GANITA,Vol.74(2), 2024, 455-460

SOLUTIONS OF PELL'S EQUATION AND EXPONENTIAL DIOPHANTINE EQUATION UTILIZING RAMANUJAN PRIME NUMBERS OF ORDER 2

C.SARANYA ✉

Abstract

The objective of this research is to identify non-trivial integer solutions to the pell's equation utilising Ramanujan Prime numbers of order 2 by employing the method of Brahmagupta lemma and the related recurrence relations are obtained. Also, the table illustrating numerous computations satisfying pell equation for Ramanujan prime numbers of order 2 along with their solutions and their corresponding recurrence relations are presented. In a way, we apply the Catalan Conjecture to attain non-negative integer solutions to the exponential diophantine problem using Ramanujan Prime numbers of order 2 under three cases. Additionally, few numerical illustrations satisfying the exponential diophantine equation and their integer solutions are analyzed and exhibited.

2010 *Mathematics subject classification:* primary 11D99.
Keywords and phrases: Ramanujan Prime numbers, prime numbers, pell's equation, Exponential Diophantine Equation, Integer solutions..

1. Introduction

The characteristics of integers have been investigated by number theorists. That being said, the first step towards constructing a shiny new mathematical theory is to speculatively explore number relationships [1, 3-7 & 12]. Pell's equation can be formulated as a diophantine equation of the form $x^2 = dy^2 + 1$, where x and y have integer solutions and n is a positive non-square integer. Pell's equation is closely associated with a few additional noteworthy mathematical topics. For a solution to this equation, nevertheless, a number of distinct steps must be fulfilled is discussed in [2 & 14].

An equation containing integer exponents has been designated as an exponential diophantine equation. The Catalan Conjecture yields solutions to an array of exponential Diophantine equations [11 & 13]. Solutions to the pell's equation utilising jarasandha numbers and the exponential diophantine equation featuring jarasandha numbers are analyzed in [9 & 10], Also Encryption Technique Involving Ramanujan Prime Numbers Using RSA Public Key Cryptography is discussed in [8]. Results from Number Theory have endless applications in math just as in reasonable applications including security, memory management, authentication, coding theory, etc. In this communication, we look for discovering non-zero unique integer solutions to the exponential diophantine equation incorporating Ramanujan Prime numbers of order 2 as well as solutions to the pell's equation using Ramanujan Prime numbers of order 2 .

2. Ramanujan Prime Number

A Ramanujan prime is a prime number that satisfies a result proved by Srinivasa Ramanujan relating to the prime counting function. The n^{th} Ramanujan prime is the least positive integer R_n for which $\pi(x) - \pi(\frac{x}{2}) \geq R_n, n \geq 1$ where $\pi(x)$ is the prime counting function (number of primes less than or equal to x). In other words, there are at least n primes between $\frac{x}{2}$ and x whenever $x \geq R_n$.
The first few numbers of this kind are: 2, 11, 17, 29, 41, 47, 59, 67, 71, 97 ...

3. Method of Analysis

Theorem 1:

The sequence of non-zero distinct integer solutions to the pell's equation

$$x^2 - dy^2 = z^2 \tag{3.1}$$

where, $d = R^2 - 1$, z is an integer and R is a Ramanujan prime number of order 2 is

$$x_s = \frac{z}{2}(\sqrt{d+1}f_s + \sqrt{d}g_s)$$
$$y_s = \frac{z}{2\sqrt{d}}(\sqrt{d}f_s + \sqrt{d+1})g_s) \qquad s = 0, 1, 2, ...$$

and their recurrence relations are

$$x_{s+2} - 2\sqrt{d+1}x_{s+1} + x_s = 0$$
$$y_{s+2} - 2\sqrt{d+1}y_{s+1} + y_s = 0, \qquad s = 0, 1, 2, ...$$

Proof:

The initial solution of (3.1) is given by, $x_0 = \sqrt{z(d+1)}$, $y_0 = \sqrt{z}$.
To find the other solutions of (3.1), consider the pell equation $x^2 = dy^2 + 1$ whose initial solution $(\tilde{x_s}, \tilde{y_s})$ is given by

$$\tilde{x_s} = \frac{1}{2}f_s$$
$$\tilde{y_s} = \frac{1}{2\sqrt{d}}g_s \qquad s = 0, 1, 2, ...$$

$$\text{where} f_s = (\sqrt{d+1} + \sqrt{d})^{s+1} + (\sqrt{d+1} - \sqrt{d})^{s+1}$$
$$g_s = (\sqrt{d+1} + \sqrt{d})^{s+1} - (\sqrt{d+1} - \sqrt{d})^{s+1} \qquad s = 0, 1, 2, ...$$

Applying Brahmagupta's lemma between the solutions (x_0, y_0) and $(\tilde{x_s}, \tilde{y_s})$, the sequence of non-zero distinct integer solutions to (3.1) are obtained as

$$x_s = \frac{z}{2}(\sqrt{d+1}f_s + \sqrt{d}g_s)$$
$$y_s = \frac{z}{2\sqrt{d}}(\sqrt{d}f_s + \sqrt{d+1})g_s) \qquad s = 0, 1, 2, ...$$

& their recurrence relations are found to be

$$x_{s+2} - 2\sqrt{d+1}x_{s+1} + x_s = 0$$
$$y_{s+2} - 2\sqrt{d+1}y_{s+1} + y_s = 0, \qquad s = 0, 1, 2, ...$$

Illustration:

The table that comes below presents numerous computations for choosing the values of R & z which satisfy the pell equation together with their solutions:

Ramanujan Prime number R	Integer z	Pell equation	Initial Solutions	Sequence of Integer solution	Recurrence Relation
11	2	$x^2 = 120y^2 + 4$	$(22, 2)$	$x_s = 11f_s + \sqrt{120}g_s$ $y_s = \frac{1}{\sqrt{120}}\left(\sqrt{120}f_s + 11g_s\right)$	$x_{s+2} - 22x_{s+1} + x_s = 0$ $y_{s+2} - 22y_{s+1} + y_s = 0$
19	3	$x^2 = 288y^2 + 9$	$(51, 3)$	$x_s = \frac{3}{2}\left(19f_s + \sqrt{288}g_s\right)$ $y_s = \frac{3}{2\sqrt{288}}\left(\sqrt{288}f_s + 19g_s\right)$	$x_{s+2} - 38x_{s+1} + x_s = 0$ $y_{s+2} - 38y_{s+1} + y_s = 0$
29	2	$x^2 = 840y^2 + 4$	$(58, 2)$	$x_s = 29f_s + \sqrt{840}g_s$ $y_s = \frac{1}{\sqrt{840}}\left(\sqrt{840}f_s + 29g_s\right)$	$x_{s+2} - 58x_{s+1} + x_s = 0$ $y_{s+2} - 58y_{s+1} + y_s = 0$
41	3	$x^2 = 1680y^2 + 9$	$(123, 3)$	$x_s = \frac{3}{2}\left(41f_s + \sqrt{1680}g_s\right)$ $y_s = \frac{3}{2\sqrt{1680}}\left(\sqrt{1680}f_s + 41g_s\right)$	$x_{s+2} - 82x_{s+1} + x_s = 0$ $y_{s+2} - 82y_{s+1} + y_s = 0$
47	2	$x^2 = 2208y^2 + 4$	$(94, 2)$	$x_s = 47f_s + \sqrt{2208}g_s$ $y_s = \frac{1}{\sqrt{2208}}\left(\sqrt{2208}f_s + 47g_s\right)$	$x_{s+2} - 94x_{s+1} + x_s = 0$ $y_{s+2} - 94y_{s+1} + y_s = 0$
59	5	$x^2 = 3480y^2 + 25$	$(295, 5)$	$x_s = \frac{5}{2}\left(59f_s + \sqrt{3480}g_s\right)$ $y_s = \frac{5}{2\sqrt{3480}}\left(\sqrt{3480}f_s + 59g_s\right)$	$x_{s+2} - 118x_{s+1} + x_s = 0$ $y_{s+2} - 118y_{s+1} + y_s = 0$
67	3	$x^2 = 4488y^2 + 9$	$(201, 3)$	$x_s = \frac{3}{2}\left(67f_s + \sqrt{4488}g_s\right)$ $y_s = \frac{3}{2\sqrt{4488}}\left(\sqrt{4488}f_s + 67g_s\right)$	$x_{s+2} - 134x_{s+1} + x_s = 0$ $y_{s+2} - 134y_{s+1} + y_s = 0$
71	5	$x^2 = 5040y^2 + 25$	$(355, 5)$	$x_s = \frac{5}{2}\left(71f_s + \sqrt{5040}g_s\right)$ $y_s = \frac{5}{2\sqrt{5040}}\left(\sqrt{5040}f_s + 71g_s\right)$	$x_{s+2} - 142x_{s+1} + x_s = 0$ $y_{s+2} - 142y_{s+1} + y_s = 0$
97	7	$x^2 = 9408y^2 + 49$	$(679, 7)$	$x_s = \frac{7}{2}\left(97f_s + \sqrt{9408}g_s\right)$ $y_s = \frac{7}{2\sqrt{9408}}\left(\sqrt{9408}f_s + 97g_s\right)$	$x_{s+2} - 194x_{s+1} + x_s = 0$ $y_{s+2} - 194y_{s+1} + y_s = 0$

Theorem 2:

$(1, 1, \frac{R+1}{2})$ is the solution of the exponential diophantine equation

$$R^x - 2^{-2y}(R-1)^{2y} = z^2 \quad (3.2)$$

where x, y, z are non-negative integers and R is a Ramanujan prime number of order 2.

Proof:

We will divide the proof under 3 cases.
Case (i):
Suppose $x = 0$, then (3.2) becomes $z^2 - 1 = 2^{-2y}(R-1)^{2y}$, if we let

$$z - 1 = 2^u(R-1)^w \quad (3.3)$$

where w, u are non-negative integers, so that

$$z + 1 = 2^{-2y-u}(R-1)^{2y-w} \quad (3.4)$$

Using (3.3) & (3.4), we get $2^{-2y-u}(R-1)^{2y-w} - 2^u(R-1)^w = 2$.
Then $u = 1, w = 0$ & $(R-1)^{2y} = 2^{2y+3}$.
This is not possible for positive values of y and values of R. Hence, $x \neq 0$.
Case (ii):
Suppose $y = 0$, then (3.2) becomes $R^x + 1 = z^2$ if we let

$$z - 1 = R^v \quad (3.5)$$

where v is a non-negative integer, so that

$$z + 1 = R^{x-v} \quad (3.6)$$

Using (3.5) & (3.6), we get $R^{x-v} - R^v = 2$.
Then $v = 0$ & $R^x = 3$.
This is not possible for positive values of x and values of R. Hence, $y \neq 0$.
Case (iii):
Rewriting (3.2) as $z^2 - 2^{-2y}(R-1)^{2y} = R^x$ and if we let

$$z - 2^{-y}(R-1)^y = R^s \quad (3.7)$$

where s is a non-negative integer, so that

$$z + 2^{-y}(R-1)^y = R^{x-s} \quad (3.8)$$

Using (3.7) & (3.8), we get $R^{x-s} - R^s = 2^{1-y}(R-1)^y$.
Then $s = 0$ & $R^x - 2^{1-y}(R-1)^y = 1$.
If $y = 1$, then $x = 1, z = \frac{R+1}{2}$ for the values of R.
Hence, $(1, 1, \frac{R+1}{2})$ is the solution of the exponential diophantine equation (3.2).

Illustration:

Few numerical demonstrations for the choice of R satisfying the exponential diophantine equation and their solutions are illustrated below in the pertaining table:

S.No	Ramanujan prime number R	Exponential Diophantine equation	Integer solutions
1.	11	$11^x + 25^y = z^2$	$(1, 1, 6)$
2.	17	$17^x + 64^y = z^2$	$(1, 1, 9)$
3.	29	$29^x + 196^y = z^2$	$(1, 1, 15)$
4.	41	$41^x + 400^y = z^2$	$(1, 1, 21)$
5.	47	$47^x + 529^y = z^2$	$(1, 1, 24)$
6.	59	$59^x + 841^y = z^2$	$(1, 1, 30)$
7.	67	$67^x + 1089^y = z^2$	$(1, 1, 34)$
8.	71	$71^x + 1296^y = z^2$	$(1, 1, 36)$
9.	97	$97^x + 2304^y = z^2$	$(1, 1, 49)$

4. Conclusion

In the present article, we have developed solutions to the exponential diophantine equation by making use of Ramanujan Prime numbers of order 2 by catalan conjecture as well as non-zero unique integer solutions to the pell's equation for Ramanujan Prime numbers of order 2 by applying the procedure of Brahmagupta lemma. Also, illustrations along with their solutions are also displayed in the corresponding table for both the equations. ln a way, it renders the ability for someone to search for further numbers with appropriate relationships.

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C.Saranya, Associate Professor, PG and Research Department of Mathematics,
Cauvery College for Women (Autonomous), (Affiliated to Bharathidasan University),
Tiruchirappalli, Tamilnadu, India
e-mail: c.saranyavinoth@gmail.com & saranyac.maths@cauverycollege.ac.in