are well defined on $[0,\nu^*]$. Now for all $\nu\in[0,\nu^*]$, we have

$$\mathcal{N}'(\nu)=\frac{(g'(\nu))^2-\delta g'(\nu)f'(\nu)+\delta f(\nu)g''(\nu)}{(g'(\nu))^2}>0$$

and

$$\mathcal{S}'(\nu)=\frac{-g'(\nu)(f'(\nu)-g'(\nu))+f(\nu)g''(\nu)}{(g'(\nu))^2}>0.$$

Thus the functions $\mathcal{N}(\nu)$ and $\mathcal{S}(\nu)$ are monotonically increasing on $[0,\nu^*]$ and hence

$$\nu_n\le\omega_n=\mathcal{N}(\nu_n)\le\nu_{n+1}=\mathcal{S}(\mathcal{N}(\nu_n))<\mathcal{S}(\nu^*)=\nu^*.$$

(b) By (2.1), (2.3) and (2.4), one can easily observe that

$$\begin{aligned}
& a\int_0^{\nu_n-\omega_{n-1}}(\nu_n-\omega_{n-1}-\nu)L(\omega_{n-1}+\nu)d\nu+\sigma(\nu_n-\omega_{n-1})^2\\
= \ & a\int_{\omega_{n-1}}^{\nu_n}(\nu_n-\theta)L(\theta)d\theta+\sigma(\nu_n-\omega_{n-1})^2\\
= \ & a\int_0^{\nu_n}(\nu_n-\theta)L(\theta)d\theta-a\int_0^{\omega_{n-1}}(\nu_n-\theta)L(\theta)d\theta+\sigma(\nu_n-\omega_{n-1})^2\\
= \ & a\int_0^{\nu_n}(\nu_n-\theta)L(\theta)d\theta-a\int_0^{\omega_{n-1}}(\omega_{n-1}-\theta)L(\theta)d\theta-a\int_0^{\omega_{n-1}}L(\theta)d\theta(\nu_n-\omega_{n-1})\\
+ \ & \sigma\nu_n^2-\sigma\omega_{n-1}^2-2\sigma\omega_{n-1}(\nu_n-\omega_{n-1})\\
= \ & f(\nu_n)-f(\omega_{n-1})-f'(\omega_{n-1})(\nu_n-\omega_{n-1})\\
= \ & f(\nu_n)-f(\omega_{n-1})-g'(\omega_{n-1})(\nu_n-\omega_{n-1})-2\sigma\omega_{n-1}(\nu_n-\omega_{n-1})\\
= \ & f(\nu_n)-2\sigma\omega_{n-1}(\nu_n-\omega_{n-1})\\
\leq \ & f(\nu_n).
\end{aligned}$$

(c) Again by (2.1), (2.3) and (2.4), we have

$$\begin{aligned}
& a\int_0^{\omega_n-\nu_n}(\omega_n-\nu_n-\nu)L(\nu_n+\nu)d\nu+\delta^2\sigma(\mu_n-\nu_n)^2+c(1-\delta)f(t_n)\\
\leq \ & a\int_{\nu_n}^{\omega_n}(\omega_n-\theta)L(\theta)d\theta+\delta^2\sigma(\mu_n-\nu_n)^2+(1-\delta)f(\nu_n)\\
= \ & a\int_0^{\omega_n}(\omega_n-\theta)L(\theta)d\theta-a\int_0^{\nu_n}(\nu_n-\theta)L(\theta)d\theta-a\int_0^{\nu_n}(\omega_n-\nu_n)L(\theta)d\theta\\
+ \ & \sigma(\omega_n-\nu_n)^2+(1-\delta)f(\nu_n)\\
= \ & a\int_0^{\omega_n}(\omega_n-\theta)L(\theta)d\theta-a\int_0^{\nu_n}(\nu_n-\theta)L(\theta)d\theta-a\int_0^{\nu_n}(\omega_n-\nu_n)L(\theta)d\theta\\
+ \ & \sigma\omega_n^2-\sigma\nu_n^2-2\sigma\nu_n(\omega_n-\nu_n)+(1-\delta)f(\nu_n)\\
= \ & f(\omega_n)-f(\nu_n)-f'(\nu_n)(\omega_n-\nu_n)+(1-\delta)f(\nu_n)\\
= \ & f(\omega_n)-f(\nu_n)-g'(\nu_n)(\omega_n-\nu_n)-2\sigma\nu_n(\omega_n-\nu_n)+(1-\delta)f(\nu_n)\\
= \ & f(\omega_n)-f(\nu_n)-\delta g'(\nu_n)(\mu_n-\nu_n)-2\sigma\nu_n(\omega_n-\nu_n)+(1-\delta)f(\nu_n)\\
= \ & f(\omega_n)-2\sigma\nu_n(\omega_n-\nu_n)\leq f(\omega_n).
\end{aligned}$$

This completes the proof. □

Definition 2.3. [1] Let X be a Banach space and $\{u_n\}$ be a sequence in X. A real sequence $\{\nu_n\}$ is said to be a majorizing sequence of $\{u_n\}$ if

$$\|u_{n+1}-u_n\|\leq\nu_{n+1}-\nu_n \quad \text{for all} \quad n\in\mathbb{N}_0.$$

Lemma 2.4. *[1] Let X be a Banach space and $\{u_n\}$ be a sequence in X and $\{\nu_n\}$ be a majorizing sequence of $\{u_n\}$. Then, we have the following:*

(a) *sequence* $\{v_n\}$ *is non-decreasing;*

(b) *if* $\{v_n\}$ *converges to* $v^* < \infty$, *then there exists* $u^* \in X$ *such that* $\lim_{n\to\infty} u_n = u^*$ *and*

$$\|u^* - u_n\| \le v^* - v_n \quad \textit{for all} \quad n \in \mathbb{N}_0.$$

Lemma 2.5. *(Rall [8], Page 50) Let X be a Banach space and* $\mathfrak{P}$ *be a bounded linear operator on X. Then,* $\mathfrak{P}^{-1}$ *exists if and only if there exists a bounded linear operator* $\mathfrak{Q}$ *on X such that* $\mathfrak{Q}^{-1}$ *exists, and*

$$\|\mathfrak{Q} - \mathfrak{P}\| \quad < \quad \frac{1}{\left\|\mathfrak{Q}^{-1}\right\|}.$$

If $\mathfrak{P}^{-1}$ *exists, then*

$$\left\|\mathfrak{P}^{-1}\right\| \le \frac{\left\|\mathfrak{Q}^{-1}\right\|}{1 - \left\|1 - \mathfrak{Q}^{-1}\mathfrak{P}\right\|} \le \frac{\left\|\mathfrak{Q}^{-1}\right\|}{1 - \left\|\mathfrak{Q}^{-1}\right\| \|\mathfrak{Q} - \mathfrak{P}\|}.$$

Lemma 2.6. *Let D be a nonempty open subset of a Banach space X and Y be another Banach space. Suppose* $\mathcal{F} : D \to Y$ *is a NFDO such that* $\mathcal{F}_{u_0}'^{-1}$ *exists for some* $u_0 \in D$. *Let* $r > 0$ *be such that* $B_r(u_0) \subset D$ *and* L_0 *a real valued non-negative non-decreasing integrable function defined on [0,r]. Assume that the following conditions hold:*

(C_1) $\|\mathcal{F}_{u_0}'^{-1}(\mathcal{F}_x' - \mathcal{F}_{u_0}')\| \le \int_0^{\|x-u_0\|} L_0(\theta)d\theta$ *for all* $x \in B_r(u_0)$;

(C_2) $\int_0^r L_0(\theta)d\theta < 1.$

Then, for each $x \in B_r(u_0)$, $\mathcal{F}_x'$ *is invertible and*

$$\|\mathcal{F}_x'^{-1}\mathcal{F}_{u_0}'\| \le \frac{1}{1 - \int_0^{\|x-u_0\|} L_0(\theta)d\theta} \le -\frac{1}{g'(\|x - u_0\|)}, \tag{2.6}$$

where function g is defined by (2.3).

Proof. For each $x \in B_r(u_0)$, we obtain

$$\begin{aligned} \|I - \mathcal{F}_{u_0}'^{-1}\mathcal{F}_x'\| &= \|\mathcal{F}_{u_0}'^{-1}(\mathcal{F}_x' - \mathcal{F}_{u_0}')\| \\ &\le \int_0^{\|x-u_0\|} L_0(\theta)d\theta < 1. \end{aligned}$$

Therefore, by Lemma 2.5, we obtain

$$\|\mathcal{F}_x'^{-1}\mathcal{F}_{u_0}'\| \le \frac{1}{1 - \int_0^{\|x-u_0\|} L_0(\theta)d\theta}. \tag{2.7}$$

For each $x \in B_r(u_0)$, we have

$$-g'(\|x - u_0\|) = 1 - a\int_0^{\|x-u_0\|} L_0(\theta)d\theta \le 1 - \int_0^{\|x-u_0\|} L_0(\theta)d\theta.$$

This implies that

$$\frac{1}{1 - \int_0^{\|x-u_0\|} L_0(\theta)d\theta} \le -\frac{1}{g'(\|x - u_0\|)}. \tag{2.8}$$

Thus (2.6) follows from (2.7) and (2.8). □

3. Main result

In this section, we discuss the SCA of our Algorithm 1.1 under weak LC (1.8). Before proving our main result, we will establish the following results:

Proposition 3.1. *Let D be a nonempty open convex subset of a Banach space X, Y be another Banach space and $u_0 \in D$. Suppose $\mathcal{F} : D \to Y$ is a NFDO such that $\mathcal{F}'^{-1}_{u_0}$ exists. Assume that $\mathcal{F}'_u$ satisfied the weak LC (1.8) and (C_1). Let $r > 0$ be such that $B_r(u_0) \subseteq D$ and let $R, R' : B_r(u_0) \to Y$ be two operators satisfy the following conditions*

$$\|\mathcal{F}'^{-1}_{u_0} R(u)\| \le \kappa'_u \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u)\|^2 \text{ and } \|\mathcal{F}'^{-1}_{u_0} R'(u)\| \le \kappa''_u \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u)\|^2,$$

where $\kappa''_u = \delta\kappa'_u$ and $0 \le \kappa'_u < 1$ such that $\sqrt{\kappa'_u}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u)\| \le 1$. Let $0 < \delta < 1$ and define $P, Q, S, T, U : B_r(u_0) \to X$ by

$$\begin{cases} T(u) = P(u) + S(u), \\ \mathcal{F}'_{P(u)}(S(u)) = -\mathcal{F}(P(u)) + R(P(u)), \\ P(u) = (1-\delta)u + \delta Q(u), \\ Q(u) = u + U(u), \\ \mathcal{F}'_u(U(u)) = -\mathcal{F}(u) + R'(u), \end{cases} \tag{3.1}$$

for all $u \in B_r(u_0)$. Then for all $u \in B_r(u_0)$ with $T(u), P(u) \in B_r(u_0)$, we obtain

(a) $\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_u\| \le 1 + \int_0^{\|u-u_0\|} L_0(\theta)d\theta$,

(b) $\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(T(u))\| \le \int_0^1 \int_{\|P(u)-u_0\|}^{\|P(u)-u_0\|+v\|T(u)-P(u)\|} L(\theta)\|T(u)-P(u)\|d\theta dv$ $+ \kappa'_{P(u)}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(P(u))\|^2$,

(c) $\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(P(T(u)))\| \le \int_0^1 \int_{\|u-u_0\|}^{\|u-u_0\|+v\|P(u)-u\|} L(\theta)\|P(u)-u\|d\theta dv + (1-\delta)\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u)\|$ $+ \delta^2\kappa'_u\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u)\|^2$.

Furthermore, if $\mathcal{F}'^{-1}_u$ exists for some $u \in B_r(u_0)$, then

(d) $\|P(u) - u\| \le \delta(1 + \sqrt{\kappa''_u})\|(\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_u)^{-1}\|\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u)\|$.

Proof. : (a) For $u \in B_r(u_0)$, we have

$$\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_u\| = \|I + \mathcal{F}'^{-1}_{u_0}(\mathcal{F}'_u - \mathcal{F}'_{u_0})\| \le 1 + \|\mathcal{F}'^{-1}_{u_0}(\mathcal{F}'_u - \mathcal{F}'_{u_0})\| \le 1 + \int_0^{\|u-u_0\|} L_0(\theta)d\theta.$$

(b) Using (1.8) and (3.1), we have

$$\begin{aligned} & \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(T(u))\| \\ = \; & \|\mathcal{F}'^{-1}_{u_0}(\mathcal{F}(T(u)) - \mathcal{F}(P(u)) - \mathcal{F}'_{P(u)}(S(u)) + R(P(u)))\| \\ = \; & \|\mathcal{F}'^{-1}_{u_0}(\mathcal{F}(T(u)) - \mathcal{F}(P(u)) - \mathcal{F}'_{P(u)}(T(u) - P(u)) + R(P(u)))\| \\ \le \; & \int_0^1 \|\mathcal{F}'^{-1}_{u_0}(\mathcal{F}'_{P(u)+v(T(u)-P(u))} - \mathcal{F}'_{P(u)})\|dv\|T(u) - P(u)\| + \|\mathcal{F}'^{-1}_{u_0}R(P(u))\| \\ \le \; & \int_0^1 \int_{\|P(u)-u_0\|}^{\|P(u)-u_0\|+v\|T(u)-P(u)\|} L(\theta)\|T(u) - P(u)\|d\theta dv + \kappa'_{P(u)}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(P(u))\|^2. \end{aligned}$$

(c) Again using (1.8) and (3.1), we have

$$\begin{aligned}
&\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(P(u))\| = \|\mathcal{F}_{u_0}'^{-1}(\mathcal{F}(P(u)) - \mathcal{F}(u) - \mathcal{F}_u' U(u) + R'(u))\| \\
= \;& \|\mathcal{F}_{u_0}'^{-1}(\mathcal{F}(P(u)) - \mathcal{F}(u) - \mathcal{F}_u'(Q(u) - u) + R'(u))\| \\
= \;& \|\mathcal{F}_{u_0}'^{-1}(\mathcal{F}(P(u)) - \mathcal{F}(u) - \mathcal{F}_u'(P(u) - u) - \mathcal{F}_u'(Q(u) - P(u)) + R'(u))\| \\
= \;& \|\mathcal{F}_{u_0}'^{-1}\int_0^1 (\mathcal{F}_{u+\nu(P(u)-u)}' - \mathcal{F}_u')(P(u) - u)d\nu + (1-\delta)\mathcal{F}(u) + \delta R'(u)\| \\
\leq \;& \int_0^1 \|\mathcal{F}_{u_0}'^{-1}(\mathcal{F}_{u+\nu(P(u)-u)}' - \mathcal{F}_u')\|d\nu\|P(u) - u\| + (1-\delta)\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(u)\| + \delta\|\mathcal{F}_{u_0}'^{-1}R'(u)\| \\
\leq \;& \int_0^1\int_{\|u-u_0\|}^{\|u-u_0\|+\nu\|P(u)-u\|} L(\theta)\|P(u) - u\|d\theta d\nu + (1-\delta)\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(u)\| + \delta\kappa_u''\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(u)\|^2 \\
= \;& \int_0^1\int_{\|u-u_0\|}^{\|u-u_0\|+\nu\|P(u)-u\|} L(\theta)\|P(u) - u\|d\theta d\nu + (1-\delta)\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(u)\| + \delta^2\kappa_u'\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(u)\|^2.
\end{aligned}$$

(d) Let $u \in B_r(u_0)$ be such that $\mathcal{F}_u'^{-1}$ exists, then

$$\begin{aligned}
\|P(u) - u\| &= \delta\|Q(u) - u\| \\
&= \delta\|\mathcal{F}_u'^{-1}(\mathcal{F}(u) - R'(u))\| \\
&\leq \delta\|(\mathcal{F}_{u_0}'^{-1}\mathcal{F}_u')^{-1}\|(\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(u)\| + \|\mathcal{F}_{u_0}'^{-1}R'(u)\|) \\
&\leq \delta\|(\mathcal{F}_{u_0}'^{-1}\mathcal{F}_u')^{-1}\|(\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(u)\| + \kappa_u''\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(u)\|^2) \\
&= \delta(1 + \sqrt{\kappa_u''})\|(\mathcal{F}_{u_0}'^{-1}\mathcal{F}_u')^{-1}\|\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(u)\|.
\end{aligned}$$

This completes the proof of proposition. □

Lemma 3.2. *Let D be a nonempty open convex subset of a Banach space X, Y be another Banach space and $u_0 \in D$. Let $\mathcal{F} : D \to Y$ be a NFDO such that $\mathcal{F}_{u_0}'^{-1}$ exists. Suppose all the conditions of Lemma 2.2, (C_1), (C_2), (1.6)-(1.8) hold and we also suppose that the following assumptions hold:*

(C_3) $\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(u_0)\| = \beta$ *for some* $\beta > 0$;

(C_4) $\sqrt{\kappa}\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(u_0)\| \leq 1$;

(C_5) $a = 1 + \sqrt{\kappa}$, $\tau = a\beta$, $\sigma = \frac{\kappa a\left(1+\int_0^r L_0(\theta)d\theta\right)^2}{(1-\sqrt{\kappa})^2}$.

Then the sequence $\{u_n\}$ defined by SIP of INM (1.5) is well defined on $B_r[u_0]$ and for each $n \in \mathbb{N}$ the following assertions hold:

(a) $a\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(u_n)\| \leq f(\nu_n)\frac{\|u_n-w_{n-1}\|}{\nu_n-\omega_{n-1}}$, $a\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(w_n)\| \leq f(\omega_n)\frac{\|u_n-w_{n-1}\|}{\nu_n-\omega_{n-1}}$,

(b) $\sqrt{\kappa}\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(u_n)\| \leq 1$, $\sqrt{\kappa}\|\mathcal{F}_{u_0}'^{-1}\mathcal{F}(w_n)\| \leq 1$

(c) $\|w_n - u_n\| \leq \frac{\omega_n-\nu_n}{\nu_n-\omega_{n-1}}\|u_n - w_{n-1}\|$ *and* $\|u_{n+1} - w_n\| \leq \frac{\nu_{n+1}-\omega_n}{\nu_n-\omega_{n-1}}\|u_n - w_{n-1}\|$, *where $\{\omega_n\}$ and $\{\nu_n\}$ are real sequences defined by (2.4).*

Proof. Set $u_{m+1} = T(u_m), w_m = P(u_m), r_m = R(w_m), r_m' = R'(u_m), \kappa_m' = \kappa_{w_m}'$ and $\kappa_m'' = \kappa_{u_m}'', m \in \mathbb{N}_0$. We will prove this Lemma by mathematical induction method and it will be complete in the following steps.

Step I Here we will prove that (a)-(c) hold for $n = 1$.
Using (2.4) and (1.5), we have

$$\begin{aligned}
\|v_0 - u_0\| &= \|\mathcal{F}'^{-1}_{u_0}(-\mathcal{F}(u_0) + r'_0)\| \\
&\leq \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_0)\| + \|\mathcal{F}'^{-1}_{u_0} r'_0\| \\
&\leq \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_0)\| + \sqrt{\kappa}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_0)\| \\
&= (1+\sqrt{\kappa})\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_0)\| \\
&= (1+\sqrt{\kappa})\beta = \tau = -\frac{f(v_0)}{g'(v_0)} = \mu_0 - v_0
\end{aligned}$$

and $\|w_0 - u_0\| = \delta\|v_0 - u_0\| \leq \delta(u_0 - v_0) = \omega_0 - v_0 = \omega_0$. Using Proposition 3.1, one can easily observe that

$$\begin{aligned}
(1+\sqrt{\kappa})\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\| &\leq (1+\sqrt{\kappa})\Big(\int_0^1\int_0^{v\|w_0-u_0\|} L_0(\theta)\|w_0-u_0\|d\theta dv \\
&+ (1-\delta)\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_0)\| + \delta\|\mathcal{F}'^{-1}_{u_0} r'_0\|\Big) \\
&\leq a\int_0^1\int_0^{v\|w_0-u_0\|} L_0(\theta)\|w_0-u_0\|d\theta dv + (1-\delta)a\beta + \delta\sqrt{\kappa}a\beta \\
&\leq a\int_0^{\omega_0}(\omega_0 - v)L(v)dv + (1-\delta)a\beta + \delta^2\kappa a\beta^2 \\
&\leq a\int_0^{\omega_0}(\omega_0 - v)L(v)dv + (1-\delta)a\beta + a\delta^2\kappa\beta^2 \\
&= a\beta - \delta a\beta + a\delta^2\kappa\beta^2 + a\int_0^{\omega_0}(\omega_0 - v)L(v)dv \\
&= \tau - \omega_0 + a\delta^2\kappa\beta^2 + a\int_0^{\omega_0}(\omega_0 - v)L(v)dv \\
&\leq \tau - \omega_0 + \sigma\omega_0^2 + a\int_0^{\omega_0}(\omega_0 - v)L(v)dv = f(\omega_0). \qquad (3.2)
\end{aligned}$$

Since f is decreasing on $[0, \omega]$. Therefore,

$$f(\omega_0) \leq f(v_0) \text{ and } \sqrt{\kappa}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\| \leq \frac{\sqrt{\kappa}\tau}{1+\sqrt{\kappa}} = \beta\sqrt{\kappa} \leq \sqrt{\kappa}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\| \leq 1.$$

From Lemma 2.6, (1.5) and (3.2), we have

$$\begin{aligned}
\|u_1 - w_0\| &= \|-\mathcal{F}'^{-1}_{w_0}(\mathcal{F}(w_0) - r_0)\| \\
&\leq \|(\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_{w_0})^{-1}\|(\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\| + \|\mathcal{F}'^{-1}_{u_0} r_0\|) \\
&\leq -\frac{1}{g'(\omega_0)}(\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\| + \sqrt{\kappa}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\|) \\
&= -\frac{(1+\sqrt{\kappa})\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\|}{g'(\omega_0)} \leq -\frac{f(\omega_0)}{g'(\omega_0)} = v_1 - \omega_0.
\end{aligned}$$

From Proposition 3.1, we have

$$\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_1)\| \leq \int_0^1 \int_{\|w_0-u_0\|}^{\|w_0-u_0\|+\nu\|u_1-w_0\|} L(\theta)\|u_1-w_0\|d\theta dt + \kappa_0'\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\|^2, \tag{3.3}$$

$$\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_{w_0}\| \leq 1 + \int_0^{\|w_0-u_0\|} L_0(\theta)d\theta \leq 1 + \int_0^r L_0(\theta)d\theta. \tag{3.4}$$

From Algorithm 1.1, we have

$$\begin{aligned}
\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_{w_0}(u_1-w_0)\| &= \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0) - \mathcal{F}'^{-1}_{u_0}r_0\| \\
&\geq \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\| - \|\mathcal{F}'^{-1}_{u_0}r_0\| \\
&\geq \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\| - \sqrt{\kappa}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\| \\
&= (1-\sqrt{\kappa})\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\|.
\end{aligned}$$

Which implies that

$$\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\| \leq \frac{\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_{w_0}\|\|u_1-w_0\|}{1-\sqrt{\kappa}} \leq \frac{1+\int_0^r L_0(\theta)d\theta}{1-\sqrt{\kappa}}\|u_1-w_0\|. \tag{3.5}$$

From (3.3), (3.4) and (3.5), we get

$$\begin{aligned}
(1+\sqrt{\kappa})\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_1)\| &\leq a\int_0^1 \int_{\|w_0-u_0\|}^{\|w_0-u_0\|+\nu\|u_1-w_0\|} L(\theta)\|u_1-w_0\|d\theta d\nu \\
&+ a\kappa\frac{\left(1+\int_0^r L_0(\theta)d\theta\right)^2}{(1-\sqrt{\kappa})^2}\|u_1-w_0\|^2 \\
&\leq \Bigg[a\int_0^1 \int_{\omega_0-\nu_0}^{\omega_0-\nu_0+\nu(\nu_1-\omega_0)} L(\theta)(\nu_1-\omega_0)d\theta d\nu \\
&+ a\kappa\frac{\left(1+\int_0^r L_0(\theta)d\theta\right)^2}{(1-\sqrt{\kappa})^2}(\nu_1-\omega_0)^2\Bigg]\frac{\|u_1-w_0\|}{\nu_1-\omega_0}
\end{aligned}$$

$$\begin{aligned}
&\leq \left[a\int_0^{\nu_1-\omega_0}(\nu_1-\omega_0-\nu)L(\omega_0+\nu)d\nu + \sigma(\nu_1-\omega_0)^2\right]\frac{\|u_1-w_0\|}{\nu_1-\omega_0} \\
&\leq f(\nu_1)\frac{\|u_1-w_0\|}{\nu_1-\omega_0}.
\end{aligned}$$

One can easily prove that $\sqrt{\kappa}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_1)\| \leq 1$. Using (2.4) and Proposition 3.1, we have

$$\begin{aligned}
\|w_1-u_1\| &\leq \delta(1+\sqrt{\kappa})\|(\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_{u_1})^{-1}\|\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_1)\| \\
&= -\delta\frac{f(\nu_1)}{g'(\nu_1)}\frac{\|u_1-w_0\|}{\nu_1-\omega_0} = \frac{\omega_1-\nu_1}{\nu_1-\omega_0}\|u_1-w_0\|.
\end{aligned}$$

Step II In this step, we will prove that (a)-(c) hold for all $n \in \mathbb{N}$.
Let $k \geq 1$ be an integer and suppose that the following hold for $n = 1, 2, 3, \cdots, k$

$$\begin{cases} a\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_{k-1})\| \leq f(\nu_{k-1})\frac{\|u_{k-1}-w_{k-2}\|}{\nu_{k-1}-\omega_{k-2}}, \\ a\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_{k-1})\| \leq f(\omega_{k-1})\frac{\|u_{k-1}-w_{k-2}\|}{\nu_{k-1}-\omega_{k-2}}, \\ \sqrt{\kappa}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_k)\| \leq 1, \quad \sqrt{\kappa}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_{k-1})\| \leq 1, \\ \|w_{k-1} - u_{k-1}\| \leq \frac{\omega_{k-1}-\nu_{k-1}}{\nu_{k-1}-\omega_{k-2}}\|u_{k-1} - w_{k-2}\|, \quad \|u_k - w_{k-1}\| \leq \frac{\nu_k-\omega_{k-1}}{\nu_{k-1}-\omega_{k-2}}\|u_{k-1} - w_{k-2}\|. \end{cases} \tag{3.6}$$

Using Proposition 3.1, we have

$$\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_k)\| \leq \int_0^1 \int_{\|w_{k-1}-u_0\|}^{\|w_{k-1}-u_0\|+\nu\|u_k-w_{k-1}\|} L(\theta)\|u_k - w_{k-1}\|d\theta d\nu + \kappa'_{k-1}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_{k-1})\|^2, \tag{3.7}$$

$$\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_{w_{k-1}}\| \leq 1 + \int_0^{\|w_{k-1}-u_0\|} L_0(\theta)d\theta \leq 1 + \int_0^r L_0(\theta)d\theta. \tag{3.8}$$

From Algorithm 1.1, we have

$$\begin{aligned} \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_{w_{k-1}}(u_k - w_{k-1})\| &= \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_{k-1}) - \mathcal{F}'^{-1}_{u_0}r_{k-1}\| \\ &\geq \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_{k-1})\| - \|\mathcal{F}'^{-1}_{u_0}r_{k-1}\| \\ &\geq \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_{k-1})\| - \sqrt{\kappa}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_{k-1})\| \\ &= (1 - \sqrt{\kappa})\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_{k-1})\|. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we have

$$\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_{k-1})\| \leq \frac{\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_{w_{k-1}}\| \, \|u_k - w_{k-1}\|}{1 - \sqrt{\kappa}} \leq \frac{1 + \int_0^r L_0(\theta)d\theta}{1 - \sqrt{\kappa}}\|u_k - w_{k-1}\|. \tag{3.10}$$

From Lemma 2.2, (3.7) and (3.10), we have

$$\begin{aligned} (1 + \sqrt{\kappa})\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_k)\| &\leq a\int_0^1 \int_{\|w_{k-1}-u_0\|}^{\|w_{k-1}-u_0\|+\nu\|u_k-w_{k-1}\|} L(\theta)\|u_k - w_{k-1}\|d\theta d\nu \\ &+ a\kappa\frac{\left(1 + \int_0^r L_0(\theta)d\theta\right)^2}{(1 - \sqrt{\kappa})^2}\|u_k - w_{k-1}\|^2 \\ &\leq \left[a\int_0^1 \int_{\omega_{k-1}-\nu_0}^{\omega_{k-1}-\nu_0+\nu(\nu_k-\omega_{k-1})} L(\theta)(\nu_k - \omega_{k-1})d\theta d\nu \right. \\ &+ \left. a\kappa\frac{\left(1 + \int_0^r L_0(\theta)d\theta\right)^2}{(1 - \sqrt{\kappa})^2}(\nu_k - \omega_{k-1})^2\right]\frac{\|u_k - w_{k-1}\|}{\nu_k - \omega_{k-1}} \\ &\leq \left[a\int_0^{\nu_k-\omega_{k-1}} (\nu_k - \omega_{k-1} - \nu)L(\omega_{k-1} + \nu)d\nu \right. \\ &+ \left. \sigma(\nu_k - \omega_{k-1})^2\right]\frac{\|u_k - w_{k-1}\|}{\nu_k - \omega_{k-1}} \\ &\leq f(\nu_k)\frac{\|u_k - w_{k-1}\|}{\nu_k - \omega_{k-1}}. \end{aligned} \tag{3.11}$$

Now from Lemma 2.2 and (3.11), we have

$$\begin{aligned}
\|w_k - u_k\| &\leq \delta(1+\sqrt{\kappa})\|(\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_{u_k})^{-1}\|\,\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_k)\| \\
&\leq -\delta\frac{f(\nu_k)}{g'(\nu_k)}\frac{\|u_k - w_{k-1}\|}{\nu_k - \omega_{k-1}} \\
&= \frac{\omega_k - \nu_k}{\nu_k - \omega_{k-1}}\|u_k - w_{k-1}\|. \qquad (3.12)
\end{aligned}$$

Again from Proposition 3.1, we have

$$\begin{aligned}
\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_k)\| &\leq \int_0^1\int_{\|u_k-u_0\|}^{\|u_k-u_0\|+\nu\|w_k-u_k\|} L(\theta)\|w_k - u_k\|d\theta d\nu \\
&+ (1-\delta)\int_0^1\int_{\|w_{k-1}-u_0\|}^{\|w_{k-1}-u_0\|+\nu\|u_k-w_{k-1}\|} L(\theta)\|u_k - w_{k-1}\|d\theta d\nu \\
&+ (1-\delta)\kappa'_{k-1}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_{k-1})\|^2 + \delta^2\kappa'_k\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_k)\|^2 \qquad (3.13)
\end{aligned}$$

and

$$\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_{u_k}\| \leq 1 + \int_0^{\|u_k-u_0\|} L_0(\theta)d\theta \leq 1 + \int_0^r L_0(\theta)d\theta. \qquad (3.14)$$

From Algorithm 1.1, we get

$$\begin{aligned}
\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_{u_k}(\nu_k - u_k)\| &= \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_k) - \mathcal{F}'^{-1}_{u_0}r'_k\| \\
&\geq \|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_k)\| - \|\mathcal{F}'^{-1}_{u_0}r'_k\| \\
&\geq (1-\sqrt{\kappa})\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_k)\|. \qquad (3.15)
\end{aligned}$$

From (3.14) and (3.15), we have

$$\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_k)\| \leq \frac{\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_{u_k}\|\,\|\nu_k - u_k\|}{1-\sqrt{\kappa}} \leq \frac{1+\int_0^r L_0(\theta)d\theta}{1-\sqrt{\kappa}}\|\nu_k - u_k\|. \qquad (3.16)$$

From Lemma 2.2, (3.10), (3.12), (3.13) and (3.16), one can easily observe that

$$\begin{aligned}
(1+\sqrt{\kappa})\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_k)\| &\leq (1+\sqrt{\kappa})\Bigg[\int_0^1\int_{\|u_k-u_0\|}^{\|u_k-u_0\|+\nu\|w_k-u_k\|} L(\theta)\|w_k - u_k\|d\theta d\nu \\
&+ (1-\delta)\int_0^1\int_{\|w_{k-1}-u_0\|}^{\|w_{k-1}-u_0\|+\nu\|u_k-w_{k-1}\|} L(\theta)\|u_k - w_{k-1}\|d\theta d\nu \\
&+ (1-\delta)\kappa\frac{\left(1+\int_0^r L_0(\theta)d\theta\right)^2\|u_k - w_{k-1}\|^2}{(1-\sqrt{\kappa})^2} + \delta^2\kappa\frac{\left(1+\int_0^r L_0(\theta)d\theta\right)^2\|\nu_k - u_k\|^2}{(1-\sqrt{\kappa})^2}\Bigg] \\
&= a\int_0^1\int_{\|u_k-u_0\|}^{\|u_k-u_0\|+\nu\|w_k-u_k\|} L(\theta)(\omega_k - \nu_k)d\theta d\nu\frac{\|w_k - u_k\|}{\omega_k - \nu_k} \\
&+ (1-\delta)a\int_0^1\int_{\|w_{k-1}-u_0\|}^{\|w_{k-1}-u_0\|+\nu\|u_k-w_{k-1}\|} L(\theta)\|u_k - w_{k-1}\|d\theta d\nu
\end{aligned}$$

$$
\begin{aligned}
&+ \quad (1-\delta)\kappa a\frac{\left(1+\int_0^r L_0(\theta)d\theta\right)^2\|u_k-w_{k-1}\|^2}{(1-\sqrt{\kappa})^2}+\kappa a\frac{\left(1+\int_0^r L_0(\theta)d\theta\right)^2\|w_k-u_k\|^2}{(1-\sqrt{\kappa})^2}\\
&\leq \quad a\int_0^1\int_{\|u_k-u_0\|}^{\|u_k-u_0\|+\nu\|w_k-u_k\|} L(\theta)(\omega_k-\nu_k)d\theta d\nu\frac{1}{\omega_k-\nu_k}\frac{\omega_k-\nu_k}{\nu_k-\omega_{k-1}}\|u_k-w_{k-1}\|\\
&+ \quad (1-\delta)a\int_0^1\int_{\|w_{k-1}-u_0\|}^{\|w_{k-1}-u_0\|+\nu\|u_k-w_{k-1}\|} L(\theta)\|u_k-w_{k-1}\|d\theta d\nu\\
&+ \quad (1-\delta)\kappa a\frac{\left(1+\int_0^r L_0(\theta)d\theta\right)^2\|u_k-w_{k-1}\|^2}{(1-\sqrt{\kappa})^2}+\kappa a\frac{\left(1+\int_0^r L_0(\theta)d\theta\right)^2(\omega_k-\nu_k)^2}{(1-\sqrt{\kappa})^2}\frac{\|u_k-w_{k-1}\|}{\nu_k-\omega_{k-1}}\\
&\leq \quad \left[a\int_0^1\int_{\nu_k-\nu_0}^{\nu_k-\nu_0+\nu(\omega_k-\nu_k)} L(\theta)(\omega_k-\nu_k)d\theta d\nu\frac{\|u_k-w_{k-1}\|}{\nu_k-\omega_{k-1}}\right.\\
&+ \quad (1-\delta)a\int_0^1\int_{\omega_{k-1}-\nu_0}^{\omega_{k-1}-\nu_0+\nu(\nu_k-\omega_{k-1})} L(\theta)(\nu_k-\omega_{k-1})d\theta d\nu\frac{\|u_k-w_{k-1}\|}{\nu_k-\omega_{k-1}}\\
&+ \quad (1-\delta)\kappa a\frac{\left(1+\int_0^r L_0(\theta)d\theta\right)^2(\nu_k-\omega_{k-1})^2}{(1-\sqrt{\kappa})^2}\frac{\|u_k-w_{k-1}\|}{\nu_k-\omega_{k-1}}\\
&+ \quad \left.\kappa a\frac{\left(1+\int_0^r L_0(\theta)d\theta\right)^2(\omega_k-\nu_k)^2}{(1-\sqrt{\kappa})^2}\frac{\|u_k-w_{k-1}\|}{\nu_k-\omega_{k-1}}\right]\\
&\leq \quad \left[a\int_0^1\int_{\nu_k-\nu_0}^{\nu_k-\nu_0+\nu(\omega_k-\nu_k)} L(\theta)(\omega_k-\nu_k)d\theta d\nu\right.\\
&+ \quad (1-\delta)a\int_0^1\int_{\omega_{k-1}-\nu_0}^{\omega_{k-1}-\nu_0+\nu(\nu_k-\omega_{k-1})} L(\theta)(\nu_k-\omega_{k-1})d\theta d\nu\\
&+ \quad (1-\delta)\kappa a\frac{\left(1+\int_0^r L_0(\theta)d\theta\right)^2(\nu_k-\omega_{k-1})^2}{(1-\sqrt{\kappa})^2}\\
&+ \quad \left.\kappa a\frac{\left(1+\int_0^r L_0(\theta)d\theta\right)^2(\omega_k-\nu_k)^2}{(1-\sqrt{\kappa})^2}\right]\frac{\|u_k-w_{k-1}\|}{\nu_k-\omega_{k-1}}\\
&\leq \quad \left[a\int_0^{\omega_k-\nu_k}(\omega_k-\nu_k-v)L(\nu_k+v)dv+a(1-\delta)\int_0^{\nu_k-\omega_{k-1}}(\nu_k-\omega_{k-1}-v)L(\omega_{k-1}+v)dv\right.\\
&+ \quad \left.(1-\delta)\sigma(\nu_k-\omega_{k-1})^2+\delta^2\sigma(u_k-\nu_k)^2\right]\frac{\|u_k-w_{k-1}\|}{\nu_k-\omega_{k-1}}\\
&\leq \quad \left[a\int_0^{\omega_k-\nu_k}(\omega_k-\nu_k-v)L(\nu_k+v)dv+(1-\delta)f(\nu_k)+\delta^2\sigma(u_k-\nu_k)^2\right]\frac{\|u_k-w_{k-1}\|}{\nu_k-\omega_{k-1}}\\
&\leq \quad f(\omega_k)\frac{\|u_k-w_{k-1}\|}{\nu_k-\omega_{k-1}}. \qquad (3.17)
\end{aligned}
$$

Since f is monotonically decreasing on $[0, s]$. Therefore,

$$f(\omega_k) \leq f(v_0) \text{ and } \sqrt{\kappa}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_k)\| \leq \frac{\sqrt{\kappa}\tau}{1+\sqrt{\kappa}} = \beta\sqrt{\kappa} \leq \sqrt{\kappa}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_0)\| \leq 1.$$

Using Lemma 2.2 and (3.17), we get

$$\begin{aligned} \|u_{k+1} - w_k\| &\leq (1+\sqrt{\kappa})\|(\mathcal{F}'^{-1}_{u_0}\mathcal{F}'_{w_k})^{-1}\|\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(w_k)\| \\ &\leq -\frac{f(\omega_k)}{f'(\omega_k)}\frac{\|u_k - w_{k-1}\|}{v_k - \omega_{k-1}} = \frac{v_{k+1} - \omega_k}{v_k - \omega_{k-1}}\|u_k - w_{k-1}\|. \end{aligned}$$

Thus (3.6) hold for $n = k+1$. Therefore, by the method of mathematical induction (3.6) is true for each $n \in \mathbb{N}_0$. Hence (a)-(c) are hold for each $n \in \mathbb{N}$. This completes the proof of Theorem. ◻

Theorem 3.3. *Assume that all the conditions of Lemma 3.2 hold and let $\tau \leq \min\{\frac{1}{\sqrt{\kappa}}, b_\omega\}$ and $B_{v^*}[u_0] \subseteq B_r(u_0)$. Then the sequence $\{u_n\}$ defined by SIP of INM (1.5) is well defined, remains in $B_{v^*}[u_0]$ and converges to the solution $u^* \in B_{v^*}[u_0]$ of operator equation $\mathcal{F}(u) = 0$. Moreover,*

$$\|u_{n+1} - u_n\| \leq v_{n+1} - v_n \ \text{ and } \ \|u^* - u_n\| \leq v^* - v_n \ \text{ for all } n \in \mathbb{N}_0,$$

where the sequence $\{v_n\}$ is real and defined by (2.4).

Proof. The condition $\tau \leq \min\{\frac{1}{\sqrt{\kappa}}, b_\omega\}$ implies that the Lemma 2.1 and Lemma 2.2 are applicable. In Lemma 3.2, we have already proved that the sequence $\{u_n\}$ defined by (1.5) is well defined and

$$\sqrt{\kappa}\|\mathcal{F}'^{-1}_{u_0}\mathcal{F}(u_n)\| \leq 1 \text{ for all } n \in \mathbb{N}_0.$$

Using Lemma 3.2, we have

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|u_{n+1} - w_n\| + \|w_n - u_n\| \\ &\leq \frac{v_{n+1} - \omega_n}{v_n - \omega_{n-1}}\|u_n - w_{n-1}\| + \frac{\omega_n - v_n}{v_n - \omega_{n-1}}\|u_n - w_{n-1}\| \\ &= \left[\frac{v_{n+1} - \omega_n}{v_n - \omega_{n-1}} + \frac{\omega_n - v_n}{v_n - \omega_{n-1}}\right]\|u_n - w_{n-1}\| \\ &= \frac{v_{n+1} - v_n}{v_n - \omega_{n-1}}\|u_n - w_{n-1}\| \leq v_{n+1} - v_n \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Now, for all $m, n \in \mathbb{N}_0$, one can easily prove that

$$\|u_{m+n} - u_n\| \leq \sum_{i=1}^{m+n-1}\|u_{i+1} - u_i\| \leq v_{m+n} - v_n. \tag{3.18}$$

But the real sequence $\{v_n\}$ is convergent and converges to v^*. Therefore, the sequence $\{u_n\}$ is a Cauchy sequence and converges to some point u^*. Taking limit $m \to \infty$ in (3.18), we obtain

$$\|u^* - u_n\| \leq v^* - v_n \text{ for all } n \in \mathbb{N}_0. \tag{3.19}$$

Taking $n = 0$ in both (3.18) and (3.19), we have

$$\|u_m - u_0\| \leq v_m \leq v^* \text{ and } \|u^* - u_0\| \leq v^*.$$

This shows that real sequence $\{v_n\}$ is a majorizing sequence of sequence $\{u_n\}$. In view of Lemma 2.4, we conclude that $\{u_n\}$ converges to $u^* \in B_{v^*}[u_0]$. □

In the special case when we take $\kappa = 0$, then $\tau = \beta, a = 1$ and $\sigma = 0$ and Theorem 3.3 reduces to the following related result of SIP of Newton-type method:

Corollary 3.4. *Let X and Y be two Banach spaces and D a nonempty open convex subset of X. Let $\mathcal{F} : D \to Y$ be a NFDO such that $\mathcal{F}'^{-1}_{u_0}$ exists for some $u_0 \in D$. Assume that the assumptions (C_1), (C_2), (C_3) and (1.8) hold. If $\beta \leq \omega - \int_0^\omega L(\theta)(\omega - \theta)d\theta$ and ω satisfying $\int_0^\omega L(\theta)d\theta \leq 1$ and $B_{v^*}[u_0] \subseteq B_\omega(u_0)$, where v^* is the solution of equation $\int_0^v L(\theta)(v - \theta)d\theta - v + \beta = 0$. Then the sequence $\{u_n\}$ defined by SIP of Newton-type method:*

$$\begin{cases} \mathcal{F}'_{w_n}(u_{n+1} - w_n) = -\mathcal{F}(w_n), \\ w_n = (1-\delta)u_n + \delta v_n, \\ \mathcal{F}'_{u_n}(v_n - u_n) = -\mathcal{F}(u_n), \end{cases} \tag{3.20}$$

for all $n \in \mathbb{N}_0$, where $\delta \in (0, 1)$ is well defined, remains in $B_{v^*}[u_0]$ and converges to the solution of operator equation $\mathcal{F}(u) = 0$. Moreover,

$$\|u_{n+1} - u_n\| \leq v_{n+1} - v_n \text{ and } \|u^* - u_n\| \leq v^* - v_n$$

for each $n \in \mathbb{N}_0$, where $\{v_n\}$ is the real sequence defined by (2.4).

Proof. The proof of above corollary follows from proof of the Theorem 3.3. □

If we take L and L_0 are positive constant such that $L_0 \leq L$ and $\kappa = 0$, then $a = 1, \tau = \beta, \sigma = 0$, $f(v) = \frac{Lv^2}{2} - v + \beta$ and $g(v) = \frac{L_0 v^2}{2} - v + \beta$ and Theorem 3.3 reduces to the following corollary:

Corollary 3.5. *Let X and Y be two Banach spaces and D a nonempty open convex subset of X. Let $\mathcal{F} : D \to Y$ be a NFDO such that $\mathcal{F}'^{-1}_{u_0}$ exists for some $u_0 \in D$. Suppose that the assumption (C_3) holds and $\mathcal{F}$ satisfies the following conditions:*

(C_6) *$\|\mathcal{F}'^{-1}_{u_0}(\mathcal{F}'_u - \mathcal{F}'_{u_0})\| \leq L_0\|u - u_0\|$ for all $u \in D$;*

(C_7) *$\|\mathcal{F}'^{-1}_{u_0}(\mathcal{F}'_u - \mathcal{F}'_v)\| \leq L\|u - v\|$ for all $u, v \in D$;*

(C_8) *$h = \beta L \leq \frac{1}{2}$ and $v^* = \frac{1-\sqrt{1-2h}}{h}\beta$.*

Then, the sequence $\{u_n\}$ defined by (3.20) is well defined, remains in $B_{v^*}[u_0]$ and converges to the solution of operator equation $\mathcal{F}(u) = 0$. Moreover,

$$\|u_{n+1} - u_n\| \leq v_{n+1} - v_n \text{ and } \|u^* - u_n\| \leq v^* - v_n$$

for each $n \in \mathbb{N}_0$, where $\{v_n\}$ is the real sequence defined by (2.4).

Proof. The proof of above corollary follows from proof of the Theorem 3.3. □

4. Semilocal convergence of Algorithm 1.1 under γ-LC

In this section, we discuss the convergence analysis of our Algorithm 1.1 under γ-LC. Let γ, τ and a be positive real numbers. The function $f_\gamma : [0, \frac{1}{\gamma}) \to \mathbb{R}$ defined by

$$f_\gamma(\nu) = \tau - \nu + \frac{a\gamma\nu^2}{1-\gamma\nu} \tag{4.1}$$

was first introduced by Wang [4, 5] for $a = 1$ and it was also used by Shen and Li [24, 25], Argyros et al. [37] and V. K. Singh [26, 27] as the majorizing function for the convergence analysis of their respective algorithms.

Define two functions $L : [0, \frac{1}{\gamma}) \to \mathbb{R}$ and $L_0 : [0, \frac{1}{\gamma_0}) \to \mathbb{R}$ by

$$L(\nu) = \frac{2\gamma}{(1-\gamma\nu)^3} \tag{4.2}$$

$$L_0(\nu) = \frac{2\gamma_0}{(1-\gamma_0\nu)^3}, \tag{4.3}$$

where γ and γ_0 are two positive real numbers such that $\gamma_0 \leq \gamma$. One can easily observe that $L_0(\nu) \leq L(\nu)$ for each $\nu \in [0, \frac{1}{\gamma})$.

Taking $\sigma = 0$ and above $L_0(\nu)$ and $L(\nu)$ in Lemma 2.1 and Lemma 2.2, we obtain the following result:

Lemma 4.1. *Let γ, γ_0, τ and a be the positive real numbers and f_γ be a real valued function defined by (4.1). Let $g_{\gamma_0} : [0, \frac{1}{\gamma_0}) \to \mathbb{R}$ be a real valued function defined by*

$$g_{\gamma_0}(\nu) = \tau - \nu + \frac{a\gamma_0\nu^2}{1-\gamma_0\nu}. \tag{4.4}$$

If

$$\tau\gamma \leq 1 + 2a - 2\sqrt{a(1+a)}. \tag{4.5}$$

Then, we have the following assertions:

(a) *Equation $f'_\gamma(\nu) = 0$ has only one solution $r^* = \left(1 - \sqrt{\frac{a}{1+a}}\right)\frac{1}{\gamma}$ in $[0, \frac{1}{\gamma})$*

(b) *Equation $f_\gamma(\nu) = 0$ has two real roots*

$$\nu^* = \frac{1+\tau\gamma - \sqrt{(1+\tau\gamma)^2 - 4\tau\gamma(1+a)}}{2(1+a)\gamma} \quad \textit{and}$$

$$\nu^{**} = \frac{1+\tau\gamma + \sqrt{(1+\tau\gamma)^2 - 4\tau\gamma(1+a)}}{2(1+a)\gamma}$$

with $\tau \leq \nu^* \leq r^* \leq \nu^{**}$,

(c) *Sequence* $\{\nu_n\}$ *defined by*

$$\begin{cases} \nu_0 = 0, \\ \mu_n = \nu_n - \frac{f_\gamma(\nu_n)}{g'_{\gamma_0}(\nu_n)}, \\ \omega_n = (1-\delta)\nu_n + \delta\mu_n, \\ \nu_{n+1} = \omega_n - \frac{f_\gamma(\omega_n)}{g'_{\gamma_0}(\omega_n)} \text{ for all } n \in \mathbb{N}_0, \end{cases} \tag{4.6}$$

where $\delta \in (0,1)$ *is non-decreasing and converges to* ν^*.

Lemma 4.2. *Let* $\{\nu_n\}$ *and* $\{\omega_n\}$ *be the real sequences defined by (4.6) and condition (4.5) hold. Then, we have the following assertions:*

$$\frac{\nu_{n+1}-\nu_n}{\nu_n-\omega_{n-1}} \le \alpha^{2^{n-1}}, \quad \frac{\nu_{n+1}-\omega_n}{\nu_n-\omega_{n-1}} \le \alpha^{2^{n-1}} \quad \text{and} \quad \frac{\nu^*-\nu_n}{\nu^*-\omega_{n-1}} \le \alpha^{2^{n-1}},$$

where ν^* *is the solution of* $f_\tau(\nu) = 0$ *and*

$$\alpha = \frac{1-\tau\gamma-\sqrt{(1+\tau\gamma)^2-4\tau\gamma(1+a)}}{1-\tau\gamma+\sqrt{(1+\tau\gamma)^2-4\tau\gamma(1+a)}}.$$

Proof. The proof of Lemma 4.2 follows from [25, Lemma 2.3] □

Definition 4.3. [27] Let X and Y be two Banach spaces, D a nonempty open convex subset of X with $u_0 \in D$ and let $\mathcal{F} : D \to Y$ be a NFDO such that $F'^{-1}_{u_0}$ exists. Let $\gamma > 0$ and $0 < r \le \frac{1}{\gamma}$ be such that $B_r(u_0) \subset D$ and L is defined by (4.2). The operator $\mathcal{F}$ is said to satisfy the γ-LC if

$$\|\mathcal{F}'^{-1}_{u_0}(\mathcal{F}'_u - \mathcal{F}'_v)\| \le \int_{\rho(u)}^{\rho(u,v)} L(\theta)d\theta \text{ for each } u \in B_r(u_0) \text{ and } v \in B_{r-\rho(u)}[u_0], \tag{4.7}$$

where $\rho(u) = \|u-u_0\|$ and $\rho(u,v) = \rho(u) + \|v-u\| \le r$.

Lemma 4.4. *Let X and Y be two Banach spaces, D a nonempty open convex subset of X with $u_0 \in D$ and let $\mathcal{F} : D \to Y$ be a NFDO such that $\mathcal{F}'^{-1}_{u_0}$ exists. Let γ_0 be a positive real number and $r = \left(1 - \frac{a}{\sqrt{1+a}}\right)\frac{1}{\gamma_0}$ be such that $B_r(u_0) \subset D$. Assume that L_0 same as in (4.3) and condition (C_1) holds. Then for all $x \in B_r(u_0)$, we have*

$$\|\mathcal{F}'^{-1}_x \mathcal{F}'_{u_0}\| \le \left(2 - \frac{1}{(1-\gamma_0\|x-u_0\|)^2}\right)^{-1} \le -\frac{1}{g'_{\tau_0}(\|x-u_0\|)},$$

where g_{γ_0} *is defined by (4.4).*

Proof. The proof of Lemma 4.4 follows from Lemma 2.6. □

Lemma 4.5. *Let X and Y be two Banach spaces, D a nonempty open convex subset of X with $u_0 \in D$ and let $\mathcal{F} : D \to Y$ be a NFDO such that $\mathcal{F}'^{-1}_{u_0}$ exists. Assume that $\mathcal{F}$ satisfies (1.6), (1.7) and γ-LC (4.7). We also assume that all the assumptions of Lemma 4.4 and Lemma 4.5 hold. Suppose that the assumptions (C_3) and (C_4) hold* *on* $B_r(u_0) \subset D$, $0 < r < \frac{1}{\gamma}$. *Let* $\tau = (1 + \sqrt{\kappa})\beta$ *and* $a = 1 + \sqrt{\kappa} + \frac{\sqrt{2}\kappa(1+\sqrt{\kappa})}{\gamma(1-\sqrt{\kappa})^2}$. *Then the sequence* $\{u_n\}$ *defined by Algorithm 1.1 is well defined. Moreover, the following assertions hold:*

(a) $\|\mathcal{F}_{u_n}^{\prime -1}\mathcal{F}_{u_0}^{\prime}\| \leq -\frac{1}{g_{\gamma_0}^{\prime}(v_n)}$;

(b) $(1+\sqrt{\kappa})\|\mathcal{F}_{u_0}^{\prime -1}\mathcal{F}(u_n)\| \leq f_\gamma(v_n)\frac{\|u_n - w_{n-1}\|}{v_n - \omega_{n-1}}$, $(1+\sqrt{\kappa})\|\mathcal{F}_{u_0}^{\prime -1}\mathcal{F}(w_n)\| \leq f_\gamma(\omega_n)\frac{\|u_n - w_{n-1}\|}{v_n - \omega_{n-1}}$;

(c) $\sqrt{\kappa}\|\mathcal{F}_{u_0}^{\prime -1}\mathcal{F}(u_n)\| \leq 1$;

(d) $\|w_n - u_n\| \leq \frac{\omega_n - v_n}{v_n - \omega_{n-1}}\|u_n - w_{n-1}\|$, $\|u_{n+1} - u_n\| \leq \frac{v_{n+1} - v_n}{v_n - \omega_{n-1}}\|u_n - w_{n-1}\|$,

where $\{v_n\}$ *and* $\{\omega_n\}$ *are real sequences defined by (4.6).*

Theorem 4.6. *Assume that all the conditions of Lemma 4.5 hold and*

$$\beta \leq \min\left\{\frac{1}{\sqrt{\kappa}}, \frac{1 + 2a - 2\sqrt{a(1+a)}}{\gamma(1+\sqrt{\kappa})}\right\}.$$

Then the sequence $\{u_n\}$ *defined by Algorithm 1.1 is well defined, remains in* $B_{v^*}[u_0]$ *and converges to the solution* $u^* \in B_{v^*}[u_0]$ *of* $\mathcal{F}(x) = 0$. *Moreover, the following error estimates hold:*

(a) $\|u_{n+1} - u_n\| \leq \alpha^{2^{n-1}}\|u_n - w_{n-1}\|$;

(b) $\|u_{n+1} - w_n\| \leq \alpha^{2^{n-1}}\|u_n - w_{n-1}\|$;

(c) $\|u^* - u_n\| \leq \frac{\alpha^{2^{n-1}}}{1-\alpha^{2^n}}\|u_1 - w_0\|$.

Proof. The proof of the Theorem 4.6 follows from the proof of Theorem 3.3 and Lemma 4.2. □

Corollary 4.7. *Let* X *and* Y *be two Banach spaces,* D *a nonempty open convex subset of* X *with* $u_0 \in D$ *and let* $\mathcal{F} : D \to Y$ *be a NFDO such that* $\mathcal{F}_{u_0}^{\prime -1}$ *exists. Assume that* $\mathcal{F}$ *satisfies the* γ*-LC (4.7). We also suppose that* $a = 1$, $\tau = \beta$ *and all the assumptions of Lemma 4.2 and 4.4 hold. Then the sequence* $\{u_n\}$ *defined by (3.20) is well defined remains in* $B_{v^*}[u_0]$ *and converges to the solution* $u^* \in B_{v^*}[u_0]$ *of* $\mathcal{F}(x) = 0$. *Moreover the following assertions hold:*

(a) $\|\mathcal{F}_{u_0}^{\prime -1}\mathcal{F}u_n\| \leq f_\gamma(v_n)\frac{\|u_n - w_{n-1}\|}{v_n - \omega_{n-1}}$, $\|\mathcal{F}_{u_0}^{\prime -1}\mathcal{F}w_n\| \leq f_\gamma(\omega_n)\frac{\|u_n - w_{n-1}\|}{v_n - \omega_{n-1}}$;

(b) $\|u_{n+1} - u_n\| \leq \alpha^{2^{n-1}}\|u_n - w_{n-1}\|$;

(c) $\|u_{n+1} - w_n\| \leq \alpha^{2^{n-1}}\|u_n - w_{n-1}\|$;

(d) $\|u^* - u_n\| \leq \frac{\alpha^{2^{n-1}}}{1-\alpha^{2^n}}\|u_1 - w_0\|$.

Proof. The proof of this Corollary follows from proof of the Theorem 4.6. □

Remark 4.8. *The result [36,* Corollary 4.1*] deals with the convergence analysis of SIP of Newton-like for approximate solution of* m $(m \geq 3)$ *times continuously NFDO equation. Our result deals with the approximate solution of only one time NFDO equation in the context of SIP of inexact Newton-like method.*

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