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EXPLICIT FORMULAS FOR RESTRICTED PARTITION FUNCTIONS IN TERMS OF DIVISOR FUNCTION

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Abstract

The objective of this paper is to establish explicit formulas in terms of divisor functions for certain restricted partition function.

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1. Introduction

For any positive integer n, a partition of n is an expression of the form

$$n = n_1 + n_2 + \cdots + n_k$$

with $n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_k$, $k \geq 1$ and the number of partition of n is denoted by $p(n)$. We also define $p(0) = 1$.

P. A. MacMahon calculated $p(n)$ upto $n = 200$. Loking the table of $p(n)$ of MacMahon, Ramanujan conjectured that

$$p(5n + 4) \equiv 0 (\text{mod } 5),$$
$$p(7n + 5) \equiv 0 (\text{mod } 7),$$

and

$$p(11n + 6) \equiv 0 (\text{mod } 11).$$

Later, all the above conjecture were proved. For details of the above proofs and history, one may refer to [7].

Motivated by the above properties of partition function, H. C. Chan [6] studied and extended similar works to the restricted partition function. In fact, he define

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{2n})}$$

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and proved that $a(3n+2) \equiv 0 (\bmod 3)$. These results motivates many mathematician to work on properties of partition functions. See for example [2],[3], [10].

In this article, our aim is to establish explicit formulas in terms of divisor functions for a restricted partition function $P_{k,a,b,c,d,e}(n)$ and $Q_{k,a,b,c}(n)$ which are defined as follows:

Let $P^e_{k,a,b,c,d,e}(n)$[or $P^o_{k,a,b,c,d,e}(n)$] denoted the number of partition of n such that

i. Parts are congruent to $0, \pm a, \pm b, \pm c, \pm d, \pm e, \pm\frac{k}{2}, k (\bmod 2k)$.
ii. Parts are congruent to $0, \pm a, \pm b, \pm c, \pm d \ (\bmod 2k)$ are distinct.
iii. Parts are congruent to $0, \pm\frac{k}{2}, k \ (\bmod 2k)$ are having two colors.
iv. Number of parts are congruent to $0, \pm c, \pm d, k \ (\bmod 2k)$ be even(odd).

Let $Q^e_{k,a,b,c}(n)$[or $Q^o_{k,a,b,c}(n)$] denoted the number of partition of n such that

i. Parts are congruent to $0, \pm a, \pm b, \pm c, \pm\frac{k}{2}, k (\bmod 2k)$.
ii. Parts are congruent to $0, \pm a, \pm b, \pm c, k (\bmod 2k)$ are distinct.
iii. Parts are congruent to $0, \pm\frac{k}{2}, k \ (\bmod 2k)$ are having two colors.
iv. Number of parts are congruent to $0, \pm c, k \ (\bmod 2k)$ be even(odd).

Let
$$P_{k,a,b,c,d,e}(n) = P^e_{k,a,b,c,d,e}(n) - P^o_{k,a,b,c,d,e}(n)$$
and
$$Q_{k,a,b,c}(n) = Q^e_{k,a,b,c}(n) - Q^o_{k,a,b,c}(n).$$
In the next section, we prove our main results. Now we shall recall certain definition which are required to prove our main results. For any complex number a and q with $|q| < 1$,
$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n).$$
For any integer n, let
$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$
Ramanujan theta function $f(a, b)$ is defined by
$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \ |ab| < 1.$$
From the Jacobi's triple product identity, we have
$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$
A special case of $f(a, b)$, $\varphi(q)$ is defined by
$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty.$$

By the above definition of $f(a,b)$ and $\varphi(q)$, one can easily see that

$$\sum_{n=0}^{\infty} P_{k,a,b,c,d,e}(n-m)q^n = q^m \psi^2(q^{\frac{k}{2}}) \frac{f(q^a,q^b)}{f(q^c,q^d)}, \text{ where } a+b=c+d=k \text{ and } e=2c. \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} Q_{k,a,b,c}(n-l)q^n = q^l \psi^2(q^{\frac{k}{2}}) \frac{f(q^a,q^b)}{f(q^c,q^d)}, \text{ where } a+b=c+d=k, c<d \tag{1.2}$$

Let $d_{x,y}(n)$ denotes the number of divisor of n which are congruent to $x (\text{mod } y)$ if n is a positive integer and $d_{x,y}(n)=0$ otherwise .

2. Main results

In this section, we establish explicit formulas in terms of divisor functions for a restricted partition function $Q_{k,a,b,c}(n)$ and $P_{k,a,b,c,d,e}(n)$.

Lemma 2.1. *For each $n \in \mathbb{N}$, we have*

$$\varphi^2(q) = \sum_{n=0}^{\infty} r_2(n) q^n$$

and

$$r_2(n) = 4\left[d_{1,4}(n) - d_{3,4}(n)\right],$$

where $r_2(n)$ is the number of representation of an integer n as a sum of two square number.

For a proof of the above lemma, one may refer to [1].

Theorem 2.2. *The following identity holds:*

$$Q_{6,1,5,2}(n-1) = d_{1,4}(n) - d_{3,4}(n) - d_{1,4}\left(\frac{n}{3}\right) + d_{3,4}\left(\frac{n}{3}\right).$$

Proof. From [5, p. 124], we have

$$4q\psi^2(q^3)\frac{f(q,q^5)}{f(q^2,q^4)} = \varphi^2(q) - \varphi^2(q^3).$$

Employing (1.2) and Lemma 2.1 in the above, we find that

$$\sum_{n=0}^{\infty} Q_{6,1,5,2}(n-1)q^n = \sum_{n=0}^{\infty}\left\{d_{1,4}(n) - d_{3,4}(n) - d_{1,4}\left(\frac{n}{3}\right) + d_{3,4}\left(\frac{n}{3}\right)\right\} q^n.$$

By comparing the coefficient of q^n in the above identity, we obtain the required result. □

Example: For n=6, we have

$Q^{e}_{6,1,5,2}(5)$	$Q^{o}_{6,1,5,2}(5)$	$Q_{6,1,5,2}(5)$
5	3_g+2, 3_r+2.	(-1)

(Table-1)

One can easily see that

$$d_{1,4}(6) = 1, d_{3,4}(6) = 1, d_{1,4}\left(\frac{6}{3}\right) = 1 \text{ and } d_{3,4}\left(\frac{6}{3}\right) = 0.$$

From Table-1 and the above, we have

$$Q_{6,1,5,2}(5) = d_{1,4}(6) - d_{3,4}(6) - d_{1,4}\left(\frac{6}{3}\right) + d_{3,4}\left(\frac{6}{3}\right) = -1.$$

This verifies the theorem for $n = 6$.

Theorem 2.3. *The following identity holds:*

$$P_{10,1,9,4,6,8}(n-2) + P_{10,3,7,2,8,4}(n-1) = d_{1,4}(n) - d_{3,4}(n) - d_{1,4}\left(\frac{n}{5}\right) + d_{3,4}\left(\frac{n}{5}\right).$$

Proof. From [5, p. 258], we have

$$4\psi^2(q^5)\left[q^2\frac{f(q,q^9)}{f(q^4,q^6)} + q\frac{f(q^3,q^7)}{f(q^2,q^8)}\right] = \varphi^2(q) - \varphi^2(q^5).$$

Employing (1.1) and Lemma 2.1 in the above, we obtain

$$\sum_{n=0}^{\infty}\{P_{10,1,9,4,6,8}(n-2) + P_{10,3,7,2,8,4}(n-1)\}\,q^n$$
$$= \sum_{n=0}^{\infty}\left\{d_{1,4}(n) - d_{3,4}(n) - d_{1,4}\left(\frac{n}{5}\right) + d_{3,4}\left(\frac{n}{5}\right)\right\}q^n.$$

By comparing the coefficient of q^n in the above identity, we obtain the required result. □

Example: For n=8, we have

$P^{e}_{10,1,9,4,6,8}(6)$	$P^{e}_{10,1,9,4,6,8}(6)$	$P^{e}_{10,1,9,4,6,8}(6)$
1+5_g, 1+5_r.	6	(1)

(Table-2)

$P^{e}_{10,3,7,2,8,4}(7)$	$P^{o}_{10,3,7,2,8,4}(7)$	$P_{10,3,7,2,8,4}(7)$
3+4, 7.	2+5_g, 2+5_r.	(0)

(Table-3)

One can easily see that

$$d_{1,4}(8) = 1, d_{3,4}(8) = 0, d_{1,4}\left(\frac{8}{5}\right) = 0 \text{ and } d_{3,4}\left(\frac{8}{5}\right) = 0.$$

From Table-2,3 and the above, we have

$$P_{10,1,9,4,6,8}(6) + P_{10,3,7,2,8,4}(7) = d_{1,4}(8) - d_{3,4}(8) - d_{1,4}\left(\frac{8}{5}\right) + d_{3,4}\left(\frac{8}{5}\right) = 1$$

This verifies the theorem for $n = 8$.

Theorem 2.4. *The following identity holds:*

$$P_{14,1,13,6,9,12}(n-3) + P_{14,3,11,4,10,8}(n-2) + P_{14,5,9,2,12,4}(n-1) = d_{1,4}(n) - d_{3,4}(n) - d_{1,4}\left(\frac{n}{7}\right) + d_{3,4}\left(\frac{n}{7}\right).$$

Proof. From [11], we have

$$4q^3\psi^2(q^7)\left[\frac{f(q,q^{13})}{f(q^6,q^8)} + \frac{f(q^3,q^{11})}{qf(q^4,q^{10})} + \frac{f(q^5,q^9)}{q^2f(q^2,q^{12})}\right] = \varphi^2(q) - \varphi^2(q^7).$$

Employing (1.1) and Lemma 2.1 in the above, we obtain

$$\sum_{n=0}^{\infty}\{P_{14,1,13,6,9,12}(n-3) + P_{14,3,11,4,10,8}(n-2) + P_{14,5,9,2,12,4}(n-1)\}\,q^n = \sum_{n=0}^{\infty}\left\{d_{1,4}(n) - d_{3,4}(n) - d_{1,4}\left(\frac{n}{7}\right) + d_{3,4}\left(\frac{n}{7}\right)\right\}q^n.$$

By comparing the coefficient of q^n in the above identity, we obtain the required result. □

Theorem 2.5. *The following identity holds:*

$$\varphi^2(q) - \varphi^2(q^9) = 4q^4\psi^2(q^9)\left[\frac{f(q,q^{17})}{f(q^8,q^{10})} + \frac{f(q^3,q^{15})}{qf(q^6,q^{12})} + \frac{f(q^5,q^{13})}{q^2f(q^4,q^{14})} + \frac{f(q^7,q^{11})}{q^3f(q^2,q^{16})}\right].$$

Proof. From [5, p. 115], we have the following identity:

$$\varphi^2(q) = 1 + 4\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}}.$$

Which implies

$$\begin{aligned}\varphi^2(q) = 1 + 4\Bigg\{&\sum_{k=1}^{\infty}\frac{q^{9k}}{1+q^{18k}} + \sum_{k=0}^{\infty}\frac{q^{9k+1}}{1+q^{18k+2}} + \sum_{k=0}^{\infty}\frac{q^{9k+2}}{1+q^{18k+4}} + \sum_{k=0}^{\infty}\frac{q^{9k+3}}{1+q^{18k+6}}\\ &+ \sum_{k=0}^{\infty}\frac{q^{9k+4}}{1+q^{18k+8}} + \sum_{k=0}^{\infty}\frac{q^{9k+5}}{1+q^{18k+10}} + \sum_{k=0}^{\infty}\frac{q^{9k+6}}{1+q^{18k+12}} + \sum_{k=0}^{\infty}\frac{q^{9k+7}}{1+q^{18k+14}}\\ &+ \sum_{k=0}^{\infty}\frac{q^{9k+8}}{1+q^{18k+16}}\Bigg\}.\end{aligned}$$

This implies

$$\varphi^2(q)-\varphi^2(q^9)=4\Bigg\{\sum_{k=-\infty}^{\infty}\frac{q^{9k+1}}{1+q^{18k+2}}+\sum_{k=-\infty}^{\infty}\frac{q^{9k+2}}{1+q^{18k+4}}+\sum_{k=-\infty}^{\infty}\frac{q^{9k+3}}{1+q^{18k+6}} \\ +\sum_{k=-\infty}^{\infty}\frac{q^{9k+4}}{1+q^{18k+8}}\Bigg\}. \tag{2.1}$$

The following is the famous Ramanujan's ${}_1\psi_1$ summation formula [5, p. 34]:

$$\sum_{k=-\infty}^{\infty}\frac{(a;q)_k}{(b;q)_k}z^k=\frac{(az;q)_\infty(q/az;q)_\infty(q;q)_\infty(b/a;q)_\infty}{(z;q)_\infty(b/az;q)_\infty(b;q)_\infty(q/a;q)_\infty},\quad |b/a|<|z|<1. \tag{2.2}$$

Setting $b=aq$ and $z=q^9$ in the above equation and then replacing q by q^{18}, we obtain

$$\sum_{k=-\infty}^{\infty}\frac{q^{9k}}{1-aq^{18k}}=\frac{f(-aq^9,-q^9/a)(q^{18};q^{18})_\infty^3}{f(-q^9,-q^9)f(-a,-q^{18}/a)}. \tag{2.3}$$

Which implies

$$\sum_{k=-\infty}^{\infty}\frac{q^{9k}}{1-aq^{18k}}=\psi^2(q^9)\frac{f(-aq^9,-q^9/a)}{f(-a,-q^{18}/a)}. \tag{2.4}$$

Employing (2.4) in (2.1) with $a=-q^2,-q^4,-q^6$, and $-q^8$, we obtain the required result. □

Theorem 2.6. *The following identity holds:*

$$P_{18,1,17,8,10,16}(n-4)+P_{18,3,15,6,12,12}(n-3)+P_{18,5,13,4,14,8}(n-2) \\ +P_{18,7,11,2,16,4}(n-1)=d_{1,4}(n)-d_{3,4}(n)-d_{1,4}\left(\frac{n}{9}\right)+d_{3,4}\left(\frac{n}{9}\right).$$

Proof. From previous theorem, we have

$$\left[\frac{f(q,q^{17})}{f(q^8,q^{10})}+\frac{f(q^3,q^{15})}{qf(q^6,q^{12})}+\frac{f(q^5,q^{13})}{q^2f(q^4,q^{14})}+\frac{f(q^7,q^{11})}{q^3f(q^2,q^{16})}\right]4q^4\psi^2(q^9)=\varphi^2(q)-\varphi^2(q^9).$$

Employing (1.1) and Lemma 2.1 in the above, we obtain

$$\sum_{n=0}^{\infty}\Big\{P_{18,1,17,8,10,16}(n-4)+P_{18,3,15,6,12,12}(n-3)+P_{18,5,13,4,14,8}(n-2) \\ +P_{18,7,11,2,16,4}(n-1)\Big\}q^n=\sum_{n=0}^{\infty}\left\{d_{1,4}(2n)-d_{3,4}(n)-d_{1,4}\left(\frac{n}{9}\right)+d_{3,4}\left(\frac{n}{9}\right)\right\}q^n.$$

By comparing the coefficient of q^n in the above identity, we obtain the required result. □

Theorem 2.7. *The following identity holds:*

$$\begin{aligned}P_{22,1,21,10,12,20}(n-5)+P_{22,3,9,8,14,16}(n-4)+P_{22,5,11,6,16,12}(n-3)\\+P_{22,7,15,4,18,8}(n-2)+P_{22,9,13,2,20,4}(n-1)\\=d_{1,4}(n)-d_{3,4}(n)-d_{1,4}\left(\frac{n}{11}\right)+d_{3,4}\left(\frac{n}{11}\right).\end{aligned}$$

Proof. From [8], we have

$$\left[\frac{f(q,q^{21})}{f(q^{10},q^{12})}+\frac{f(q^3,q^{19})}{qf(q^8,q^{14})}+\frac{f(q^5,q^{17})}{q^2f(q^6,q^{16})}+\frac{f(q^7,q^{15})}{q^3f(q^4,q^{18})}+\frac{f(q^9,q^{13})}{q^4f(q^2,q^{20})}\right]\\ \times 4q^5\psi^2(q^{11})=\varphi^2(q)-\varphi^2(q^{11}).$$

Employing (1.1) and Lemma 2.1 in the above, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\Big\{P_{22,1,21,10,12,20}(n-5)+P_{22,3,9,8,14,16}(n-4)+P_{22,5,11,6,16,12}(n-3)\\+P_{22,7,15,4,18,8}(n-2)+P_{22,9,13,2,20,4}(n-1)\Big\}q^n\\=\sum_{n=0}^{\infty}\left\{d_{1,4}(n)-d_{3,4}(n)-d_{1,4}\left(\frac{n}{11}\right)+d_{3,4}\left(\frac{n}{11}\right)\right\}q^n.\end{aligned}$$

By comparing the coefficient of q^n in the above identity, we obtain the required result. □

Theorem 2.8. *The following identity holds:*

$$\varphi^2(q)-\varphi^2(q^{13})=\psi^2(q^{13})q^6\times\\ \left[\frac{f(q,q^{25})}{f(q^{12},q^{14})}+\frac{f(q^3,q^{23})}{qf(q^{10},q^{16})}+\frac{f(q^5,q^{21})}{q^2f(q^8,q^{18})}+\frac{f(q^7,q^{19})}{q^3f(q^6,q^{20})}+\frac{f(q^9,q^{17})}{q^4f(q^4,q^{22})}+\frac{f(q^{11},q^{15})}{q^5f(q^2,q^{24})}\right].$$

Proof. From [5, 115], we have the following identity:

$$\varphi^2(q)=1+4\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}}.$$

Which implies

$$\begin{aligned}\varphi^2(q)=1+4\Big\{&\sum_{k=1}^{\infty}\frac{q^{13k}}{1+q^{26k}}+\sum_{k=0}^{\infty}\frac{q^{13k+1}}{1+q^{26k+2}}+\sum_{k=0}^{\infty}\frac{q^{13k+2}}{1+q^{26k+4}}+\sum_{k=0}^{\infty}\frac{q^{13k+3}}{1+q^{26k+6}}\\&+\sum_{k=0}^{\infty}\frac{q^{13k+4}}{1+q^{26k+8}}+\sum_{k=0}^{\infty}\frac{q^{13k+5}}{1+q^{26k+10}}+\sum_{k=0}^{\infty}\frac{q^{13k+6}}{1+q^{26k+12}}+\sum_{k=0}^{\infty}\frac{q^{13k+7}}{1+q^{26k+14}}\\&+\sum_{k=0}^{\infty}\frac{q^{13k+8}}{1+q^{26k+16}}+\sum_{k=0}^{\infty}\frac{q^{13k+9}}{1+q^{26k+18}}+\sum_{k=0}^{\infty}\frac{q^{13k+10}}{1+q^{26k+20}}+\sum_{k=0}^{\infty}\frac{q^{13k+11}}{1+q^{26k+22}}\\&.+\sum_{k=0}^{\infty}\frac{q^{13k+12}}{1+q^{26k+24}}\Big\}.\end{aligned}$$

This implies

$$\varphi^2(q) - \psi^2(q^{13}) = 4\Bigg\{\sum_{k=-\infty}^{\infty} \frac{q^{13k+1}}{1+q^{26k+2}} + \sum_{k=-\infty}^{\infty} \frac{q^{13k+2}}{1+q^{26k+4}} + \sum_{k=-\infty}^{\infty} \frac{q^{13k+3}}{1+q^{26k+6}} + \sum_{k=-\infty}^{\infty} \frac{q^{13k+4}}{1+q^{26k+8}} + \sum_{k=-\infty}^{\infty} \frac{q^{13k+5}}{1+q^{26k+10}} + \sum_{k=-\infty}^{\infty} \frac{q^{13k+6}}{1+q^{26k+12}}\Bigg\} \tag{2.5}$$

Setting $b = aq$ and $z = q^{13}$ in the Ramanujan's ${}_1\psi_1$ summation formula (2.2) and then replacing q by q^{26}, we obtain

$$\sum_{k=-\infty}^{\infty} \frac{q^{13k}}{1-aq^{26k}} = \frac{f(-aq^{13}, -q^{13}/a)(q^{26};q^{26})_\infty^3}{f(-q^{13}, -q^{13})f(-a, -q^{26}/a)}. \tag{2.6}$$

Which implies

$$\sum_{k=-\infty}^{\infty} \frac{q^{13k}}{1-aq^{26k}} = \psi^2(q^{13})\frac{f(-aq^{13}, -q^{13}/a)}{f(-a, -q^{26}/a)}. \tag{2.7}$$

Employing (2.7) in (2.5) with $a = -q^2, -q^4, -q^6, -q^8, -q^{10}$, and $-q^{12}$, we obtain the required result. □

THEOREM 2.9. *The following identity holds:*

$$P_{26,1,25,12,14,24}(n-6) + P_{26,3,23,10,16,20}(n-5) + P_{26,5,21,8,18,16}(n-4) + P_{26,7,19,6,20,12}(n-3) + P_{26,9,17,4,22,8}(n-2) + P_{26,11,15,2,24,4}(n-1) = d_{1,4}(n) - d_{3,4}(n) - d_{1,4}\left(\frac{n}{13}\right) + d_{3,4}\left(\frac{n}{13}\right).$$

PROOF. From previous theorem, we have

$$\left[\frac{f(q,q^{25})}{f(q^{12},q^{14})} + \frac{f(q^3,q^{23})}{qf(q^{10},q^{16})} + \frac{f(q^5,q^{21})}{q^2f(q^8,q^{18})} + \frac{f(q^7,q^{19})}{q^3f(q^6,q^{20})} + \frac{f(q^9,q^{17})}{q^4f(q^4,q^{22})} + \frac{f(q^{11},q^{15})}{q^5f(q^2,q^{24})}\right] \times q^6\psi^2(q^{13}) = \varphi^2(q) - \varphi^2(q^{13})$$

Employing (1.1) and Lemma 2.1 in the above, we obtain

$$\sum_{n=0}^{\infty}\Big\{P_{26,1,25,12,14,24}(n-6) + P_{26,3,23,10,16,20}(n-5) + P_{26,5,21,8,18,16}(n-4) + P_{26,7,19,6,20,12}(n-3) + P_{26,9,17,4,22,8}(n-2) + P_{26,11,15,2,24,4}(n-1)\Big\}q^n = \sum_{n=0}^{\infty}\left\{d_{1,4}(n) - d_{3,4}(n) - d_{1,4}\left(\frac{n}{13}\right) + d_{3,4}\left(\frac{n}{13}\right)\right\}q^n.$$

By comparing the coefficient of q^n in the above identity, we obtain the required result. □

Theorem 2.10. *The following identity holds:*

$$\begin{aligned}\varphi^2(q)&-\varphi^2(q^{15})\\ &=4\psi^2(q^{15})q^7\Big[\frac{f(q,q^{29})}{f(q^{14},q^{16})}+\frac{f(q^3,q^{27})}{qf(q^{12},q^{18})}+\frac{f(q^5,q^{25})}{q^2f(q^{10},q^{20})}\\ &\quad+\frac{f(q^7,q^{23})}{q^3f(q^8,q^{22})}+\frac{f(q^9,q^{21})}{q^4f(q^6,q^{24})}+\frac{f(q^{11},q^{19})}{q^5f(q^4,q^{26})}+\frac{f(q^{13},q^{17})}{q^6f(q^2,q^{28})}\Big].\end{aligned}$$

Proof. From [5, p. 115], we have the following identity:

$$\varphi^2(q)=1+4\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}}.$$

Which implies

$$\begin{aligned}\varphi^2(q^2)=1+4\Big\{&\sum_{k=1}^{\infty}\frac{q^{15k}}{1+q^{30k}}+\sum_{k=0}^{\infty}\frac{q^{15k+1}}{1+q^{30k+3}}+\sum_{k=0}^{\infty}\frac{q^{15k+2}}{1+q^{30k+4}}+\sum_{k=0}^{\infty}\frac{q^{15k+3}}{1+q^{30k+6}}\\ &+\sum_{k=0}^{\infty}\frac{q^{15k+4}}{1+q^{30k+8}}+\sum_{k=0}^{\infty}\frac{q^{15k+5}}{1+q^{30k+10}}+\sum_{k=0}^{\infty}\frac{q^{15k+6}}{1+q^{30k+12}}+\sum_{k=0}^{\infty}\frac{q^{15k+7}}{1+q^{30k+14}}\\ &+\sum_{k=0}^{\infty}\frac{q^{15k+8}}{1+q^{30k+16}}+\sum_{k=0}^{\infty}\frac{q^{15k+9}}{1+q^{30k+18}}+\sum_{k=0}^{\infty}\frac{q^{15k+10}}{1+q^{30k+20}}+\sum_{k=0}^{\infty}\frac{q^{15k+11}}{1+q^{30k+22}}\\ &.+\sum_{k=0}^{\infty}\frac{q^{15k+12}}{1+q^{30k+24}}+\sum_{k=0}^{\infty}\frac{q^{15k+13}}{1+q^{30k+26}}+\sum_{k=0}^{\infty}\frac{q^{15k+14}}{1+q^{30k+28}}\Big\}.\end{aligned}$$

This implies

$$\begin{aligned}\varphi^2(q^2)-\varphi^2(q^{15})=4\Big\{&\sum_{k=-\infty}^{\infty}\frac{q^{15k+1}}{1+q^{30k+2}}+\sum_{k=-\infty}^{\infty}\frac{q^{15k+2}}{1+q^{30k+4}}+\sum_{k=-\infty}^{\infty}\frac{q^{15k+3}}{1+q^{30k+6}}\\ &+\sum_{k=-\infty}^{\infty}\frac{q^{15k+4}}{1+q^{30k+8}}+\sum_{k=-\infty}^{\infty}\frac{q^{15k+5}}{1+q^{30k+10}}+\sum_{k=-\infty}^{\infty}\frac{q^{15k+6}}{1+q^{30k+12}}\\ &+\sum_{k=-\infty}^{\infty}\frac{q^{15k+7}}{1+q^{30k+14}}\Big\}.\end{aligned}\tag{2.8}$$

Setting $b=aq$ and $z=q^{15}$ in the Ramanujan's ${}_1\psi_1$ summation formula (2.2) and then replacing q by q^{30}, we obtain

$$\sum_{k=-\infty}^{\infty}\frac{q^{15k}}{1-aq^{30k}}=\frac{f(-aq^{15},-q^{15}/a)(q^{30};q^{30})_\infty^3}{f(-q^{15},-q^{15})f(-a,-q^{30}/a)}.\tag{2.9}$$

Which implies

$$\sum_{k=-\infty}^{\infty}\frac{q^{15k}}{1-aq^{30k}}=\psi^2(q^{15})\frac{f(-aq^{15},-q^{15}/a)}{f(-a,-q^{30}/a)}.\tag{2.10}$$

Employing (2.10) in (2.8) with $a = -q^2, -q^4, -q^6, -q^8, -q^{10}, -q^{12}$ and $-q^{14}$, we obtain the required result.

□

Theorem 2.11. *The following identity holds:*

$$P_{30,1,29,14,16,28}(n-7) + P_{30,3,27,12,18,24}(n-6) + P_{30,5,25,10,20,20}(n-5) + P_{30,7,23,8,22,16}(n-4) + P_{30,9,21,6,24,12}(n-3) + P_{30,11,19,4,26,8}(n-2) + P_{30,13,17,2,28,4}(n-1) = d_{1,4}(n) - d_{3,4}(n) + d_{1,4}\left(\frac{n}{15}\right) - d_{3,4}\left(\frac{n}{15}\right).$$

Proof. From previous theorem, we have

$$\Bigg[\frac{f(q,q^{29})}{f(q^{14},q^{16})} + \frac{f(q^3,q^{27})}{qf(q^{12},q^{18})} + \frac{f(q^5,q^{25})}{q^2f(q^{10},q^{20})} + \frac{f(q^7,q^{23})}{q^3f(q^8,q^{22})} + \frac{f(q^9,q^{21})}{q^4f(q^6,q^{24})} + \frac{f(q^{11},q^{19})}{q^5f(q^4,q^{26})} + \frac{f(q^{13},q^{17})}{q^6f(q^2,q^{28})}\Bigg] \times 4q^7\psi^2(q^{15}) = \varphi^2(q) - \varphi^2(q^{15})$$

Employing (1.1) and Lemma 2.1 in the above, we obtain

$$\sum_{n=0}^{\infty}\Big\{P_{30,1,29,14,16,28}(n-7) + P_{30,3,27,12,18,24}(n-6) + P_{30,5,25,10,20,20}(n-5) + P_{30,7,23,8,22,16}(n-4) + P_{30,9,21,6,24,12}(n-3) + P_{30,11,19,4,26,8}(n-2) + P_{30,13,17,2,28,4}(n-1)\Big\}q^n = \sum_{n=0}^{\infty}\left\{d_{1,4}(2n+1) - d_{3,4}(2n+1) + d_{1,4}\left(\frac{n}{15}\right) - d_{3,4}\left(\frac{n}{15}\right)\right\}q^n.$$

By comparing the coefficient of q^n in the above identity , we obtain the required result. □

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