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NEW PERSPECTIVES ON CARTESIAN AND RESTRICTED CARTESIAN PRODUCTS IN THE CONTEXT OF SOFT DISEMIGRAPHS

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Abstract

Soft set theory provides a systematic approach for handling imprecision and uncertainty by categorizing elements of a set based on specific parameters. In semigraph theory, soft semigraphs utilize this approach, offering a parameterized perspective that has significantly advanced the field through effective parameter management. Building on this foundation, disemigraphs extend semigraphs by incorporating directional relationships among vertices, making them ideal for modelling scenarios where the sequence and direction of connections are crucial. In this paper, we introduce and explore the Cartesian product and restricted Cartesian product of soft disemigraphs. We prove that these products of soft disemigraphs are also soft disemigraphs.

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1. Introduction

Most traditional methods of formal modeling, reasoning, and computation are known for their clarity, determinism, and precision. However, many complex problems in various fields require data that may not be inherently precise. Due to the various types of uncertainty present in these problem areas, conventional methods may not always be applicable. One possible reason for this is the insufficiency of parameterisation tools. Fuzzy set theory deals with a specific type of uncertainty called 'fuzziness,' which arises from the partial membership of elements in a set. While fuzzy set theory effectively manages uncertainties related to vague or partially belonging elements, it does not capture all forms of uncertainties present in diverse real-world problems. As a result, the search for new theories has continued. In 1999, Molodtsov[33] introduced the concept of soft set theory, a mathematical tool designed to address uncertainty. Soft set theory provides a more practical and versatile approach, offering better parameterisation tools than fuzzy set theory. Maji et al.[31, 32], Ali et al.[8], and Saleh et al.[35, 36] further explored soft sets in their studies.

The concept of soft graphs was initially introduced by Thumbakara and George [30, 42–44], with subsequent modifications by Akram and Nawas in 2015 [1, 2].

Further advancements by Akram and Nawas include the introduction and exploration of various types of fuzzy soft graphs and trees[3, 4, 6, 7], along with their properties and applications. Additionally, Akram and Shahzadi proposed a decision-making approach using Pythagorean Dombi fuzzy soft graphs [5]. Contributions to the study of soft graphs have been made by Thenge, Jain, and Reddy [39–41], highlighting their growing significance within graph theory. George, Thumbakara, and Jose extended the concept of soft graphs to include soft hypergraphs[9, 16], soft directed graphs[23, 25], soft directed hypergraphs[21], and soft disemigraphs[22], studying their properties and product operations [10–12, 24, 26–28].

E. Sampathkumar[37, 38] introduced semigraphs to broaden traditional graph theory by incorporating more complex relationships than those depicted by simple binary connections. Semigraphs extend the classic graph model, preserving their visual and structural characteristics when drawn on a plane. This extension enables many graph-related concepts and results to be applied in a more general context, overcoming the limitations of binary graph structures. A hypergraph is a graph generalisation in which any subset of the vertex set can serve as an edge. Semigraphs differ from hypergraphs in that the vertices within each edge have a specific order.

Expanding on semigraph theory, Sampathkumar developed the concept of disemigraphs or directed semigraphs. Disemigraphs incorporate directional relationships among vertices, which is crucial for scenarios where the order and direction of connections are important. This framework allows for a detailed representation of relationships, capturing both the presence and direction of connections. Disemigraphs share some properties with undirected semigraphs, such as planarity, fundamental circuits, and cut-sets, while possessing unique features like in-degree, out-degree, and strong connectedness specific to directed structures. These characteristics make disemigraphs particularly valuable in applications such as network analysis, scheduling, and group comparisons, where the directionality and order of connections play a significant role. The appeal and flexibility of disemigraphs lie in their ability to model complex relational structures that surpass the capabilities of traditional graphs and hypergraphs. Sampathkumar's pioneering contributions have spurred extensive research, with numerous graph-theoretic concepts yet to be explored within semigraphs and disemigraphs. Soft semigraphs, introduced by George, Thumbakara, and Jose in 2022, apply soft set principles to semigraphs, with defined operations and associated degrees, graphs, and matrices [13–15, 17–20]. Also, they[22] introduced soft disemigraph as a fusion of disemigraph and soft set. In this paper, we introduce some operations on soft disemigraphs and investigate some of their properties.

2. Preliminaries

In this preliminary section, we lay the foundation for comprehending soft sets, disemigraphs, and soft disemigraphs.

2.1. Disemigraph The concepts of directed semigraph (or disemigraph) were contributed to semigraph theory by Sampathkumar and Pushpalatha. They [24],[25] defined a disemigraph as follows. "A *directed semigraph* (or *disemigraph*) Λ is a pair (V, A) where V is a nonempty set whose elements are called *vertices* of Λ, and A is a set of ordered n-tuples of distinct vertices in V for various $n \geq 2$, called *arcs* of Λ. The arcs directed in the sense that if $\vec{a} = (w_1, w_2, \ldots, w_n)$ is an arc, then $(w_n, w_{n-1}, \ldots, w_1)$ need not be an arc. Further, if both $(w_1, w_2, \ldots, w_n)$ and $(w_n, w_{n-1}, \ldots, w_1)$ are arcs, they are different arcs, and in this case these two arcs together form a *symmetric arc*. The arcs in a disemigraph Λ satisfy the following condition: If $\vec{a_1}$ and $\vec{a_2}$ are arcs in Λ, then they have at most one vertex in common or they form a symmetric arc. If $\vec{a} = (w_1, w_2, \ldots, w_n)$ is an arc, then w_1, w_n are *end vertices* of $\vec{a}$, w_1 is the *tail* or *initial vertex* of $\vec{a}$, w_n is the *head* or *terminal vertex* of $\vec{a}$ and $w_i, 2 \leq i \leq n-1$ are *middle vertices* of $\vec{a}$. We use small circles to represent middle vertices and thick dots to represent end vertices while representing a disemigraph Λ in the plane. Consider an arc $\vec{a} = (w_1, w_2, \ldots, w_n)$. We say that for $1 \leq i < j \leq n$, w_i is *adjacent to* w_j and w_j is *adjacent from* w_i. Thus, each w_i is adjacent to each w_j, $1 \leq i < j \leq n$ and each w_j is adjacent from each w_i, $1 \leq i < j \leq n$. The *out-degree* of a vertex w of a disemigraph Λ is the number of vertices of Λ that are adjacent from w and is denoted as $od(v)$. The *in-degree* $id(v)$ of w is the number of vertices of Λ that are adjacent to w. The *degree* of w is defined as the sum of the in-degree and out-degree of w and is denoted by $deg(v)$. Let w be a vertex in a disemigraph Λ. A *ca-vertex* of w is a vertex that is consecutively adjacent to w. The *consecutive adjacency in-degree* of w, denoted by $ca\ id(v)$ is the number of ca-vertices adjacent to w. Similarly, the *consecutive adjacency out-degree* of w, denoted by $ca\ od(v)$ is the number of ca-vertices adjacent from w. Let $\vec{a} = (z_1, z_2, \ldots, z_n)$ be an arc in a disemigraph Λ. A *subarc* of $\vec{a}$ is an r-tuple $(z_{i_1}, z_{i_2}, \ldots, z_{i_r})$ where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$. A *partial arc* of $\vec{a}$ is a $(j-i+1)$-tuple $(z_i, z_{i+1}, \ldots, z_j)$ where $1 \leq i < j \leq n$. Every arc is a subarc (partial arc) of itself, and a proper subarc (partial arc) is not an arc of Λ. Also, a partial arc is a subarc, but not conversely. $\Lambda' = (V', A')$ is a *partial disemigraph* of a disemigraph Λ if $V' \subseteq V$ and the arcs of Λ' are partial arcs of some arcs in Λ. ".

2.2. Soft Sets In 1999, Molodtsov[33] introduced the notion of soft sets, defined as follows: "Let U be an initial universe set and let F be a set of parameters. A pair (R, F) is called a soft set (over U) if and only if R is a mapping of F into the set of all subsets of the set U. That is, $R : F \to \mathcal{P}(U)$. Soft set theory provides a parameterized point of view for uncertainty modelling and soft computing. By means of parameterization, a soft set produces a series of approximate descriptions of a complicated object being perceived from various points of view."

2.3. Soft Disemigraph George, Jose and Thumbakara [12] introduced soft disemigraph as follows. "Let V be the vertex set of a disemigraph Λ^*. Consider a subset V_1 of V. Then a partial arc formed by some or all vertices of V_1 is said to be a *maximum partial arc* or *mp arc* if it is not a partial arc of any other partial arc formed by some or all vertices of V_1. Let $\Lambda^* = (V, A)$ be a disemigraph having vertex set V and arc set A.

Let A_p be the collection of all partial arcs of the disemigraph Λ^* and Θ be a nonempty set. Let a subset $\mathcal{R}$ of $\Theta \times V$ be an arbitrary relation from Θ to V. We define a mapping Φ from Θ to $\mathcal{P}(V)$ by $\Phi(\theta) = \{w \in V \mid w\mathcal{R}\theta\}, \forall\theta \in \Theta$, where $\mathcal{P}(V)$ denotes the power set of V. Then the pair (Φ, Θ) is a soft set over V. Also define a mapping Ω from Θ to $\mathcal{P}(A_p)$ by $\Omega(\theta) = \{mp\ arcs\langle\Phi(\theta)\rangle\}$, where $\{mp\ arcs\langle\Phi(\theta)\rangle\}$ denotes the set of all mp arcs that can be formed by some or all vertices of $\Phi(\theta)$ and $\mathcal{P}(A_p)$ denotes the power set of A_p. The pair (Ω, Θ) is a soft set over A_p. Then we can define a soft disemigraph as follows:

The 4-tuple $\Lambda = (\Lambda^*, \Phi, \Omega, \Theta)$ is called a *soft disemigraph* of Λ^* if the following conditions are satisfied:

1. $\Lambda^* = (V, A)$ is a disemigraph having vertex set V and arc set A,
2. Θ is the nonempty set of parameters,
3. (Φ, Θ) is a soft set over V,
4. (Ω, Θ) is a soft set over A_p,
5. $\Upsilon(\theta) = (\Phi(\theta), \Omega(\theta))$ is a partial disemigraph of $\Lambda^*, \forall\theta \in \Theta$.

Let $\Lambda^* = (V, A)$ be a disemigraph and $\Lambda = (\Lambda^*, \Phi, \Omega, \Theta)$ be a soft disemigraph of Λ^* which is also given by $\{\Upsilon(\theta) : \theta \in \Theta\}$. Then the partial disemigraph $\Upsilon(\theta)$ corresponding to any parameter θ in Θ is called a *d-part* of the soft disemigraph Λ^*. An arc present in a soft disemigraph Λ of Λ^* is called an *f-arc*. It may be a partial arc of some arc in Λ^* or an arc in Λ^*.

3. Cartesian Product of Soft Disemigraphs

In this section, we introduce the concept of the Cartesian product of soft disemigraphs and then provide a concrete example to illustrate this concept. Following the definition and example, we present a theorem related to the Cartesian product of soft disemigraphs. This theorem establishes the foundational properties of the Cartesian product in the context of soft disemigraphs.

Definition 3.1. Let $\Lambda_1^* = (V_1, A_1)$ and $\Lambda_2^* = (V_2, A_2)$ be two disemigraphs and $\Lambda_1 = (\Lambda_1^*, \Phi_1, \Omega_1, \Theta_1) = \{\Upsilon_1(\theta) : \theta \in \Theta_1\}$ and $\Lambda_2 = (\Lambda_2^*, \Phi_2, \Omega_2, \Theta_2) = \{\Upsilon_2(\theta) : \theta \in \Theta_2\}$ be two soft disemigraphs of Λ_1^* and Λ_2^* respectively. Then the *cartesian product* of Λ_1 and Λ_2 denoted by $\Lambda_1 \times \Lambda_2$ is defined as $\Lambda_1 \times \Lambda_2 = \{\Upsilon_1(\theta_1) \times \Upsilon_2(\theta_2) : (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2\}$. Here $\Upsilon_1(\theta_1) \times \Upsilon_2(\theta_2)$ denotes the cartesian product of the d-parts $\Upsilon_1(\theta_1)$ of Λ_1 and $\Upsilon_2(\theta_2)$ of Λ_2 which is defined as follows: $\Upsilon_1(\theta_1) \times \Upsilon_2(\theta_2)$ is a disemigraph with vertex set $V = \Phi_1(\theta_1) \times \Phi_2(\theta_2)$ and arc set A, where the vertex (w_1, w_1') is consecutively adjacent to the vertex (w_2, w_2') if and only if

1. $w_1 = w_2$ and there exists a partial arc or an arc (w_1', w_2') in $\Omega_2(\theta_2)$ or
2. $w_1' = w_2'$ and there exists a partial arc or an arc (w_1, w_2) in $\Omega_1(\theta_1)$

and a vertex (z, w) in $\Upsilon_1(\theta_1) \times \Upsilon_2(\theta_2)$ is a middle vertex if and only if z is a middle vertex in Λ_1 and w is a middle vertex in Λ_2; otherwise the vertex (z, w) is an end vertex in $\Upsilon_1(\theta_1) \times \Upsilon_2(\theta_1)$.

Example 3.2. *Let* $\Lambda_1^* = (V_1, A_1)$ *be a disemigraph given in Figure 1 having vertex set* $V_1 = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ *and the arc set* $A_1 = \{(w_1, w_2, w_3), (w_3, w_4), (w_5, w_1, w_6), (w_6, w_1, w_5)\}$.

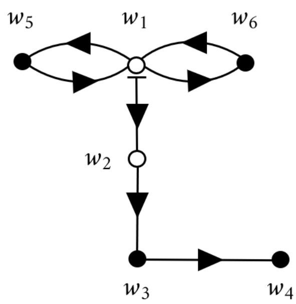

Figure 1. Disemigraph $\Lambda_1^* = (V_1, A_1)$

Let the parameter set be $\Theta_1 = \{w_3, w_4\} \subseteq V_1$. *Define* $\Phi_1 : \Theta_1 \to \mathcal{P}(V_1)$ *by* $\Phi_1(\theta) = \{y \in V_1 | \theta = y \text{ or } \theta \text{ is adjacent to } y \text{ or } \theta \text{ is adjacent from } y\}, \forall \theta \in \Theta_1$. *Also define* $\Omega_1 : \Theta_1 \to \mathcal{P}(A_{1p})$ *by* $\Omega_1(\theta) = \{mp\ arcs\langle \Phi_1(\theta) \rangle\}, \forall \theta \in \Theta_1$. *That is,* $\Phi_1(w_3) = \{w_1, w_2, w_3, w_4\}$ *and* $\Phi_1(w_4) = \{w_3, w_4\}$. *Also* $\Omega_1(w_3) = \{(w_1, w_2, w_3), (w_3, w_4)\}$ *and* $\Omega_1(w_4) = \{(w_3, w_4)\}$. *Then* (Φ_1, Θ_1) *is a soft set over* V_1 *and* (Ω_1, Θ_1) *is a soft set over* A_{1p}. *Here* $\Upsilon_1(w_3) = (\Phi_1(w_3), \Omega_1(w_3))$ *and* $\Upsilon_1(w_4) = (\Phi_1(w_4), \Omega_1(w_4))$ *are partial disemigraphs of* Λ_1^* *as shown in Figure 2. Hence* $\Lambda_1 = \{\Upsilon_1(w_3), \Upsilon_1(w_4)\}$ *is a soft disemigraph of* Λ_1^*.

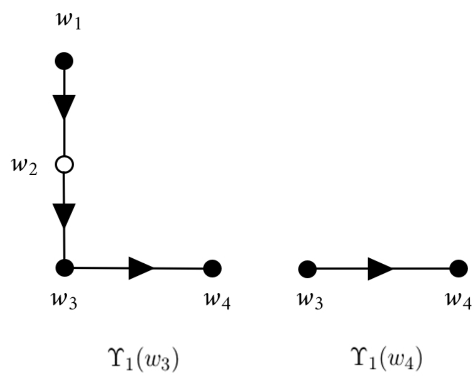

Figure 2. Soft Disemigraph $\Lambda_1 = \{\Upsilon_1(w_3), \Upsilon_1(w_4)\}$

Let $\Lambda_2^* = (V_2, A_2)$ *be a disemigraph given in Figure 3 having vertex set* $V_2 = \{z_1, z_2, z_3, z_4, z_5, z_6\}$ *and the arc set* $A_2 = \{(z_1, z_2, z_3), (z_3, z_2, z_1), (z_3, z_4, z_5, z_6), (z_2, z_5)\}$.

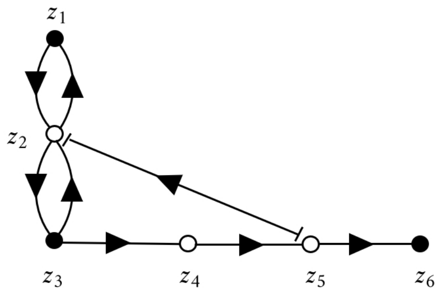

FIGURE 3. Disemigraph $\Lambda_2^* = (V_2, A_2)$

Let $\Theta_2 = \{z_1\} \subseteq V_2$ be a parameter set. Define $\Phi_2 : \Theta_2 \to \mathcal{P}(V_2)$ by $\Phi_2(\theta) = \{y \in V_2 | \theta = y$ or θ is adjacent to y or θ is adjacent from $y\}, \forall \theta \in \Theta_2$ and $\Omega_2 : \Theta_2 \to \mathcal{P}(A_{2p})$ by $\Omega_2(\theta) = \{mp\ arcs\langle\Phi_2(\theta)\rangle\}, \forall\theta \in \Theta_2$. That is, $\Phi_2(z_1) = \{z_1, z_2, z_3\}$ and $\Omega_2(z_1) = \{(z_1, z_2, z_3), (z_3, z_2, z_1)\}$. Then (Φ_2, Θ_2) is a soft set over V_2 and (Ω_2, Θ_2) is a soft set over A_{2p}. Here $\Upsilon_2(z_1) = (\Phi_2(z_1), \Omega_2(z_1))$ is a partial disemigraph of Λ_2^* as shown in Figure 4. Hence $\Lambda_2 = \{\Upsilon_2(z_1)\}$ is a soft disemigraph of Λ_2^*.

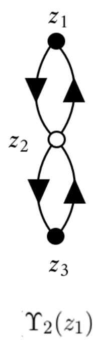

FIGURE 4. Soft Disemigraph $\Lambda_2 = \{\Upsilon_2(z_1)\}$

The cross product of the soft disemigraphs Λ_1 and Λ_2 is given by $\Lambda_1 \times \Lambda_2 = \{\Upsilon_1(w_3) \times \Upsilon_2(z_1), \Upsilon_1(w_4) \times \Upsilon_2(z_1)\}$ and is given in Figure 5.

THEOREM 3.3. *Let $\Lambda_1^* = (V_1, A_1)$ and $\Lambda_2^* = (V_2, A_2)$ be two disemigraphs and Λ_1 and Λ_2 be two soft disemigraphs of Λ_1^* and Λ_2^* respectively. Then the cartesian product of Λ_1 and Λ_2 denoted by $\Lambda_1 \times \Lambda_2$ is a soft disemigraph of $\Lambda_1^* \times \Lambda_2^*$.*

PROOF. Let $\Lambda_1 = (\Lambda_1^*, \Phi_1, \Omega_1, \Theta_1) = \{\Upsilon_1(\theta) : \theta \in \Theta_1\}$ be a soft disemigraph of $\Lambda_1^* = (V_1, A_1)$ and $\Lambda_2 = (\Lambda_2^*, \Phi_2, \Omega_2, \Theta_2) = \{\Upsilon_2(\theta) : \theta \in \Theta_2\}$ be a soft disemigraph

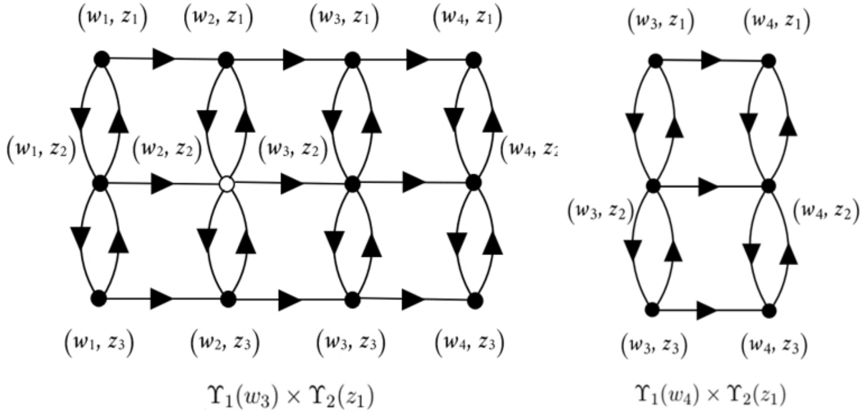

FIGURE 5. $\Lambda_1 \times \Lambda_2 = \{\Upsilon_1(w_3) \times \Upsilon_2(z_1), \Upsilon_1(w_4) \times \Upsilon_2(z_1)\}$

of $\Lambda_2^* = (V_2, A_2)$. Then the cartesian product $\Lambda_1 \times \Lambda_2$ is defined as $\Lambda_1 \times \Lambda_2 = \{\Upsilon_1(\theta_1) \times \Upsilon_2(\theta_2) : (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2\}$. Here $\Upsilon_1(\theta_1) \times \Upsilon_2(\theta_2)$ denotes the cartesian product of the d-parts $\Upsilon_1(\theta_1)$ of Λ_1 and $\Upsilon_2(\theta_2)$ of Λ_2 which is defined as follows: $\Upsilon_1(\theta_1) \times \Upsilon_2(\theta_2)$ is a disemigraph with vertex set $V(\Upsilon_1(\theta_1) \times \Upsilon_2(\theta_2)) = \Phi_1(\theta_1) \times \Phi_2(\theta_2)$ and arc set $A(\Upsilon_1(\theta_1) \times \Upsilon_2(\theta_2))$ where the vertex (w_1, w_1') is consecutively adjacent to the vertex (w_2, w_2') if and only if

1. $w_1 = w_2$ and there exists a partial arc or an arc (w_1', w_2') in $\Omega_2(\theta_2)$ or
2. $w_1' = w_2'$ and there exists a partial arc or an arc (w_1, w_2) in $\Omega_1(\theta_1)$

and a vertex (z, w) in $\Upsilon_1(\theta_1) \times \Upsilon_2(\theta_2)$ is a middle vertex if and only if z is a middle vertex in Λ_1 and w is a middle vertex in Λ_2; otherwise the vertex (z, w) is an end vertex in $\Upsilon_1(\theta_1) \times \Upsilon_2(\theta_1)$.

The cartesian product $\Lambda_1^* \times \Lambda_2^*$ of the two disemigraphs Λ_1^* and Λ_2^* is a disemigraph with vertex set $V(\Lambda_1^* \times \Lambda_2^*) = V_1 \times V_2$ and arc set $A(\Lambda_1^* \times \Lambda_2^*)$ where where the vertex (w_1, w_1') is consecutively adjacent to the vertex (w_2, w_2') if and only if

1. $w_1 = w_2$ and there exists a partial arc or an arc (w_1', w_2') in A_2 or
2. $w_1' = w_2'$ and there exists a partial arc or an arc (w_1, w_2) in A_1

and a vertex (z, w) in $\Lambda_1^* \times \Lambda_2^*$ is a middle vertex if and only if both z and w are middle vertices in Λ_1^* and Λ_2^* respectively; otherwise the vertex is an end vertex in $\Lambda_1^* \times \Lambda_2^*$. Also, the arc between (w_1, w_1') and (w_2, w_2') will be a partial arc if any of these two vertices is a middle vertex. Let the parameter set be $\Theta_{\Lambda_1 \times \Lambda_2} = \Theta_1 \times \Theta_2$. Define a mapping $\Phi_{\Lambda_1 \times \Lambda_2}$ from $\Theta_{\Lambda_1 \times \Lambda_2}$ to $\mathcal{P}[V(\Lambda_1^* \times \Lambda_2^*)]$ by $\Phi_{\Lambda_1 \times \Lambda_2}(\theta_1, \theta_2) = \Phi_1(\theta_1) \times \Phi_2(\theta_2), \forall (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$ where $\mathcal{P}[V(\Lambda_1^* \times \Lambda_2^*)]$ denotes the power set of $V(\Lambda_1^* \times \Lambda_2^*)$. Then $(\Phi_{\Lambda_1 \times \Lambda_2}, \Theta_{\Lambda_1 \times \Lambda_2})$ is a soft set over $V(\Lambda_1^* \times \Lambda_2^*)$. Also define a mapping $\Omega_{\Lambda_1 \times \Lambda_2}$ from $\Theta_{\Lambda_1 \times \Lambda_2}$ to $\mathcal{P}[A(\Lambda_1^* \times \Lambda_2^*)_p]$ by $\Omega_{\Lambda_1 \times \Lambda_2}(\theta_1, \theta_2) = \{\text{mp arcs}\langle \Phi_{\Lambda_1 \times \Lambda_2}(\theta_1, \theta_2)\rangle\} = \{\text{mp arcs}\langle \Phi_1(\theta_1) \times \Phi_2(\theta_2)\rangle\}, \forall (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$, where $A(\Lambda_1^* \times \Lambda_2^*)_p$ denotes the collection of all partial arcs of the disemigraph $\Lambda_1^* \times \Lambda_2^*$ and $\mathcal{P}[A(\Lambda_1^* \times \Lambda_2^*)_p]$ denotes the power set of $A(\Lambda_1^* \times \Lambda_2^*)_p$. Then $(\Omega_{\Lambda_1 \times \Lambda_2}, \Theta_{\Lambda_1 \times \Lambda_2})$ is a soft set over $A(\Lambda_1^* \times \Lambda_2^*)_p$. Also if we denote $(\Phi_{\Lambda_1 \times \Lambda_2}(\theta_1, \theta_2), \Omega_{\Lambda_1 \times \Lambda_2}(\theta_1, \theta_2))$ by $\Upsilon_{\Lambda_1 \times \Lambda_2}(\theta_1, \theta_2)$, then $\Upsilon_{\Lambda_1 \times \Lambda_2}(\theta_1, \theta_2)$ is a partial disemigraph of $\Lambda_1^* \times \Lambda_2^*$, $\forall (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$, since $\Phi_1(\theta_1) \times \Phi_2(\theta_2) \subseteq V_1 \times V_2$ and any arc in $\Omega_{\Lambda_1 \times \Lambda_2}(\theta_1, \theta_2)$ is in $A(\Lambda_1^* \times \Lambda_2^*)$ or it is a partial arc of an arc in $A(\Lambda_1^* \times \Lambda_2^*)$. Then $\Lambda_1 \times \Lambda_2$ can be represented by the 4-tuple $(\Lambda_1^* \times \Lambda_2^*, \Phi_{\Lambda_1 \times \Lambda_2}, \Omega_{\Lambda_1 \times \Lambda_2}, \Theta_{\Lambda_1 \times \Lambda_2})$ and also by $\{\Upsilon_{\Lambda_1 \times \Lambda_2}(\theta_1, \theta_2) : (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2\}$ and $\Lambda_1 \times \Lambda_2$ is a soft disemigraph of $\Lambda_1^* \times \Lambda_2^*$ since all the conditions for soft disemigraphs are satisfied. □

4. Restricted Cartesian Product of Soft Disemigraphs

In this section, we define the restricted Cartesian product of disemigraphs and extend this concept to soft disemigraphs. We start by introducing the formal definition of the restricted Cartesian product for two disemigraphs and then provide an illustrative example to clarify this concept. Following the definition and example, we present and prove a key theorem that establishes the foundational properties of the restricted Cartesian product in the context of soft disemigraphs.

Definition 4.1. Let $\Lambda^* = (V, A)$ be a disemigraph and $\Lambda_1 = (\Lambda^*, \Phi_1, \Omega_1, \Theta_1) = \{\Upsilon_1(\theta) : \theta \in \Theta_1\}$ and $\Lambda_2 = (\Lambda^*, \Phi_2, \Omega_2, \Theta_2) = \{\Upsilon_2(\theta) : \theta \in \Theta_2\}$ be two soft disemigraphs of Λ^* such that $\Theta_1 \cap \Theta_2 \neq \phi$. Then the *restricted cartesian product* of Λ_1 and Λ_2 denoted by $\Lambda_1 \otimes \Lambda_2$ is defined as $\Lambda_1 \otimes \Lambda_2 = \{\Upsilon_1(\theta) \times \Upsilon_2(\theta) : \theta \in \Theta_1 \cap \Theta_2\}$. Here $\Upsilon_1(\theta) \times \Upsilon_2(\theta)$ denotes the cartesian product of the d-parts $\Upsilon_1(\theta)$ of Λ_1 and $\Upsilon_2(\theta)$ of Λ_2 which is defined as follows: $\Upsilon_1(\theta) \times \Upsilon_2(\theta)$ is a disemigraph with vertex set $V = \Phi_1(\theta) \times \Phi_2(\theta)$ and arc set A, where the vertex (w_1, w_1') is consecutively adjacent to the vertex (w_2, w_2') if and only if

1. $w_1 = w_2$ and there exists a partial arc or an arc (w_1', w_2') in $\Omega_2(\theta)$ or
2. $w_1' = w_2'$ and there exists a partial arc or an arc (w_1, w_2) in $\Omega_1(\theta)$

and a vertex (z, w) in $\Upsilon_1(\theta) \times \Upsilon_2(\theta)$ is a middle vertex if and only if z is a middle vertex in Λ_1 and w is a middle vertiex in Λ_2 ; otherwise the vertex (z, w) is an end vertex in $\Upsilon_1(\theta) \times \Upsilon_2(\theta)$.

Example 4.2. *Let $\Lambda^* = (V, A)$ be a disemigraph given in Figure 6 where $V = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11}, w_{12}\}$ and $X = \{(w_1, w_2, w_3), (w_6, w_5, w_1), (w_1, w_7), (w_3, w_8, w_7), (w_6, w_7), (w_7, w_6), (w_4, w_3), (w_7, w_{10}), (w_{10}, w_9, w_4), (w_4, w_{12}, w_{11}), (w_{11}, w_{12}, w_4), (w_{10}, w_{11})\}$.*
Let $\Theta_1 = \{w_4, w_6\} \subseteq V$ be a parameter set. Define $\Phi_1 : \Theta_1 \to \mathcal{P}(V)$ by $\Phi_1(\theta) = \{y \in V | \theta = y$ or θ is adjacent to y or θ is adjacent from $y\}, \forall \theta \in \Theta_1$ and $\Omega_1 : \Theta_1 \to \mathcal{P}(A_p)$ by $\Omega_1(\theta) = \{mp\ arcs\langle \Phi_1(\theta) \rangle\}, \forall \theta \in \Theta_1$. That is, $\Phi_1(w_4) = \{w_3, w_4, w_9, w_{10}, w_{11}, w_{12}\}$ and $\Phi_1(w_6) = \{w_1, w_5, w_6, w_7\}$. Also, $\Omega_1(w_4) = \{(w_{10}, w_9, w_4), (w_4, w_3), (w_4, w_{12}, w_{11}), (w_{11}, w_{12}, w_4), (w_{10}, w_{11})\}$ and $\Omega_1(w_6) = \{(w_6, w_5, w_1), (w_6, w_7), (w_7, w_6), (w_1, w_7)\}$. Then (Φ_1, Θ_1) is a soft set over V and (Ω_1, Θ_1) is a soft set over A_p. Here $\Upsilon_1(w_4) = (\Phi_1(w_4), \Omega_1(w_4))$ and $\Upsilon_1(w_6) = (\Phi_1(w_6), \Omega_1(w_6))$ are partial disemigraphs of Λ^* as shown in Figure 7. Hence $\Lambda_1 = \{\Upsilon_1(w_4), \Upsilon_1(w_6)\}$ is a

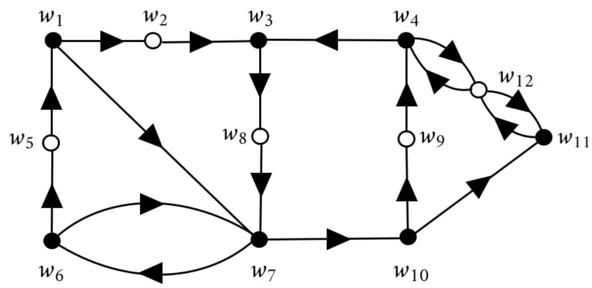

Figure 6. Disemigraph $\Lambda^* = (V, A)$

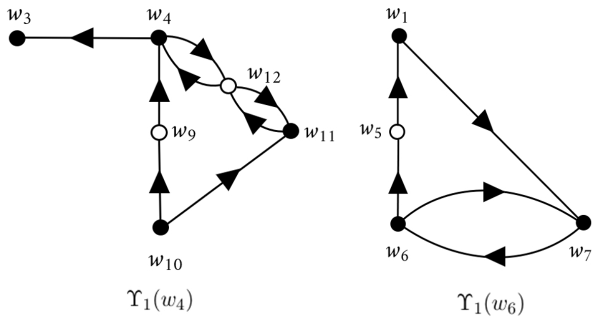

Figure 7. Soft Disemigraph $\Lambda_1 = \{\Upsilon_1(w_4), \Upsilon_1(w_6)\}$

soft disemigraph of Λ^*.
Let $\Theta_2 = \{w_6, w_{10}\} \subseteq V$ *be another parameter set. Define* $\Phi_2 : \Theta_2 \to \mathcal{P}(V)$ *by* $\Phi_2(\theta) = \{y \in V | \theta = y$ *or* θ *is consecutively adjacent to* y *or* θ *is consecutively adjacent from* $y\}, \forall \theta \in \Theta_2$ *and* $\Omega_2 : \Theta_2 \to \mathcal{P}(A_p)$ *by* $\Omega_2(\theta) = \{mp\ arcs\langle\Phi_2(\theta)\rangle\}, \forall \theta \in \Theta_2$. *That is,* $\Phi_2(w_6) = \{w_5, w_6, w_7\}$ *and* $\Phi_2(w_{10}) = \{w_7, w_9, w_{10}, w_{11}\}$. *Also,* $\Omega_2(w_6) = \{(w_6, w_5), (w_6, w_7), (w_7, w_6)\}$ *and* $\Omega_2(w_{10}) = \{(w_7, w_{10}), (w_{10}, w_9), (w_{10}, w_{11})\}$. *Then* (Φ_2, Θ_2) *is a soft set over* V *and* (Ω_2, Θ_2) *is a soft set over* A_p. *Here* $\Upsilon_2(w_6) = (\Phi_2(w_6), \Omega_2(w_6))$ *and* $\Upsilon_2(w_{10}) = (\Phi_2(w_{10}), \Omega_2(w_{10}))$ *are partial disemigraphs of* Λ^* *as shown in Figure 8. Hence* $\Lambda_2 = \{\Upsilon_2(w_6), \Upsilon_2(w_{10})\}$ *is a soft disemigraph of* Λ^*.
The restricted cartesian product of the soft disemigraphs Λ_1 *and* Λ_2 *is given by* $\Lambda_1 \otimes \Lambda_2 = \{\Upsilon_1(w_6) \times \Upsilon_2(w_6)\}$ *and is given in Figure 9.*

Theorem 4.3. *Let* $\Lambda^* = (V, A)$ *be a disemigraph and* $\Lambda_1 = (\Lambda^*, \Phi_1, \Omega_1, \Theta_1) = \{\Upsilon_1(\theta) : \theta \in \Theta_1\}$ *and* $\Lambda_2 = (\Lambda^*, \Phi_2, \Omega_2, \Theta_2) = \{\Upsilon_2(\theta) : \theta \in \Theta_2\}$ *be two soft disemigraphs of* Λ^* *such that* $\Theta_1 \cap \Theta_2 \neq \phi$. *Then the restricted cartesian product of* Λ_1 *and* Λ_2 *denoted by* $\Lambda_1 \otimes \Lambda_2$ *is a soft disemigraph of* $\Lambda^* \times \Lambda^*$.

Proof. Let $\Lambda_1 = (\Lambda^*, \Phi_1, \Omega_1, \Theta_1) = \{\Upsilon_1(\theta) : \theta \in \Theta_1\}$ and $\Lambda_2 = (\Lambda^*, \Phi_2, \Omega_2, \Theta_2) =$

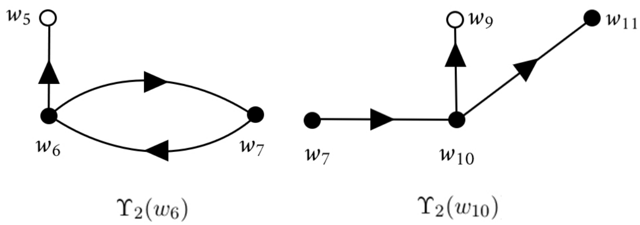

FIGURE 8. Soft Disemigraph $\Lambda_2 = \{\Upsilon_2(w_6), \Upsilon_2(w_{10})\}$

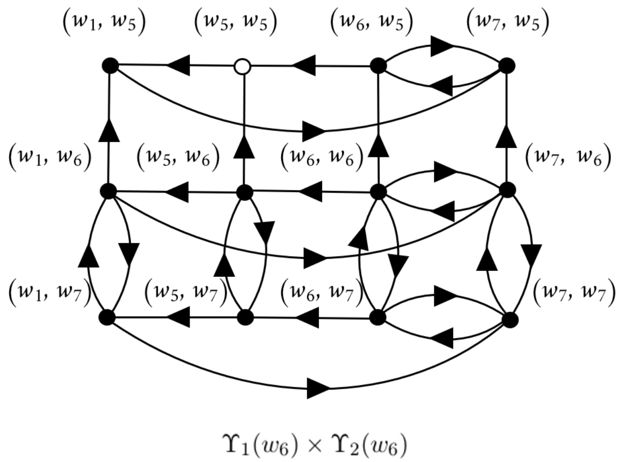

FIGURE 9. $\Lambda_1 \otimes \Lambda_2 = \{\Upsilon_1(w_6) \times \Upsilon_2(w_6)\}$

$\{\Upsilon_2(\theta) : \theta \in \Theta_2\}$ be soft disemigraphs of $\Lambda^* = (V, A)$ such that $\Theta_1 \cap \Theta_2 \neq \phi$. Then the restricted cartesian product $\Lambda_1 \otimes \Lambda_2$ is defined as $\Lambda_1 \otimes \Lambda_2 = \{\Upsilon_1(\theta) \times \Upsilon_2(\theta) : \theta \in \Theta_1 \cap \Theta_2\}$. Here $\Upsilon_1(\theta) \times \Upsilon_2(\theta)$ denotes the cartesian product of the d-parts $\Upsilon_1(\theta)$ of Λ_1 and $\Upsilon_2(\theta)$ of Λ_2 which is defined as follows: $\Upsilon_1(\theta) \times \Upsilon_2(\theta)$ is a disemigraph with vertex set $V = \Phi_1(\theta) \times \Phi_2(\theta)$ and arc set A, where the vertex (w_1, w_1') is consecutively adjacent to the vertex (w_2, w_2') if and only if

1. $w_1 = w_2$ and there exists a partial arc or an arc (w_1', w_2') in $\Omega_2(\theta)$ or
2. $w_1' = w_2'$ and there exists a partial arc or an arc (w_1, w_2) in $\Omega_1(\theta)$

and a vertex (z, w) in $\Upsilon_1(\theta) \times \Upsilon_2(\theta)$ is a middle vertex if and only if z is a middle vertex in Λ_1 and w is a middle vertex in Λ_2; otherwise the vertex (z, w) is an end vertex in $\Upsilon_1(\theta) \times \Upsilon_2(\theta)$.

The cartesian product $\Lambda^* \times \Lambda^*$ is a disemigraph with vertex set $V(\Lambda^* \times \Lambda^*) = V \times V$ and arc set $A(\Lambda^* \times \Lambda^*)$ where the vertex (w_1, w_1') is consecutively adjacent to the vertex

(w_2, w_2') if and only if

1. $w_1 = w_2$ and there exists a partial arc or an arc (w_1', w_2') in X or
2. $w_1' = w_2'$ and there exists a partial arc or an arc (w_1, w_2) in X

and a vertex (z, w) in $\Lambda^* \times \Lambda^*$ is a middle vertex if and only if both z and w are middle vertices in Λ^*; otherwise the vertex is an end vertex in $\Lambda^* \times \Lambda^*$. Also, the arc between (w_1, w_1') and (w_2, w_2') will be a partial arc if any of these two vertices is a middle vertex. Let the parameter set be $\Theta_{\Lambda_1 \otimes \Lambda_2} = \Theta_1 \cap \Theta_2$. Define a mapping $\Phi_{\Lambda_1 \otimes \Lambda_2}$ from $\Theta_{\Lambda_1 \otimes \Lambda_2}$ to $\mathcal{P}[V(\Lambda^* \times \Lambda^*)]$ by $\Phi_{\Lambda_1 \otimes \Lambda_2}(\theta) = \Phi_1(\theta) \times \Phi_2(\theta), \forall \theta \in \Theta_1 \cap \Theta_2$ where $\mathcal{P}[V(\Lambda^* \times \Lambda^*)]$ denotes the power set of $V(\Lambda^* \times \Lambda^*)$. Then $(\Phi_{\Lambda_1 \otimes \Lambda_2}, \Theta_{\Lambda_1 \otimes \Lambda_2})$ is a soft set over $V(\Lambda^* \times \Lambda^*)$. Also define a mapping $\Omega_{\Lambda_1 \otimes \Lambda_2}$ from $\Theta_{\Lambda_1 \otimes \Lambda_2}$ to $\mathcal{P}[A(\Lambda^* \times \Lambda^*)_p]$ by $\Omega_{\Lambda_1 \otimes \Lambda_2}(\theta) = \{\text{mp arcs}\langle \Phi_{\Lambda_1 \otimes \Lambda_2}(\theta) \rangle\} = \{\text{mp arcs}\langle \Phi_1(\theta) \times \Phi_2(\theta) \rangle\}, \forall \theta \in \Theta_1 \cap \Theta_2$, where $A(\Lambda^* \times \Lambda^*)_p$ denotes the collection of all partial arcs of the disemigraph $\Lambda^* \times \Lambda^*$ and $\mathcal{P}[A(\Lambda^* \times \Lambda^*)_p]$ denotes the power set of $A(\Lambda^* \times \Lambda^*)_p$. Then $(\Omega_{\Lambda_1 \otimes \Lambda_2}, \Theta_{\Lambda_1 \otimes \Lambda_2})$ is a soft set over $A(\Lambda^* \times \Lambda^*)_p$. Also if we denote $(\Phi_{\Lambda_1 \otimes \Lambda_2}(\theta), \Omega_{\Lambda_1 \otimes \Lambda_2}(\theta))$ by $\Upsilon_{\Lambda_1 \otimes \Lambda_2}(\theta)$, then $\Upsilon_{\Lambda_1 \otimes \Lambda_2}(\theta)$ is a partial disemigraph of $\Lambda^* \times \Lambda^*, \forall \theta \in \Theta_1 \cap \Theta_2$, since $\Phi_1(\theta) \times \Phi_2(\theta) \subseteq V \times V$ and any arc in $\Omega_{\Lambda_1 \otimes \Lambda_2}(\theta)$ is in $A(\Lambda^* \times \Lambda^*)$ or it is a partial arc of an arc in $A(\Lambda^* \times \Lambda^*)$. Then $\Lambda_1 \otimes \Lambda_2$ can be represented by the 4-tuple $(\Lambda^* \times \Lambda^*, \Phi_{\Lambda_1 \otimes \Lambda_2}, \Omega_{\Lambda_1 \otimes \Lambda_2}, \Theta_{\Lambda_1 \otimes \Lambda_2})$ and also by $\{\Upsilon_{\Lambda_1 \otimes \Lambda_2}(\theta) : \theta \in \Theta_1 \cap \Theta_2\}$ and $\Lambda_1 \otimes \Lambda_2$ is a soft disemigraph of $\Lambda^* \times \Lambda^*$ since all the conditions for soft disemigraphs are satisfied. □

5. Conclusion

The integration of soft set principles into disemigraphs to develop soft disemigraphs marks a notable advancement in graph theory. This innovative method provides a strong foundation for parameterizing disemigraphs, enabling a more refined representation and analysis of intricate relationships. This research paper systematically explores and expands the theoretical framework of soft disemigraphs by introducing and analyzing several product operations in depth. Starting with the definition of the Cartesian product of soft disemigraphs, it includes examples and theorems that establish its key properties. The study then moves on to the restricted Cartesian product. We proved that both of these products of soft disemigraphs are again soft disemigraphs. Through these discussions, the theoretical foundations of soft disemigraphs have been significantly broadened. This work lays the groundwork for future research and possible applications in various fields, showcasing the flexibility and richness of soft disemigraph theory.

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