*GANITA,Vol.74(2), 2024, 443-454*

# NUMERICAL SOLUTION TO DISCRITISED SPACE-TIME MODEL OF FRACTIONAL RADON DIFFUSION EQUATION

**S. V. LATHKAR and R. N. INGLE**

**Abstract**

This paper focuses on numerical solution of Fractional Radon diffusion equation. The partial differential diffusion equation is converted to fractional form. The Radon concentration is the dependent variable of study. The Radon concentration is function of time and distance. The Partial derivatives with respect to $t$ and $x$ both are both treated in fractional order manner. The Caputo fractional derivative is taken for time variable and right shifted Grunwald-Letnikov derivative is taken for the space variable. The paper includes the entire discussion on choice of derivatives, discretisation of diffusion equation, analysis of boundary conditions, and interpretation of numerical solution. We have chosen Implicit finite difference scheme for obtaining the discretisation of STFRDE (Space -Time Fractional Diffusion Equation). The graphical representation of solution in 1 $D$ and 3 $D$ simplifies the interpretation of solution obtained. The solution is found to be positively stable and convergent.

2010 *Mathematics subject classification:* primary 35R10; secondary 65M06.
*Keywords and phrases:* Fractional, diffusion Space-, Time, Discretization, Finite difference method, Implicit.

## 1. Introduction

There is a lot of research happening across the globe on fractional differential equations, its solution and nature, by applying various tools, transforms, techniques, and methods. The existence of fractional order type of derivative and its significance has been appreciated for its valuable contribution in the existing study as compared to integer order derivatives and differential equation. The micro level analysis of the variable under study is possible because of research in fractional differential equation. The solution to the Radon diffusion equation as a fractional partial differential equation 1in time and space can serve the purpose of measurement of propagation of Radon specially in air medium.

The aims and objectives of this research paper hence are, to study the Radon concentration through air medium, to study the space-time fractional Radon diffusion equation, to find the numerical solution by applying implicit finite difference algorithm.

## 2. Literature Review

Various forms of time and space fractional heat or diffusion equations are discussed in [1, 2, 6], that includes three-dimensional fractional diffusion and kinetic equations both time and space fractional general forms [15, 11, 12], Variable orders of fractional derivatives and equations related to it are discussed in [27]. The importance of fractional derivatives is increasing, and so we need to focus on a scheme providing numerical solution that can be comparatively modest and wide-ranging to handle efficiently with variety of fractional differential equation forms. Few numerical methods are there to simplify fractional differential equations in partial derivatives. Investigation of soil radon diffusion coefficient measurements has been studied in two medium cylinder in [21]. In [22] the numerical solution of the diffusion equation describing the flow of radon through concrete has been discussed in detail. We recall the different methods used in the literature for finite difference schemes, Time fractional and space fractional differential equations, stability, uniqueness and convergence of solution for various diffusion equations In this paper we have applied the Time-Space fractional derivative to the Radon diffusion equation and solved it for approximate solution by using the Implicit finite difference scheme.

## 3. Radon Diffusion Equation

The radon concentration through air medium is the outcome of the second order Radon diffusion equation which is the prime interest in this paper.

$$\frac{\partial v(x,t)}{\partial t} = D\,\frac{\partial^2 v(x,t)}{\partial x^2} - \lambda v(x,t)$$

where $\lambda$ is decay constant of radon, $D$ is diffusion coefficient depending on the medium.
We apply the fractional order derivative to the partial order differential equation in time and space. $\frac{\partial^\alpha v(x,t)}{\partial t^\alpha}$ is the caputo fractional derivative of function $V(x,t)$ with respect to $t$. $\alpha$ is the time fractional order such that, $0 \le \alpha \le 1$.
$\frac{\partial^\beta v(x,t)}{\partial x^\beta}$ is the space fractional derivative of the function $V(x,t)$ with respect to $x$ and $\beta$ is the space fractional order such that $1 \le \beta \le 2$.
We consider the following equation which is time fractional diffusion equation,

$$\begin{aligned} \frac{\partial^\alpha v(x,t)}{\partial t^\alpha} &= D\,\frac{\partial^\beta v(x,t)}{\partial x^\beta} - \lambda v(x,t) \\ IC : v(x,0) &= 0, 0 < x < l \\ B.C. : v(0,t) &= v_0, t \ge 0 \quad \text{and} \\ \frac{\partial\, v(x,t)}{\partial\, x} &= 0, t \ge 0. \quad 0 \le \alpha \le 1 \end{aligned}$$

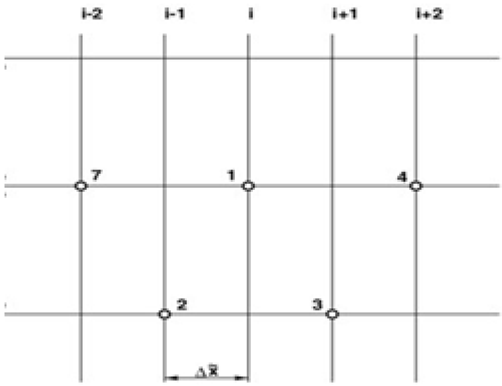



FIGURE 1. **Time-Space fractional derivative grid**

The Caputo time fractional order derivative is given by:
The time fractional derivative is approximated in Caputo sense is given by,

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\xi)^\alpha} \frac{\partial v(x_i,t)}{\partial \xi} \partial\xi, & 0 \le \alpha \le 1 \\ \frac{\partial\, v(x_i,t)}{\partial\, t}, & \alpha = 1 \end{cases}$$

The space time fractional derivative is given by :

$$\frac{\partial^\beta v(x,t)}{\partial x^\beta} = \begin{cases} \frac{1}{\Gamma(-\beta)} \lim_{n\to\infty} \frac{1}{h^\beta} \sum_{j=0}^{N} \frac{\Gamma(j-\beta)}{\Gamma(j+1)} v(x-(j-1)h,t), & \text{when} \quad 1 < \beta < 2 \\ \frac{\partial^2 v(x,t)}{\partial x^2}, & \text{when} \quad \beta = 2 \end{cases}$$

## 4. Implicit Finite difference for Discretization

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = D\frac{\partial^\beta v(x,t)}{\partial x^\beta} - \lambda v(x,t) \tag{4.1}$$

$$IC : v(x,0) = 0, 0 < x < l \tag{4.2}$$

$$B.C : v(0,t) = v_0 t \ge 0 \text{ and } \frac{\partial v(x,t)}{\partial x} = 0, t \ge 0.\ \ 0 \le \alpha \le 1 \tag{4.3}$$

To convert the time fractional derivative in discrete form, we use $t_k = k\tau$, and $x_i = ih, \tau = \dfrac{T}{N}$,
$h = \dfrac{1}{N}$.
Let $v(x_i, t_k), i = 0, 1, 2 \cdots\cdots, M$ and $k = 0, 1, 2, \cdots\cdots, N$ be the exact solution of TSFRDE from (4.1) – (4.3) at the mesh point $(x_i, t_k)$. Let $u_i^k$ be the numerical approximation of the point $v(x_j, t_k) = v(ih, k\tau)$.
The time fractional derivative is approximated in Caputo sense is given by,

$$\frac{\partial^\alpha v(x_i, t_{k+1})}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{k+1}} \frac{1}{(t_{k+1}-\xi)^\alpha} \frac{\partial v(x_i, \xi)}{\partial \xi} d\xi$$

$$\frac{\partial^\alpha v(x_i, t_{k+1})}{\partial t^\alpha} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[v_i^{k+1} - v_i^k\right][b_0] + 0(\tau) + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} \frac{(v_i^{k-j+1} - v_i^{k-j})}{\tau} [b_j]$$

where, $b_j = (j+1)^{(1-\alpha)} - (j)^{(1-\alpha)}, j = 0, 1, 2, \cdots N$ but $b_0 = 1$, so we have;
Now we discretise the spatial $\beta$ order fractional derivative by using the Grunwald-Letnikov finite order formula at all time values. The standard Grunwald formula produces unstable results by all finite difference schemes applied to it. Hence, we apply the right shifted Grunwald formula for stable results in the equation after discretisation.

$$\frac{\partial^\beta v(x,t)}{\partial x^\beta} = \begin{cases} \dfrac{1}{\Gamma(-\beta)} \lim\limits_{n\to\infty} \dfrac{1}{h^\beta} \sum\limits_{j=0}^{N} \dfrac{\Gamma(j-\beta)}{\Gamma(j+1)} v(x-(j-1)h, t), & \text{where} \quad 1 < \beta < 2 \\ \dfrac{\partial^2 v(x,t)}{\partial x^2}, & \text{when} \quad \beta = 2 \end{cases}$$

Where $N$ is the positive integer, $h = \dfrac{L}{N}$, $t_k = k\tau$, and $x_i = ih$, $\tau = \dfrac{T}{N}$, $h = \dfrac{1}{N} \cdot \Delta x = h \geq 0$, $0 \leq \tau \leq T$. These dimensions define the grid size for the solution.
The normalized Grunwald weights are given by $g_{\beta j} = \frac{\Gamma(1-\beta}{\Gamma(j+1)\Gamma(-\beta)}$, for $j = 0, 1, 2, 3,$.
For $D_t^\alpha\, v(x_i, t_{k+1})$ the time fractional caputo derivative and $D_x^\beta\, v(x_i, t_{k+1})$ the right shifted Grunwald formula, defines the numerical approximation equation. The resulting time-space fractional approximation using implicit FDM to equation (4.1 – 4.3), is as follows:

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[v_i^{k+1} - v_i^k\right] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} [b_j] \left(v_i^{k-j+1} - v_i^{k-j}\right) = D\delta_{\beta,x} - \lambda v_i^k$$

So, $\left[v_i^{k+1} - v_i^k\right] + \sum\limits_{j=1}^{k} [b_j] \left(v_i^{k-j+1} - v_i^{k-j}\right) = D\, D\delta_{\beta,x} - \Gamma(2-\alpha)\, \lambda v_i^k.$

Let $r = D\, \frac{\Gamma(2-\alpha)\tau^\alpha}{h^\beta}$ and $\mu = \Gamma(2-\alpha)\, \lambda\tau^\alpha$, $b_j = (j+1)^{(1-\alpha)} - (j)^{(1-\alpha)},\ j = 0, 1, 2, \cdots N.$

$$\left[v_i^{k+1} - v_i^k\right] + \sum_{j=1}^{k} [b_j] \left(v_i^{k-j+1} - v_i^{k-j}\right) = \Gamma[\delta_{\beta,x}] \qquad (4.4)$$

Where $\delta_{\beta,x} v_i^k = \frac{1}{h^\beta} \sum_{j=0}^{i+1} g_{\beta j} v_{i-j+1}^k$

Now the initial and boundary value conditions in discretised as $v_i^0 = 0$; $i = 0, 1, 2, \cdots m$;
The boundary conditions $x_0$, and $x_m$, the discritization scheme implements as: $v_0^k = 0$ and $\frac{v_{m+1}^{k+1} - v_{m-1}^{k+1}}{2h} = 0$; implies $v_{m+1}^{k+1} - v_{m-1}^{k+1}$ from (4.4.)

$$\left[v_i^{k+1} - v_i^k\right] + b_1 v_i^k - \sum_{j=1}^{k-1} (b_j - b_{j+1}) v_i^{k-j} - b_k v_i^0 = -r\beta v_i^{K+1} + r \sum_{j=0}^{i+1} g_{\beta j} v_{i-j+1}^{k+1} - \mu v_i^k$$

$$(1 + r\beta) v_i^{k+1} - r \sum_{j=0}^{i+1} g_{\beta j} v_{i-j+1}^{k+1} = (1 - b_{1-\mu}) v_i^k + \sum_{j=1}^{k-1} (b_j - b_{j+1}) v_i^{k-j} + b_k v_i^0 \tag{4.5}$$

where, $b_j = (j+1)^{1-\alpha} - (j)^{1-\alpha}; i = 0, 1, 2, \cdots m; k = 0, 1, 2, \cdots n$
For $k = 0$, the fractional approximation IBVP looks like, (from 4.5) $i = 1$
$(1 + r\beta) v_i^1 - r \sum_{j=0}^{1+1} g_{\beta j} v_{1-j+1}^1 = (1 - \mu) v_1^0$, same can be further written as
$(1 + r\beta) v_1^1 - r \left[g_{\beta 0} v_2^1 - g_{\beta 2} v_0^1\right] = (1 - \mu) v_i^0$ similarly, for $i = 2$
$(1 + r\beta) v_2^1 - r \left[g_{\beta 0} v_3^1 + g_{\beta 2} v_1^{1+} - g_{\beta 3} v_0^1\right] = (1 - \mu) v_2^0$, likewise, $i = N$
$(1 + r\beta) v_N^1 - r \left[g_{\beta 0} v_{N+1}^1 + g_{\beta 2} v_{N+}^1 - g_{\beta 3} v_{n-1}^1 - \cdots, \cdots g_{\beta N+1} v_0^1\right] = (1 - \mu) v_N^0$
This system of equations derived from (4.5) for $k = 0$ and various values of $i = 1, 2, 3 \cdots N$, we generate a system of linear equations, which can be written in matrix form as:

$$AV^1 = (1 - \mu) v_1^0 + B \cdots\cdots\cdots \tag{4.6}$$

Where, the Matrix $A$ of coefficients is given by :

$$A = \begin{bmatrix} (1 + r\beta) & -rg_{\beta 0} & 0 & & 0 & 0 & 0 & \\ -rg_{\beta 2} & (1 + r\beta) & -rg_{\beta 0} & & 0 & 0 & 0 & \\ -rg_{\beta 3} & -rg_{\beta 2} & (1 + r\beta) & & -rg_{\beta 0} & 0 & 0 & \\ \cdot & \cdot & \cdot & \cdot & & \cdots & & 0 \\ -rg_{\beta N-1} & -rg_{\beta N-2} & & (1 + r\beta) & & -rg_{\beta 0} & & 0 \\ -rg_{\beta N} & -rg_{\beta N-1} & \cdot & \cdot & & -r(-rg_{\beta 0} - rg_{\beta 3}) & & (1 + r\beta) \end{bmatrix}$$

$$V^1 = \begin{bmatrix} v_1^1 \\ v_2^1 \\ \cdot \\ v_{m-1}^1 \\ v_m^1 \end{bmatrix}, \ (1 - \mu) v_1^0 = (1 - \mu) \begin{bmatrix} v_1^0 \\ v_2^0 \\ \cdot \\ v_{m-1}^0 \\ v_m^0 \end{bmatrix}$$

$$B = \left[rg_{\beta 2} V_l \ rg_{\beta 3} V_l \cdots\cdots g_{\beta N+1} V_l\right]'$$

For $k \geq 0$, from 4.5, At $i = 1$

$$(1 + r\beta)\, v_1^{k+1} - r \sum_{j=0}^{i+1} g_{\beta j}\, v_{1-j+1}^{k+1} = (1 - b_{1-\mu})\, v_1^k + \sum_{j=1}^{k-1} (b_j - b_{j+1})\, v_1^{k-j} + b_k v_1^0$$

$$(1 + r\beta)\, v_1^{k+1} - r\left[g_{\beta 0}\, v_2^{k+1} - g_{\beta 2}\, v_0^{k+1}\right]$$

$$= (1 - b_{1-\mu})\, v_1^k + \sum_{j=1}^{k-1} (b_j - b_{j+1})\, v_1^{k-j} + b_k v_1^0$$

At $i = 2$,

$$(1 + r\beta)\, v_2^{k+1} - r\left[g_{\beta 0}\, v_3^{k+1} + g_{\beta 2}\, v_1^{k+1} - g_{\beta 3}\, v_0^{k+1}\right]$$

$$= (1 - b_{1-\mu})\, v_2^k + \sum_{j=1}^{k-1} (b_j - b_{j+1})\, v_1^{k-j} + b_k\, v_2^0$$

At $i = N$,

$$(1 + r\beta)\, v_N^{k+1} - r\left[g_{\beta 0}\, v_{N+1}^{k+1} + g_{\beta 2}\, v_{N+}^{k+1} - g_{\beta 3}\, v_{N-1}^{k+1} - \cdots\cdots g_{\beta N+1}\, v_0^{k+1}\right]$$

$$= (1 - b_{1-\mu})\, v_N^k + \sum_{j=1}^{k-1} (b_j - b_{j+1})\, v_N^{k-j} + b_k\, v_N^0$$

This system of equations derived from (4.5) for $k \geq 0$ and various values of $i = 1, 2, 3 \cdots N$, we generate a system of linear equations, which can be written in matrix form as:

$$AV^{k+1} = (1 - b_{1-\mu})\, V^k + \sum_{j=1}^{k-1} (b_j - b_{j+1})\, v_i^{k-j} + b_k\, v_N^0 + B \cdots \tag{4.7}$$

Where, the Matrix $A$ of coefficients is given by :

$$A = \begin{bmatrix} (1 + r\beta) & -rg_{\beta 0} & 0 & & 0 & 0 & 0 & \\ -rg_{\beta 2} & (1 + r\beta) & -rg_{\beta 0} & & 0 & 0 & 0 & \\ -rg_{\beta 3} & -rg_{\beta 2} & (1 + r\beta) & & -rg_{\beta 0} & 0 & 0 & \\ \cdot & \cdot & \cdot & \cdot & & \cdots & & 0 \\ -rg_{\beta N-1} & -rg_{\beta N-2} & & (1 + r\beta) & & -rg_{\beta 0} & & 0 \\ -rg_{\beta N} & -rg_{\beta N-1} & \cdot & \cdot & & -r(-rg_{\beta 0} - rg_{\beta 3}) & & (1 + r\beta) \end{bmatrix}$$

$$V^{k+1} = \begin{bmatrix} v_1^{k+1} \\ v_2^{k+1} \\ \cdot \\ v_{m-1}^{k+1} \\ v_m^{k+1} \end{bmatrix},\ (1 - \mu)\, v_1^0 = (1 - \mu) \begin{bmatrix} v_1^0 \\ v_2^0 \\ \cdot \\ v_{m-1}^0 \\ v_m^0 \end{bmatrix},\ v_i^{k-j} = \begin{bmatrix} v_1^{k-j} \\ v_2^{k-j} \\ \cdot \\ v_{m-1}^{k-j} \\ v_m^{k-j} \end{bmatrix}$$

$$B = \left[rg_{\beta 2}\, V_l\ rg_{\beta 3}\, V_l \cdots\cdots g_{\beta N+1}\, V_l\right]'$$

With initial conditions, $v = 0,\ i = 0, 1, 2 \cdots\cdots m \cdots\cdots$ (4.8)

and boundary condition

$$\text{so } v_{m+1}^{k+1} = v_{m-1}^{k+1} \tag{4.9}$$

## 5. The analytical estimation of radon diffusion in air in Time -space fractional derivative

The secondary data for different parameters included in the Radon diffusion in air medium :

- (•) The diffusion coefficient of radon in air. $D_a = 1 \times \dfrac{10^{-5}m^2}{s} = 0 \cdot 1cm^2/s.$
- (•) The radon concentration in ambient air $v_0 = 200Bq/m^3$
- (•) The radon absorption coefficient $k = \dfrac{1m^3}{kg}$ and $\rho = \frac{1g}{cm^2}$
- (•) The length of cylinder for measurement $l = 1m$
- (•) The volume of cylinder for measurement $v = 1m^3$
- (•) Radius of cylinder used for measurement is, $R = \dfrac{1}{\sqrt{\pi}}\, m.$
- (•) The experiment for measurement of Radon diffusion was conducted for 72 hours duration, for saturation of radon activity in air.
- (•) $v(0,t) = k\rho C_0 = 1 \times 1 \times 200 = 200 \times 10^6 Bq/cm^3 \cdot \lambda = 2 \cdot 1 \times 10^{-6}/s$ is the decay coefficient of Radon.
- (•) $r = D\,\frac{\Gamma(2-\alpha)\,\tau^{\alpha}}{h^{\beta}}$
- (•) $\mu = \Gamma(2-\alpha)\,\lambda$
- (•) surface area $S = \pi r^2$
- (•) $\beta = 1 \cdot 1, 1 \cdot 8$
- (•) $g_{\beta j} = \dfrac{\Gamma(1-\beta)}{\Gamma(j+1)\Gamma(-\beta)}$, for $j = 0, 1, 2, \cdots$
- (•) $\alpha = 0 \cdot 9, 0 \cdot 8$

The estimates are used to solve the system of equations obtained in matrix form finding the relation between the radon concentration as function of distance $x$ and time $t$. Similarly solving it for fractional order derivative $0 \leq \alpha \leq 1,\ \alpha = 0 \cdot 8,\ 1 \leq \beta \leq 2,\ \beta = 1 \cdot 8$ we see the solution interpreted graphically by using ‘Mathematica’.

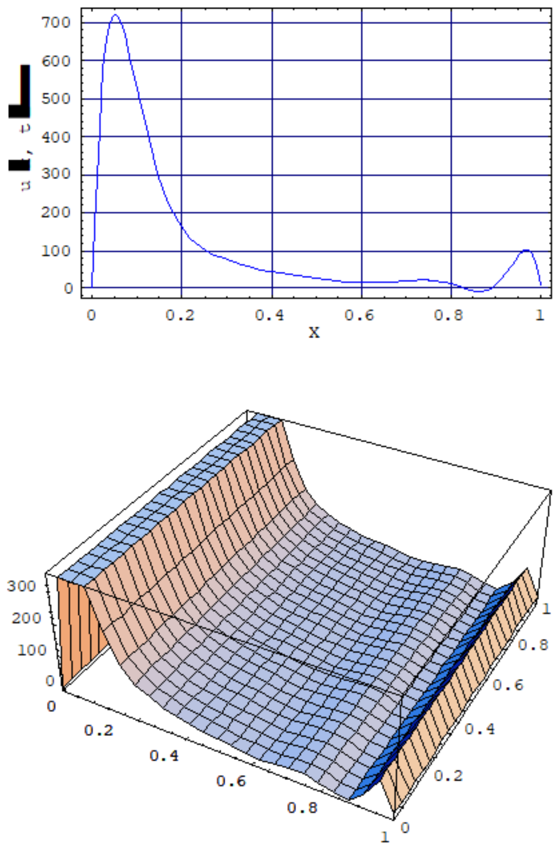
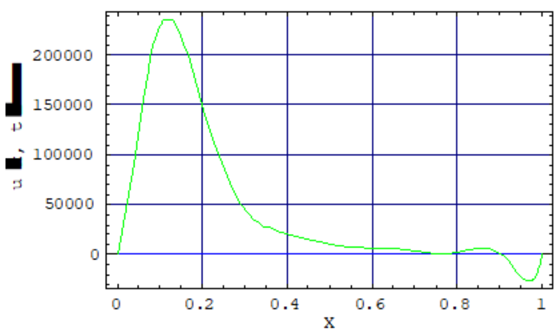



FIGURE 2. Radon Concentration at $\alpha = 0.9, \beta = 1.1$ fractional order derivative



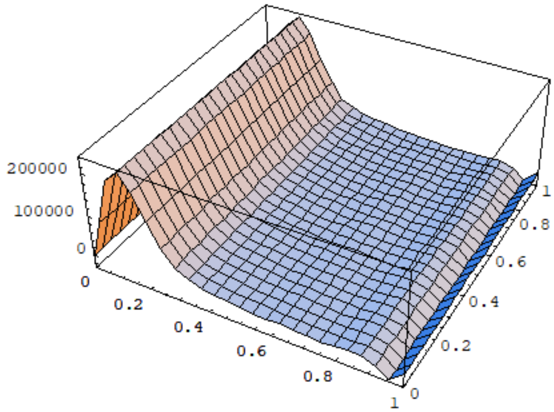



FIGURE 3. Radon Concentration at $\alpha = 0.8, \beta = 1.8$ fractional order derivative

## 6. Stability and Convergence:

**Theorem 5.1:** The solution of the discretised scheme (4.6)-(4.9), for the Time-Space fractional Radon diffusion equation obtained from (4.1)-(4,3), is unconditionally stable, when solved by implicit finite difference scheme.

**Proof:** For $i, j, k$ changing from $i = 1, 2, 3, \cdots, N,\ j = 0, 1, 2, \cdots, N, k = 0, 1, 2, \cdots$ the coefficients $g_{\beta j} = \dfrac{\Gamma(1-\beta)}{\Gamma(j+1)\Gamma(-\beta)}$, follows certain criteria, $g_{\beta 0} - 1 \cdot g_{\beta 1} = -\beta < 0, g_{\beta j} > 0$ for $j \neq 1$. With the same $\sum_{j=0}^{\infty} g_{\beta j} = 0, \sum_{j=0}^{i+1} g_{\beta j} < 0.$

The exact solution $\overline{v_l^k}$ of the TSFRDE $(4.1-4.3)$ and $v_i^k$, is the approximate solution of the equation by Implicit FDM method. Then $E_i^k = \overline{v_l^k} - v_i^k$,
Satisfies the matrix equations $AV^1 = (1-\mu)\, v_1^0 + b$ and

$$AV^{k+1} = (1 - b_{1-\mu}\, V^k + \sum_{j=1}^{k-1} (b_j - b_{j+1})\, v_i^{k-j} + b_k\, v_N^0 + B$$

Which could be written for the Component of $E$ as

$$AE^1 = E^0 + B$$
$$AE^{k+1} = a_1 E^k + a_2 E^{k-1} + a_3 E^{k-3} + \cdots + a_k E^0 + B$$

Where, $E^0 = 0$ and $E^k = (\epsilon_1^k, \epsilon_1^k, \epsilon_2^k, \epsilon_3^k \cdots\cdots \epsilon_{N-1}^k)^T$ and $B$ includes all the boundary value conditions. From (4.7)

$$|E_l^1| = \left|(1 + r\beta)e_i^1 - r \sum_{j=0}^{i+1} g_{\beta j} \in_{i-j+1}^1\right|$$

$$|E_l^1| \le | \in_i^0 | \le |E_i^0| = ||E^0||_\alpha$$

From (3.8)

$$\begin{aligned}|E_i^{k+1}| &= \left|(1 + r\beta) \ \in_i^{k+1} - r \sum_{j=0}^{i+1} g_{\beta j} \ \in_{i-j+1}^{k+1}\right| \\ &\le \left|(1 - b_1) \ \in_i^k + \sum_{j=1}^{k-1} (b_i - b_{j+1}) \ \in_i^{k-j} + b_k \ \in_i^0\right| \\ &\le (1 - b_1)| \in_i^k | + (b_1 - b_{j+1})| \in_i^{k-j} | + b_k| \in_i^0 | \\ &\le (1 - b_1 + b_1 = b_k + b_k) \left||E_l^k|\right| \\ ||E_i^{k+1}||_\infty &\le ||E_l^k|| = ||E^k||_\infty \le ||E^0||_\infty\end{aligned}$$

So, the solution is unconditionally stable.
Also, we prove the convergence for the approximate solution towards the exact solution of (4.1) – (4.3).
If $|e_l^k| = \max_{1\le i\le N-1} | \in_i^k | = ||E^k||_\infty$, then $|T_i^k| = \max_{1\le i\le N} |T_i^k|$
From (4.7)

$$(1 + r\beta)|e_l^1| = \left|(1 + r\beta)\, e_l^1 - r \sum_{j=0}^{i+1} g_{\beta j}\, e_{i-j+1}^{k+1}\right| \le ||e_i^0|| + |T_i^1|$$

$$|e_l^k| \le \frac{1}{(1 + r\beta)} |e_i^0| + \frac{1}{(1 + r)} |T_i^1| \le |e_i^0| + c|T_i^1| \le |e_i^0| + ch^2\{0(\tau + h^2)\}$$

$$||E^1||_\infty \le ||E^0||_\infty + ch^2\{0(\tau + h^2)\}$$

$$||E^k||_\infty \le ||E^0||_\infty + ch^2\{0(\tau + h^2)\}$$

Similarly from (4.8)

$$||E^{k_1}||_\infty \le ||E^0||_\infty + ch^2\{0(\tau + h^2)\}$$

So, the approximate solution is approaching the exact solution of (4.1) – (4.3). So, the convergence is proved.

## 7. Conclusion:

The radon diffusion equation (4.1) − (4.3) has been solved by discritizing the equation in Time-Space fractional form. The Implicit finite difference scheme is applied for approximation. The approximate numerical solution is attained for radon

diffusion equation which is studied in air medium with specified boundary conditions. The solution is represented graphically by using 'Mathematica'. At $\alpha = 0 \cdot 9$, $\beta = 1 \cdot 1$ and, $\alpha = 0 \cdot 8$, $\beta = 1 \cdot 8$, the numerical solutions are analysed at $t = 0 \cdot 05$ by taking into consideration the terms $\tau = 0 \cdot 005$, $h = 0 \cdot 1$.
Stable and convergent numerical solution is obtained for the TSFRDE under analysis. The Radon transport in a cylinder of air medium is considered with the concentration at various levels. The solution obtained by Implicit finite difference scheme for time-space fractional radon diffusion equation is in agreement with actual solution.

## References


[1] A. Chechkin, V. Gonchar, R. Gorenflo, N. Korabel, I. Sokolov, *Generalized fractional diffusion equations for accelerating subdiffusion and truncated levy flights*, Physical Review **78** (2008) Article 021111.

[2] A. Chechkin, R. Gorenflo, I. Sokolov, *Fractional diffusion in inhomogeneous medi*, Journal of Physics A: Mathematical and General **38(42)** (2005)

[3] Dongfang Li, *A linear finite difference scheme for generalized time fractional Burgers equation*, Analysis **16** (2016) http://dx.doi.org/10.1016/j.apm.2016.01.043 S0307- 904X.

[4] Dosimetry, , *The Numerical Solution of the Diffusion Equation Describing the Flow of Radon Through Cracks in a Concrete Slab*, Radiation Protection, **11(4)**, (1985), 229-236

[5] Dosimetry, *Radon transport through concrete and determination of its diffusion coefficient*, Radiation Protection Dosimetry, **104(1)** (2003) 65-70 ·

[6] F. Liu, V. Anh, I. Turner, *Numerical solution of the space fractional Fokker-Planck equation*, Journal of Computational and Applied Mathematics.

[7] Huynh NguyenPhong ThuabNguyenVan ThangaLe CongHaoab, *The effects of some soil characteristics on radon emanation and diffusion*, Journal of Environmental Radioactivity , **216**, (2020) 106189.

[8] K. B. Oldham, J. Spanier, *The Fractional Calculu*s, Academic Press, New York, (1974).

[9] K. C. Takale, V. R. Nikam and A. S. Shinde, *Finite Difference Scheme for Space Fractional Diffusion Equation with Mixed Boundary Conditions*, American Jr. of Mathematics and Sciences, **2** (2013)

[10] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley, New York .(1993)

[11] M. M. Meerschaert, H.-P. Scheffler, C. Tadjeran, *Finite difference methods for two-dimensional fractional dispersion equation*, Journal of Computational Physics **2**

[12] M. Meerschaert, C. Tadjeran, *Finite difference approximations for twosided space-fractional partial differential equations*, Applied Numerical Mathematics **56** (2006) 80-90.

[13] N. Heymans, I. Podlubny, *Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives*, Rheologica Acta **45(5)** (2006) 765-771.

[14] R. Gorenflo, E. Abdel-Rehim, *Discrete models of time-fractional diffusion in a potential well*, Fractional Calculus and Applied Analysis **8(2)** (2005) 173-200.

[15] R. Gorenflo, F. Mainardi, D. Moretti, G. Pagnini, P. Paradisi, *Discrete random walk models for space-time fractional diffusion*, Chemical Physics **284(1-2)** (2002) 521-541.

[16] S. Shen, F.Liu,, *Error Analysis of an explicit Finite Difference Approximation for the Space Fractional Diffusion equation with insulated ends*,‖ ANZIAM **46(E)**, ( 2005) C871 - C887.

[17] S. V. Lathkar and R. N. Ingle ; *Adomian Decomposition Method for solving Non Linear Three variable*, Parabolic Fractional Partial Differential Equation,6(1) 2019

[18] S. V. Lathkar and R. N. Ingle ; *Adomian Decomposition Method for solving Non Linear Three variable*, Parabolic Fractional Partial Differential Equation,6(1) 2019 www.ijrar.org (E- ISSN 2348-1269, P-ISSN 2349-5138).

[19] S. V. Lathkar and R. N. Ingle; *Computational and analytical measurement of radon concentration in air medium using implicit finite difference scheme for time fractional radon diffusion equation*, Malaya Journal of Matematik, 9(1), (2021) 1000-1005 https://doi.org/10.26637/MJM0901/0176

[20] S. V. Lathkar and R. N, Ingle ; *Comparative Analysis of Radon Concentration in Air and Charcoal Medium through Time Fractional Radon Diffusion Equation*, Science, Technology and Development, **X(XII)** (2021) 104-112.

[21] Sasaki, T.,Gunji,Y.,Okuda,T., *Investigation of soil radon diffusion coefficient measurements and methodology*.RAE-50344-001-1, URS Corporation, Salt Lake. , (2006)

[22] Savovic', S., Djordjevich, A.,, *Numerical solution of the diffusion equation describing the flow of radon through concrete SEQ CHAPTER, 11*, Applied Radiation and Isotopes **66** (2008) 552-555

[23] Schery, S.D., Whittlestone,S. *Description of radon at the earth's surface*. J.Geophys.Res.**94(D15)** (1989) 18297-18303.

[24] T. Langlands, B. Henry, *The accuracy and stability of an implicit solution method for the fractional diffusion equation*, Journal of Computational Physics **205(2)** (2005) 719-736

[25] Tomozo SASAKI , Yasuyoshi GUNJI & Takao IIDA *Transient- Diffusion Measurements of Radon: Practical Interpretation of Measured Data*, Journal of Nuclear Science and Technology, **44(7)**,(2007) 1032-1037.

[26] V.C. Rogers,etal, Radon Flux Measurement And Computational Methodologies, (1984)

[27] Y. Lin, C. Xu, , *Finite difference/spectral approximations for the time-fractional diffusion equation*, J. Compute. Phys. **225**, (2007) 1533-1552.

S. V. Lathkar, Research Scholar, Dept of Mathematics, NES Science College Nanded, Maharashtra, India
e-mail: seemlathkar@gmail.com

R. N. Ingle, Dept of Mathematics, Bahirji smarak Maha Vidyalaya Vasmat, Maharashtra, India
e-mail: ingleraju11@gmail.com