GANITA, Vol.74(2), 2024, 473-479

PÁL TYPE (1; 0) INTERPOLATION PROCESS

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Abstract

The aim of this paper is to study a Pál type interpolation problem when derivatives are prescribed on the zeros of certain polynomial, say $W(x)$, and function values are prescribed on the zeros of the polynomial $aW(x) + bW'(x)$, where a, b are constants and $W'(x)$ is the derivative of $W(x)$.

2010 *Mathematics subject classification:* primary 41A05.
Keywords and phrases: Zeros, interpolation, Pál type interpolation.

1. Introduction

In 1975, L G Pál [8] introduced an interpolation process when function values and derivatives are prescribed on $\{x_k\}_{k=1}^{n}$ and $\{x'_k\}_{k=1}^{n-1}$ respectively which are interscaled such that

$$-\infty < x_n < x'_{n-1} < x_{n-2} < \ldots < x_2 < x'_1 < x_1 < \infty$$

and are the zeros of the polynomial

$$W_n(x) = \Pi_{k=1}^{n}(x - x_k) \tag{1.1}$$

and

$$W'_n(x) = n\Pi_{k=1}^{n-1}(x - x'_k) \tag{1.2}$$

respectively. After which many mathematicians like [1, 3, 6, 9, 14] considered this problem on different sets of inter scaled points which are the zeros of certain classical orthogonal polynomials. They proved the existence, uniqueness of the interpolatory polynomial and also obtained its explicit representation. This Pál type interpolation problem was also considered [4, 5] when the nodes corresponding to the function value and first derivative were interchanged, that is when function values were prescribed on the zeros of $W'_n(x)$ and derivatives were prescribed on the zeros of $W_n(x)$. The explicit representation of the corresponding interpolation polynomial was also obtained.

In 1995, Szabó and Joó [2], showed that there exists a unique polynomial, say $R(x)$, of degree $\leq n + n^* - 1$ when function values and derivatives are prescribed on the zeros of (1.1) and $aW(x)+bW'(x)$ respectively, a and b are constants.They also obtained the explicit representation of the fundamental polynomials associated with $R(x)$.

Now a natural question arises: Do there exist a unique interpolatory polynomial of minimum possible degree if the nodes, x_k and x_k^*, are interchanged, that is, if function values and derivatives are prescribed on x_k^* and x_k respectively.

In this paper, we answer this question in affirmative. Precisely, we obtain the explicit representation of the uniquely determined interpolatory polynomial, say $R_{n+n^*-1}(x)$ of degree $\leq n+n^*-1$ satisfying the conditions:

$$\begin{cases} R_{n+n^*-1}(x_k^*) = y_k^*, & k = 1,2,\cdots,n^* \\ R'_{n+n^*-1}(x_k) = y'_k, & k = 1,2,\cdots,n. \end{cases} \tag{1.3}$$

where $\{y_k\}_{k=1}^{n^*}$ and $\{y_k\}_{k=1}^{n}$ are arbitrary given real numbers.

2. Preliminaries

Suppose that the polynomial $W(x)$ satisfy the differential equation

$$\Omega(x) = bW'(x) + aW(x) = (a_1x+b_1)(a_nx+b_n)W''(x) \tag{2.1}$$

where a_1, a_n, b_1, b_n are non-zero constants such that $a_1b_n - b_1a_n \neq 0$. Thus, $x_k^*, k = 1,2,\cdots,n^*$ are also the zeros of $(a_1x+b_1)(a_nx+b_n)W''(x)$. In particular, if $b=0$, then $R_{n+n^*-1}(x)$ is a Hermite interpolation polynomial, whereas, when $a=0$, then $R_{n+n^*-1}(x)$ is a Pál type $(1;0)$ interpolation. Here, we consider the case when, a_1, a_n, b_1, b_n are non-zero.

3. Explicit representation of the Fundamental Polynomials

The interpolatory polynomial, $R_{n+n^*-1}(x)$, of the lowest possible degree satisfying the conditions (1.3) can be explicitly represented as

$$R_{n+n^*-1}(x) = \sum_{k=1}^{n} y_k^* A_k(x) + \sum_{k=1}^{n^*} y'_k B_k(x) \tag{3.1}$$

where $\{A_k(x)\}_{k=1}^{n}$ and $\{B_k(x)\}_{k=1}^{n^*}$ are fundamental polynomials each of degree $\leq n+n^*-1$ and satisfying the conditions

$$\begin{cases} A_k(x_j^*) = \delta_{k,j}, & j,k = 1,2,\cdots,n^* \\ A'_k(x_j) = 0, & k,j = 1,2,\cdots,n \end{cases} \tag{3.2}$$

and

$$\begin{cases} B_k(x_j^*) = 0, & j,k = 1,2,\cdots,n^* \\ B'_k(x_j) = \delta_{k,j}, & k,j = 1,2,\cdots,n \end{cases} \tag{3.3}$$

where $\delta_{k,j}$ is the Kronecker's symbol.

3.1. Explicit representaion of $B_k(x)$ First we construct the polynomial $B_k(x), k = 1, 2, \cdots, n^*$, let

$$B_k(x) = \Omega(x)\eta_k(x) \tag{3.4}$$

where $\eta_k(x)$ is a polynomial of degree $\leq n-1$ such that $\eta_k(x_j^*) \neq 0, j = 1, 2, \cdots, n^*$. Obviously,

$$B_k(x_j^*) = 0, \quad j, k = 1, 2, \cdots, n^*. \tag{3.5}$$

Differentiating the equation (3.4), we have,

$$\begin{aligned} B_k'(x) &= \Omega'(x)\eta_k(x) + \Omega(x)\eta_k'(x) \\ &= \{aW'(x) + bW''(x)\}\eta_k(x) + \{aW(x) + bW'(x)\}\eta_k'(x) \end{aligned} \tag{3.6}$$

Also, as $B_k'(x_j) = \delta_{k,j}$ for $k, j = 1, 2, \cdots, n$, it can also be represented as

$$B_k'(x) = \frac{W(x)}{x - x_k} V_k(x)$$

where $V_k(x)$ is a polynomial. Thus

$$\{aW'(x) + bW''(x)\}\eta_k(x) + \{aW(x) + bW'(x)\}\eta_k'(x) = \frac{W(x)}{x - x_k} V_k(x) \tag{3.7}$$

We know that, a linear combination of $W(x)$ and $W'(x)$, with the help of the second order linear differential equation satisfied by $W(x)$, can be written in the terms of $W''(x)$, that is,

$$\Omega(x) = aW(x) + bW'(x) = (a_1x + b_1)(a_2x + b_2)W''(x) \tag{3.8}$$

where a, b, a_1, a_2, b_1, b_2 are constants. Hence substituting the value of $W''(x)$ from (3.8) in (3.7), we get

$$\left\{aW'(x) + \frac{b\,(aW(x) + bW'(x))}{(a_1x + b_1)(a_2x + b_2)}\right\}\eta_k(x) + (aW(x) + bW'(x))\,\eta_k'(x) = \frac{W(x)}{x - x_k} V_k(x)$$

Thus,

$$W'(x)\left[\left\{a(x - x_k) + \frac{b^2(x - x_k)}{(a_1x + b_1)(a_2x + b_2)}\right\}\eta_k(x) + b(x - x_k)\eta_k'(x)\right] =$$

$$W(x)\left[V_k(x) - a(x - x_k)\eta_k'(x) - \frac{ab(x - x_k)}{(a_1x + b_1)(a_2x + b_2)}\eta_k(x)\right].$$

Since $(W(x), W'(x)) = 1$ (the root of $W(x)$ are simple) therefore,

$$b(x - x_k)\eta_k'(x) + \left\{a(x - x_k) + \frac{b^2(x - x_k)}{(a_1x + b_1)(a_2x + b_2)}\right\}\eta_k(x) = W(x)\lambda_k(x) \tag{3.9}$$

and

$$V_k(x) - a(x - x_k)\eta_k'(x) - \frac{ab(x - x_k)}{(a_1x + b_1)(a_2x + b_2)}\eta_k(x) = W'(x)\lambda_k(x). \tag{3.10}$$

where $\lambda_k(x)$ is a polynomial such that $\lambda_k(x_k) \neq 0$. From equation (3.9), we have,

$$\eta_k'(x) + \left[\left\{\frac{a}{b} + \frac{b}{(a_1x+b_1)(a_2x+b_2)}\right\}\right]\eta_k(x) = \frac{W(x)}{b(x-x_k)}\lambda_k(x) \tag{3.11}$$

which is a first order liner differential equation having the solution as

$$\eta_k(x) = \frac{1}{b}\left\{\frac{(a_1x+b_1)}{(a_2x+b_2)}\right\}^C e^{\left(\frac{ax}{b}\right)} \int e^{\left(-\frac{ax}{b}\right)} \left\{\frac{(a_1x+b_1)}{(a_2x+b_2)}\right\}^{-C} \frac{W(x)}{x-x_k}\lambda_k(x)dx \tag{3.12}$$

Therefore from equation (3.4), we have

$$B_k(x) = \frac{\Omega(x)e^{\left(\frac{ax}{b}\right)}}{b}\left\{\frac{(a_1x+b_1)}{(a_2x+b_2)}\right\}^C \int e^{\left(-\frac{ax}{b}\right)} \left\{\frac{(a_1x+b_1)}{(a_2x+b_2)}\right\}^{-C} \frac{W(x)}{x-x_k}\lambda_k(x)dx \tag{3.13}$$

Now we need to determine $\lambda_k(x)$. Since $B_k'(x_k) = 1$, hence by equation (3.6) and (3.11), we have

$$1 = b\eta_k'(x_k) + \left\{a + \frac{b^2}{(a_1x_k+b_1)(a_2x_k+b_2)}\right\}\eta_k(x_k) \tag{3.14}$$

$$= W'(x_k)\lambda_k(x_k) \tag{3.15}$$

thus, $\lambda_k(x_k) = 1/(W'(x_k))$. For the polynomial of minimum possible degree, we consider

$$\lambda_k(x) = \frac{1}{W'(x_k)}, k = 1, 2, \cdots, n. \tag{3.16}$$

Thus,

$$B_k(x) = \frac{\Omega(x)e^{\left(\frac{ax}{b}\right)}}{b}\left\{\frac{(a_1x+b_1)}{(a_2x+b_2)}\right\}^C \int e^{\left(-\frac{ax}{b}\right)} \left\{\frac{(a_1x+b_1)}{(a_2x+b_2)}\right\}^{-C} \frac{W(x)}{(x-x_k)W'(x_k)}dx.$$

3.2. Explicit representaion of $A_k(x)$ Now, we construct the polynomial $\{A_k(x)\}, k = 1, 2, \cdots, n$.
Let

$$A_k(x) = \left(\frac{\Omega(x)}{x-x_k^*}\right)S_k(x) \tag{3.17}$$

where $S_k(x)$ is a polynomial of degree $\leq n$ such that $S_k(x_j^*) \neq 0, j = 1, 2, \cdots, n$. Differentiating equation (3.17), we get

$$A_k'(x) = \left[\frac{\Omega'(x)}{(x-x_k^*)} - \frac{\Omega(x)}{(x-x_k^*)^2}\right]S_k(x) + \frac{\Omega(x)}{(x-x_k^*)}S_k'(x) \tag{3.18}$$

Also, since $A_k'(x_j) = 0$ for $k, j = 1, 2, \cdots, n$, it can be represented as

$$A_k'(x) = W(x)T_k(x) \tag{3.19}$$

where $T_k(x)$ is polynomial. Thus, we have

$$\left[\frac{\Omega'(x)}{(x-x_k^*)} - \frac{\Omega(x)}{(x-x_k^*)^2}\right] S_k(x) + \frac{\Omega(x)}{(x-x_k^*)} S'_k(x) = W(x)T_k(x) \tag{3.20}$$

which, by (3.8), implies

$$\left[(x-x_k^*)\{aW'(x)+bW''(x)\} - \{aW(x)+bW'(x)\}\right] S_k(x) + (aW(x)+bW'(x))(x-x_k^*)S'_k(x) = (x-x_k^*)^2 W(x)T_k(x)$$

$$\left[(x-x_k^*)\left\{aW'(x) + b\frac{aW(x)+bW'(x)}{(a_1x+b_1)(a_2x+b_2)}\right\} - \{aW(x)+bW'(x)\}\right] S_k(x) + +(x-x_k^*)(aW(x)+bW'(x))S'_k(x) = (x-x_k^*)^2 W(x)T_k(x).$$

Rearranging the terms in the above equation, we get

$$W'(x)\left[\left\{a(x-x_k^*) + \frac{b^2}{(a_1x+b_1)(a_2x+b_2)} - b\right\} S_k(x) + b(x-x_k^*)S'_k(x)\right] = W(x)\left[(x-x_k^*)^2 T_k(x) - \left\{\frac{ab(x-x_k^*)}{(a_1x+b_1)(a_2x+b_2)} - a\right\} S_k(x) - a(x-x_k^*)S'_k(x)\right].$$

Since, $(W(x), W'(x)) = 1$,we have

$$\left[a(x-x_k^*) + \frac{b^2(x-x_k^*)}{(a_1x+b_1)(a_2x+b_2)} - b\right] S_k(x) + b(x-x_k^*)S'_k(x) = W(x)\nu_k(x) \tag{3.21}$$

$$(x-x_k^*)^2 T_k(x) - \left\{\frac{ab(x-x_k^*)}{(a_1x+b_1)(a_2x+b_2)} - a\right\} S_k(x) - a(x-x_k^*)S'_k(x) = W'(x)\nu_k(x) \tag{3.22}$$

where $\nu_k(x)$ is a polynomial. From (3.21), we have

$$\frac{S'_k(x)}{(x-x_k^*)} - \frac{S_k(x)}{(x-x_k^*)^2} + \left[\frac{a}{b} + \frac{b}{(a_1x+b_1)(a_2x+b_2)}\right] \frac{S_k(x)}{(x-x_k^*)} = \frac{1}{b(x-x_k^*)^2} W(x)\lambda_k(x)$$

$$\left(\frac{S_k(x)}{(x-x_k^*)}\right)' + \left[\frac{a}{b} + \frac{b}{(a_1x+b_1)(a_2x+b_2)}\right] \frac{S_k(x)}{(x-x_k^*)} = \frac{W(x)}{b(x-x_k^*)^2} \nu_k(x)$$

which is a first order liner differential equation having the solution

$$\left(\frac{S_k(x)}{(x-x_k^*)}\right) e^{-\frac{ax}{b}} \left\{\frac{(a_1x+b_1)}{(a_2x+b_2)}\right\}^{-C} = \frac{1}{b}\int e^{-\frac{ax}{b}} \left\{\frac{(a_1x+b_1)}{(a_2x+b_2)}\right\}^{-C} \frac{1}{(x-x_k^*)^2} W(x)\nu_k(x)dx$$

where $C = -\frac{b}{(b_2a_1-b_1a_2)}$. Thus

$$A_k(x) = \frac{1}{b} e^{\frac{ax}{b}} \left\{\frac{(a_1x+b_1)}{(a_2x+b_2)}\right\}^{C} \Omega(x) \int e^{-\frac{ax}{b}} \left\{\frac{(a_1x+b_1)}{(a_2x+b_2)}\right\}^{-C} \frac{W(x)}{(x-x_k^*)^2} \nu_k(x)dx \tag{3.23}$$

Since, $A_k(x_k^*) = 1$ thus by (3.17), we have

$$\Omega'(x_k^*)S_k(x_k^*) = 1$$

Also by (3.21), we have

$$bS_k(x_k^*) = -W(x_k^*)\nu_k(x_k^*)$$

thus

$$\nu_k(x_k^*) = -\frac{b}{\Omega'(x_k^*)W(x_k^*)}. \tag{3.24}$$

We choose,

$$\nu_k(x) = -\frac{b}{\Omega'(x_k^*)W(x_k^*)} \tag{3.25}$$

thus

$$A_k(x) = -e^{\frac{ax}{b}}\left\{\frac{(a_1x+b_1)}{(a_2x+b_2)}\right\}^C \frac{\Omega(x)}{\Omega'(x_k^*)}\int e^{-\frac{ax}{b}}\left\{\frac{(a_1x+b_1)}{(a_2x+b_2)}\right\}^{-C}\frac{W(x)}{(x-x_k^*)^2W(x_k^*)}dx.$$

which completely determines the polynomial $R_(n + n * -1)$.

Conflicts of Interest: The authors declare no conflict of interest.

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