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LEVERAGE CENTRALITY ANALYSIS ON GENERALIZED TRANSFORMATION OF PATH GRAPH

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Abstract

This paper explores generalized transformation graphs denoted as G^{ab}, deriving expressions for node leverage centrality within these graphs. Our primary emphasis lies in the transformation of path graph $P_n, n \geq 5$ and we establish precise formulas for the leverage centrality across various forms of path transformations P_n^{++}, P_n^{+-}, P_n^{-+}, and P_n^{--} for $n \geq 5$.

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1. Introduction

Research in network and graph theory, focusing on structural properties, is a rapidly evolving field with increasing relevance. In this context, the structure of chemical compounds or networks is represented as graphs, where nodes represent atoms, and edges signify chemical bonds. A key approach in exploring these structural properties involves deriving quantitative metrics that encapsulate the entire network's structural information as real numbers. The centrality index of a vertex is one such numeric measure derived from the graphical representation of these structures. Joyce K.E. et al. [3] introduced the concept of leverage centrality of nodes in a graph. This article delves into the study of leverage centrality for vertices in generalized transformations of path graph P_n where $n \geq 5$. Leverage centrality serves as a valuable tool for probing the physical and chemical characteristics of structural entities and predicting relationships between molecules. Anam Rani, Muhammad Imran, Asima Razzaque, and Usman Ali [4] assert that indices associated with a vertex's degree play a significant role in the domains of theoretical chemistry and nanotechnology. Actually, the computation of degree-related indices has become a successful and significant area of research [1], [6], [10]. In contrast to the widely-used degree centrality, which assesses a vertex's importance solely based on its degree, leverage centrality takes into account the vertex's connectivity relative to the connectivities (degrees) of its neighbors.

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The paper follows this structure: Section 2 serves as the preliminary section, encompassing essential definitions and notations employed throughout the paper. In Section 3, we delve into the basic properties and propositions regarding generalized transformation graphs and leverage centrality. Section 4 presents our findings regarding the leverage centrality of vertices within the generalized transformation graphs derived from the path $P_n, n \geq 5$. Finally, in Section 5, we provide a summary of our work and offer insights into potential future research directions.

2. Preliminaries

Definition 2.1. [9] The degree of a vertex v is the number of edges incident to v and is denoted by $deg(v)$.

The formal definition of leverage centrality is as follows:

Definition 2.2. [8] Leverage centrality is a measure of the relationship between the degree of a given node v and the degree of each of its neighbors v_i averaged over all neighbors N_v and is defined as:

$$l(v) = \frac{1}{deg(v)} \sum_{v_i \in N_v} \frac{deg(v) - deg(v_i)}{deg(v) + deg(v_i)}$$

The definition highlights the uniqueness of this measure compared to existing ones, as it considers not just the degree of a given node but also takes into account the degrees of its neighbors.

Definition 2.3. [7] The leverage center of a graph is defined as the set of nodes having the highest leverage centrality in the graph.

Definition 2.4. [7] Bicentric leverage graphs are defined as graphs with exactly two leverage centers.

In a graph G, we represent its point set and line set as $V(G)$ and $E(G)$, respectively. A line within G, connecting points e_1 and e_2, is denoted as $e_1\ e_2$.

E. Sampathkumar and S.B.Chikkodimath [5] defined the semitotal graphs $T_1(G)$ and $T_2(G)$ of a given graph G as follows:

Definition 2.5. [5] The semitotal (line graph) $T_1(G)$ of G is the graph whose point set is $V(G) \cup E(G)$ where two points are adjacent if and only if

1. they are adjacent lines of G or
2. one is a point of G and other is a line of G incident with it.

Definition 2.6. [5] The semitotal (point graph) $T_2(G)$ of G is the graph whose point set is $V(G) \cup E(G)$ where two points are adjacent if and only if

1. they are adjacent points of G or
2. one is a point of G and other is a line of G incident with it.

Basavanagoud B., Gutman Ivan and Desai Veena [1] generalized the concept of semitotal-point graph by introducing some new graphical transformations: Let $G = (V, E)$ be a graph, and let α, β be two elements of $V(G) \cup E(G)$. The associativity of α and β is + if they are adjacent or incident in G, otherwise is −. Let ab be a 2-permutation of the set $\{+, -\}$. Here α and β correspond to the first term a of ab if both α and β are in $V(G)$, whereas α and β correspond to the second term b of ab if one of α and β is in $V(G)$ and the other is in $E(G)$. The generalized transformation graph G^{ab} of G is defined on the vertex set $V(G) \cup E(G)$. Two vertices α and β of G^{ab} are joined by an edge if and only if their associativity in G is consistent with the corresponding term of ab.

Thus we can obtain four distinct graphical transformations of graphs corresponding to the four distinct 2-permutations of $\{+, -\}$. Here G^{++} is just the semitotal point graph $T_2(G)$ of G, whereas the other generalized transformation graphs are G^{+-}, G^{-+} and G^{--}.

In other words, the generalized transformation graph G^{ab} is a graph whose vertex set is $V(G) \cup E(G)$ and $\alpha, \beta \in V(G^{ab})$. Then α and β are adjacent in G^{ab} if and only if the following hold:

1. $\alpha, \beta \in V(G)$, α, β are adjacent in G if $a = +$ and α, β are not adjacent in G if $a = -$.
2. $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $b = +$ and α, β are not incident in G if $b = -$.

The vertex v_i of G^{ab} corresponding to a vertex v_i of G is referred to as a point vertex. The vertex e_i of G^{ab} corresponding to an edge e_i of G is referred to as a line vertex.

In this paper, we obtain expressions for the leverage centralities of vertices of the generalized transformation path graphs P_n^{++}, P_n^{+-}, P_n^{-+}, and P_n^{--} for $n \geq 5$.

3. Some Basic Propositions

Proposition 3.1. *[1]*

Let G be an (n, m) graph. Then the degrees of point and line vertices in G^{ab} are

1. $deg_{G^{++}}(v_i) = 2deg_G(v_i)$ *and* $deg_{G^{++}}(e_i) = 2$
2. $deg_{G^{+-}}(v_i) = m$ *and* $deg_{G^{+-}}(e_i) = n - 2$
3. $deg_{G^{-+}}(v_i) = n - 1$ *and* $deg_{G^{-+}}(e_i) = 2$
4. $deg_{G^{--}}(v_i) = n + m - 1 - 2deg_G(v_i)$ *and* $deg_{G^{--}}(e_i) = n - 2$

Proposition 3.2. *[8] Let G be a graph with n vertices. For any vertex v,* $|l(v)| \leq 1 - \dfrac{2}{n}$. *Furthermore, these bounds are tight in the cases of stars and complete graphs.*

Proposition 3.3. *[8] For any graph G,* $\sum_{v \in G} l(v) \leq 0$.

Proposition 3.4. *[8]* $l(v) = 0$ *for every vertex* $v \in G$ *if and only if G is a regular graph.*

Theorem 3.5. *[8] In a graph G of order n, the maximum number of vertices with positive leverage centrality is* $n - 1$.

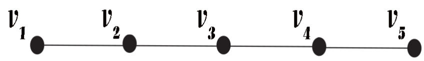

FIGURE 1. Path P_5

4. Leverage centrality of nodes in generalized transformation of Path graph

Let the vertices and the edges of the path $P_n(n \geq 5)$ be labeled as $\{v_1, v_2, ..., v_n\}$ and $\{e_1, e_2, ..., e_m\}$ respectively. Here the vertices v_1 and v_n are of degree 1 and all the other vertices are of degree 2. The vertices in the path $P_n(n \geq 5)$ can be classified as follows:

- Type I: $deg(v) = 1$
- Type II: $deg(v) = 2$ and is adjacent to Type I
- Type III: $deg(v) = 2$ and is adjacent only to degree 2 nodes

The leverage centrality of vertices in a path P_n, $n \geq 5$ is given as follows.

THEOREM 4.1. *[2] Let $G = P_n(n \geq 5)$ of order n. Then, for $v \in G$,*

$$l(v) = \begin{cases} \frac{-1}{3}, & \text{if } v = v_1, v_n \\ \frac{1}{6}, & \text{if } v = v_2, v_{n-1} \\ 0, & \text{if } v_3 \leq v \leq v_{n-2} \end{cases}$$

Thus, $P_n(n \geq 5)$ can be characterized as a bicentric leverage graph, where both v_2 and v_{n-1} serve as leverage centers.

Now we investigate the leverage centrality of vertices in the generalized transformation path graphs P_n^{++}, P_n^{+-}, P_n^{-+}, and P_n^{--}.

4.1. Leverage centrality of nodes in P_n^{++} : We state our first theorem as follows.

THEOREM 4.2. *Let P_n be a path, $n \geq 5$. Then for the line vertices $e_i \in P_n^{++}$*

$$l(e_i) = \begin{cases} \frac{-1}{6}, & \text{if } i = 1, m \\ \frac{-1}{3}, & \text{if } i = 2, 3, ..., m-1 \end{cases}$$

Proof: From the above proposition, we have $deg_{P_n^{++}}(e_i) = 2$ and $deg_{P_n^{++}}(v_i) = 2deg_{P_n}(v_i)\ \forall i$.

For $i = 1, m$, we have $N(e_i) = \{v_i, v_{i+1}\}$. Here for $i = 1, v_i$ is a Type I and v_{i+1} is a Type II vertex in P_n. Also, for $i = m, v_i$ is a Type II and v_{i+1} is a Type I vertex in P_n.

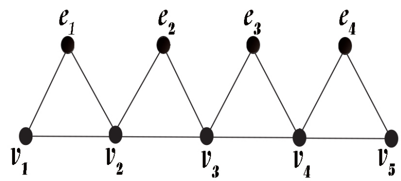

FIGURE 2. Transformation P_5^{++}

Hence

$$\begin{aligned} l(e_i) &= \frac{1}{2} \sum_{v_j \in N(e_i)} \left(\frac{2 - deg_{P_n^{++}}(v_j)}{2 + deg_{P_n^{++}}(v_j)} \right) \\ &= \frac{1}{2} \sum_{v_j \in N(e_i)} \left(\frac{2 - 2deg_{P_n}(v_j)}{2 + 2deg_{P_n}(v_j)} \right) \\ &= \frac{1}{2} \left(\frac{2 - 2deg_{P_n}(v_i)}{2 + 2deg_{P_n}(v_i)} + \frac{2 - 2deg_{P_n}(v_{i+1})}{2 + 2deg_{P_n}(v_{i+1})} \right) \\ &= \frac{1}{2} \left(\frac{2-4}{6} \right) \\ &= \frac{-1}{6} \end{aligned}$$

Now for $2 \leq i \leq m-1$, we have $N(e_i) = \{v_i, v_{i+1}\}$. Here $v_i's$ are either Type II or Type III vertices in P_n. Hence

$$\begin{aligned} l(e_i) &= \frac{1}{2} \sum_{v_j \in N(e_i)} \left(\frac{2 - deg_{P_n^{++}}(v_j)}{2 + deg_{P_n^{++}}(v_j)} \right) \\ &= \frac{1}{2} \sum_{v_j \in N(e_i)} \left(\frac{2 - 2deg_{P_n}(v_j)}{2 + 2deg_{P_n}(v_j)} \right) \\ &= \frac{1}{2} \left(\frac{2 - 2deg_{P_n}(v_i)}{2 + 2deg_{P_n}(v_i)} + \frac{2 - 2deg_{P_n}(v_{i+1})}{2 + 2deg_{P_n}(v_{i+1})} \right) \\ &= \frac{2-4}{6} \\ &= \frac{-1}{3} \end{aligned}$$

which completes the proof.

Now the analogous result for the leverage centrality of point vertices in P_n^{++} as follows:

THEOREM 4.3. *Let P_n be a path, $n \geq 5$. Then for the point vertices $v_i \in P_n^{++}$*

$$l(v_i) = \begin{cases} \frac{-1}{6}, & \text{if } i = 1, n \\ \frac{1}{4}, & \text{if } i = 2, n-1 \\ \frac{1}{6}, & \text{if } i = 3, ..., n-2 \end{cases}$$

Proof: Since $deg_{P_n^{++}}(v_i) = 2deg_{P_n}(v_i)\ \forall i$ and for $i = 1, n$ we have $deg_{P_n^{++}}(v_i) = 2$. $N(v_i) = \{e_i, v_{i+1}\}$ or $N(v_i) = \{e_{i-1}, v_{i-1}\}$ according as $i = 1$ or n. In either case, we have $deg_{P_n^{++}}(e_j)= 2$, $1 \leq j \leq m$ and the vertices $v_j's$ are of Type II in P_n. Therefore

$$\begin{aligned} l(v_i) &= \frac{1}{2} \sum_{a_j \in N(v_i)} \left(\frac{2 - deg_{P_n^{++}}(a_j)}{2 + deg_{P_n^{++}}(a_j)} \right) \\ &= \frac{1}{2} \left(\frac{2 - deg_{P_n^{++}}(e_j)}{2 + deg_{P_n^{++}}(e_j)} + \frac{2 - 2deg_{P_n}(v_j)}{2 + 2deg_{P_n}(v_j)} \right) \\ &= \frac{1}{2} \left(\frac{2-4}{6} \right) \\ &= \frac{-1}{6} \end{aligned}$$

Now for $i = 2, n-1$, $N(v_i) = \{e_{i-1}, e_i, v_{i-1}, v_{i+1}\}$. Here we note that one of the $v_j's$ are of Type I and the other is of Type III in P_n. Hence for these vertices in P_n^{++}

$$\begin{aligned} l(v_i) &= \frac{1}{4} \sum_{a_j \in N(v_i)} \left(\frac{4 - deg_{P_n^{++}}(a_j)}{4 + deg_{P_n^{++}}(a_j)} \right) \\ &= \frac{1}{4} \left[\sum_{e_j \in N(v_i)} \left(\frac{4 - deg_{P_n^{++}}(e_j)}{4 + deg_{P_n^{++}}(e_j)} \right) + \sum_{v_j \in N(v_i)} \left(\frac{4 - deg_{P_n^{++}}(v_j)}{4 + deg_{P_n^{++}}(v_j)} \right) \right] \\ &= \frac{1}{4} \left[\sum_{e_j \in N(v_i)} \left(\frac{4 - deg_{P_n^{++}}(e_j)}{4 + deg_{P_n^{++}}(e_j)} \right) + \sum_{v_j \in N(v_i)} \left(\frac{4 - 2deg_{P_n}(v_j)}{4 + 2deg_{P_n}(v_j)} \right) \right] \\ &= \frac{1}{4} \left(\frac{4-2}{6} \right) 3 \\ &= \frac{1}{4} \end{aligned}$$

Finally for $i = 3, ..., n-2$, $N(v_i) = \{e_{i-1}, e_i, v_{i-1}, v_{i+1}\}$. Here the $v_j's$ are of Type II or Type III in P_n. Hence

$$\begin{aligned} l(v_i) &= \frac{1}{4}\sum_{a_j\in N(v_i)}\left(\frac{4-deg_{P_n^{++}}(a_j)}{4+deg_{P_n^{++}}(a_j)}\right)\\ &= \frac{1}{4}\left[\sum_{e_j\in N(v_i)}\left(\frac{4-deg_{P_n^{++}}(e_j)}{4+deg_{P_n^{++}}(e_j)}\right)+\sum_{v_j\in N(v_i)}\left(\frac{4-deg_{P_n^{++}}(v_j)}{4+deg_{P_n^{++}}(v_j)}\right)\right]\\ &= \frac{1}{4}\left[\sum_{e_j\in N(v_i)}\left(\frac{4-deg_{P_n^{++}}(e_j)}{4+deg_{P_n^{++}}(e_j)}\right)+\sum_{v_j\in N(v_i)}\left(\frac{4-2deg_{P_n}(v_j)}{4+2deg_{P_n}(v_j)}\right)\right]\\ &= \frac{1}{4}\left(\frac{4-2}{6}\right)2\\ &= \frac{1}{6} \end{aligned}$$

which completes the proof.

COROLLARY 4.4. *The leverage centers of P_n^{++} and P_n are the same.*

4.2. Leverage centrality of nodes in P_n^{+-}

THEOREM 4.5. *Let P_n be a path, $n \geq 5$. Then for the line vertices $e_i \in P_n^{+-}$*

$$l(e_i) = \frac{n-m-2}{n+m-2} \quad, 1 \leq i \leq m.$$

Proof: From the proposition, we have $deg_{P_n^{+-}}(e_i) = n-2$ and $deg_{P_n^{+-}}(v_i) = m$. Also, $N(e_i) = \{v_i : v_i$ is not incident with e_i in $P_n\}$. Hence

$$\begin{aligned} l(e_i) &= \frac{1}{n-2}\sum_{v_j\in N(e_i)}\left(\frac{n-2-deg_{P_n^{+-}}(v_j)}{n-2+deg_{P_n^{+-}}(v_j)}\right)\\ &= \frac{1}{n-2}\left(\frac{n-2-m}{n-2+m}\right)n-2\\ &= \frac{n-m-2}{n+m-2} \end{aligned}$$

which completes the proof.

THEOREM 4.6. *Let P_n be a path, $n \geq 5$. Then for the point vertices $v_i \in P_n^{+-}$*

$$l(v_i) = \begin{cases} \dfrac{m-1}{m}\left(\dfrac{m-n+2}{m+n-2}\right), & \text{if } i = 1, n \\ \dfrac{m-2}{m}\left(\dfrac{m-n+2}{m+n-2}\right), & \text{if } i = 2, 3, ..., n-1 \end{cases}$$

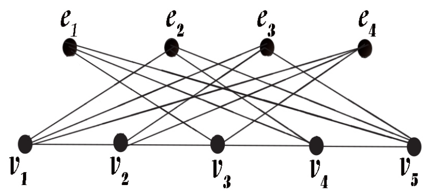

FIGURE 3. Transformation P_5^{+-}

Proof: For $i = 1, n$ we have $N(v_i) = \left\{v_{i+1}, e_j : 2 \le j \le m\right\}$ or $N(v_i) = \left\{v_{i-1}, e_j : 1 \le j \le m-1\right\}$ according as $i = 1$ or n. Hence

$$\begin{aligned} l(v_i) &= \frac{1}{m} \sum_{a_j \in N(v_i)} \left(\frac{m - deg_{P_n^{+-}}(a_j)}{m + deg_{P_n^{+-}}(a_j)} \right) \\ &= \frac{1}{m} \left(\frac{m - deg_{P_n^{+-}}(e_j)}{m + deg_{P_n^{+-}}(e_j)} + \frac{m - deg_{P_n^{+-}}(v_j)}{m + deg_{P_n^{+-}}(v_j)} \right) \\ &= \frac{1}{m} \left(\frac{m-(n-2)}{m+n-2} \right) m - 1 \\ &= \frac{m-1}{m} \left(\frac{m-n+2}{m+n-2} \right) \end{aligned}$$

Now for $i = 2, 3, ..., n-1$, $N(v_i) = \left\{v_{i-1}, v_{i+1}, e_j : j \neq i-1 \text{ and } j \neq i\right\}$. Therefore

$$\begin{aligned} l(v_i) &= \frac{1}{m} \sum_{a_j \in N(v_i)} \left(\frac{m - deg_{P_n^{+-}}(a_j)}{m + deg_{P_n^{+-}}(a_j)} \right) \\ &= \frac{1}{m} \left(\frac{m - deg_{P_n^{+-}}(e_j)}{m + deg_{P_n^{+-}}(e_j)} + \frac{m - deg_{P_n^{+-}}(v_j)}{m + deg_{P_n^{+-}}(v_j)} \right) \\ &= \frac{1}{m} \left(\frac{m-(n-2)}{m+n-2} \right) m - 2 \\ &= \frac{m-2}{m} \left(\frac{m-n+2}{m+n-2} \right) \end{aligned}$$

which completes the proof.

COROLLARY 4.7. *The leverage centers of P_n^{+-} and P_n are different, even though both are of the same leverage type (bicentric).*

4.3. Leverage centrality of nodes in P_n^{-+}

THEOREM 4.8. *Let P_n be a path, $n \geq 5$. Then for the line vertices $e_i \in P_n^{-+}$*

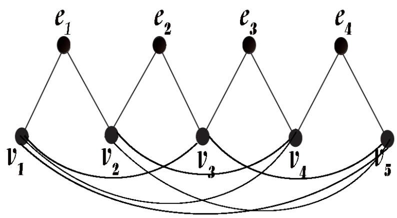

FIGURE 4. Transformation P_5^{-+}

$$l(e_i) = \frac{-(n-3)}{n+1} \quad , 1 \leq i \leq m.$$

Proof: We have $deg_{P_n^{-+}}(e_i) = 2$ and $deg_{P_n^{-+}}(v_i) = n-1$.
Also, $N(e_i) = \{v_i, v_{i+1}\}$, $1 \leq i \leq m$. Hence

$$\begin{aligned} l(e_i) &= \frac{1}{2} \sum_{v_j \in N(e_i)} \left(\frac{2 - deg_{P_n^{-+}}(v_j)}{2 + deg_{P_n^{-+}}(v_j)} \right) \\ &= \frac{1}{2} \left(\frac{2-(n-1)}{2+(n-1)} \right) 2 \\ &= \frac{-(n-3)}{n+1} \end{aligned}$$

which completes the proof.

THEOREM 4.9. *Let P_n be a path, $n \geq 5$. Then for the point vertices $v_i \in P_n^{-+}$*

$$l(v_i) = \begin{cases} \frac{1}{n-1} \left(\frac{n-3}{n+1} \right), & \text{if } i = 1, n \\ \frac{2}{n-1} \left(\frac{n-3}{n+1} \right), & \text{if } i = 2, 3, ..., n-1 \end{cases}$$

Proof:

For $i = 1, n$ we have

$$N(v_i) \ = \ \left\{ v_j : v_j \text{ is not adjacent to } v_i \text{ in } P_n \right\} \ \cup \ \left\{ e_j : e_j \text{ is incident with } v_i \text{ in } P_n \right\}.$$

Hence

$$
\begin{aligned}
l(v_i) &= \frac{1}{n-1}\sum_{a_j\in N(v_i)}\left(\frac{n-1-deg_{P_n^{-+}}(a_j)}{n-1+deg_{P_n^{-+}}(a_j)}\right)\\
&= \frac{1}{n-1}\left(\frac{n-1-deg_{P_n^{-+}}(e_j)}{n-1+deg_{P_n^{-+}}(e_j)}+\frac{n-1-deg_{P_n^{-+}}(v_j)}{n-1+deg_{P_n^{-+}}(v_j)}\right)\\
&= \frac{1}{n-1}\left(\frac{n-1-2}{n-1+2}\right)\\
&= \frac{1}{n-1}\left(\frac{n-3}{n+1}\right)
\end{aligned}
$$

Again, for $i = 2, 3, ..., n-1$,

$$N(v_i) \;=\; \left\{v_j : v_j \text{ is not adjacent to } v_i \text{ in } P_n\right\} \;\cup\; \left\{e_j : e_j \text{ is incident with } v_i \text{ in } P_n\right\}.$$

Hence

$$
\begin{aligned}
l(v_i) &= \frac{1}{n-1}\sum_{a_j\in N(v_i)}\left(\frac{n-1-deg_{P_n^{-+}}(a_j)}{n-1+deg_{P_n^{-+}}(a_j)}\right)\\
&= \frac{1}{n-1}\left(\frac{n-1-deg_{P_n^{-+}}(e_j)}{n-1+deg_{P_n^{-+}}(e_j)}+\frac{n-1-deg_{P_n^{-+}}(v_j)}{n-1+deg_{P_n^{-+}}(v_j)}\right)\\
&= \frac{1}{n-1}\left(\frac{n-1-2}{n-1+2}\right)2\\
&= \frac{2}{n-1}\left(\frac{n-3}{n+1}\right)
\end{aligned}
$$

which completes the proof.

Corollary 4.10. *The leverage centers of P_n^{-+} and P_n are different with distinct leverage types.*

4.4. Leverage centrality of nodes in P_n^{--}

Theorem 4.11. *Let P_n be a path, $n \geq 5$. Then for the line vertices $e_i \in P_n^{--}$*

$$
l(e_i) = \begin{cases}
\dfrac{1}{n-2}\left(\dfrac{-1}{3}-\dfrac{(n-3)(n-4)}{3n-8}\right) & \text{if } i = 1, m\\[2ex]
\dfrac{1}{n-2}\left(\dfrac{-2}{3}-\dfrac{(n-4)^2}{3n-8}\right), & \text{if } i = 2, 3, ..., m-1
\end{cases}
$$

Proof: We have $deg_{P_n^{--}}(v_i) = n + m - 1 - 2deg_{P_n}(v_i)$ and $deg_{P_n^{--}}(e_i) = n - 2$

Also, $N(e_i) = \left\{v_j : v_j \text{ is not incident with } e_i \text{ in } P_n\right\}$. Hence for $i = 1, m$

$$l(e_i) = \frac{1}{deg_{P_n^{--}}(e_i)} \sum_{v_j \in N(e_i)} \left(\frac{deg_{P_n^{--}}(e_i) - deg_{P_n^{--}}(v_j)}{deg_{P_n^{--}}(e_i) + deg_{P_n^{--}}(v_j)} \right)$$
$$= \frac{1}{n-2} \left(\frac{n-2-(2n-4)}{n-2+2n-4} + \left[\frac{n-2-(2n-6)}{n-2+2n-6} \right] (n-3) \right)$$
$$= \frac{1}{n-2} \left(\frac{-1}{3} - \frac{(n-3)(n-4)}{3n-8} \right)$$

Now for the remaining line vertices $e_i, 2 \leq i \leq m-1$

$$l(e_i) = \frac{1}{deg_{P_n^{--}}(e_i)} \sum_{v_j \in N(e_i)} \left(\frac{deg_{P_n^{--}}(e_i) - deg_{P_n^{--}}(v_j)}{deg_{P_n^{--}}(e_i) + deg_{P_n^{--}}(v_j)} \right)$$
$$= \frac{1}{n-2} \left(\left[\frac{n-2-(2n-4)}{n-2+2n-4} \right] 2 + \left[\frac{n-2-(2n-6)}{n-2+2n-6} \right] (n-4) \right)$$
$$= \frac{1}{n-2} \left(\frac{-2}{3} - \frac{(n-4)^2}{3n-8} \right)$$

which completes the proof.

THEOREM 4.12. *Let P_n be a path, $n \geq 5$. Then for the point vertices $v_i \in P_n^{--}$*

$$l(v_i) = \begin{cases} \frac{1}{2(n-2)} \left(\frac{n-2}{3} + \frac{n-3}{2n-5} \right), & \text{if } i = 1, n \\ \frac{1}{2(n-3)} \left(\frac{(n-3)(n-4)}{3n-8} - \frac{1}{2n-5} \right), & \text{if } i = 2, n-1 \\ \frac{1}{2(n-3)} \left(\frac{(n-3)(n-4)}{3n-8} - \frac{2}{2n-5} \right), & \text{if } i = 3, 4, ..., n-2 \end{cases}$$

Proof:

In this case, we have $deg_{P_n^{--}}(v_i) = n + m - 1 - 2deg_{P_n}(v_i)$ and $deg_{P_n^{--}}(e_i) = n - 2$

For $i = 1, n$ we have

$N(v_i) = \left\{v_j : v_j \text{ is not adjacent to } v_i \text{ in } P_n\right\} \cup \left\{e_j : e_j \text{ is not incident with } v_i \text{ in } P_n\right\}$. Hence

$$l(v_i) = \frac{1}{deg_{P_n^{--}}(v_i)} \sum_{a_j \in N(v_i)} \left(\frac{deg_{P_n^{--}}(v_i) - deg_{P_n^{--}}(a_j)}{deg_{P_n^{--}}(v_i) + deg_{P_n^{--}}(a_j)} \right)$$
$$= \frac{1}{2n-4} \left(\frac{2n-4-deg_{P_n^{--}}(e_j)}{2n-4+deg_{P_n^{--}}(e_j)} + \frac{2n-4-deg_{P_n^{--}}(v_j)}{2n-4+deg_{P_n^{--}}(v_j)} \right)$$
$$= \frac{1}{2n-4} \left(\left[\frac{2n-4-(n-2)}{2n-4+(n-2)} \right] (n-2) + \left[\frac{2n-4-(2n-6)}{2n-4+(2n-6)} \right] (n-3) \right)$$
$$= \frac{1}{2(n-2)} \left(\frac{n-2}{3} + \frac{n-3}{2n-5} \right)$$

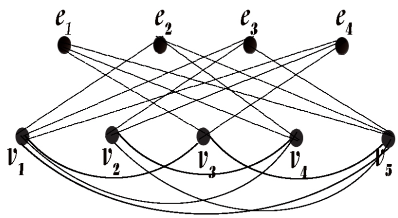

FIGURE 5. Transformation P_5^{--}

Now, for $i = 2, n-1$

$N(v_i) = \left\{v_j : v_j \text{ is not adjacent to } v_i \text{ in } P_n\right\} \cup \left\{e_j : e_j \text{ is not incident with } v_i \text{ in } P_n\right\}$.
Hence

$$\begin{aligned} l(v_i) &= \frac{1}{n+m-1-2deg_{P_n}(v_i)} \sum_{a_j \in N(v_i)} \left(\frac{n+m-1-2deg_{P_n}(v_i) - deg_{P_n^{--}}(a_j)}{n+m-1-2deg_{P_n}(v_i) + deg_{P_n^{--}}(a_j)} \right) \\ &= \frac{1}{2n-6} \left(\frac{2n-6-deg_{P_n^{--}}(e_j)}{2n-6+deg_{P_n^{--}}(e_j)} + \frac{2n-6-deg_{P_n^{--}}(v_j)}{2n-6+deg_{P_n^{--}}(v_j)} \right) \\ &= \frac{1}{2n-6} \left(\left[\frac{2n-6-(n-2)}{2n-6+n-2} \right](n-3) + \left[\frac{2n-6-(2n-4)}{2n-6+2n-4} \right] \right) \\ &= \frac{1}{2(n-3)} \left(\frac{(n-3)(n-4)}{3n-8} - \frac{1}{2n-5} \right) \end{aligned}$$

Now, for $i = 3, 4, ..., n-2$

$N(v_i) = \left\{v_j : v_j \text{ is not adjacent to } v_i \text{ in } P_n\right\} \cup \left\{e_j : e_j \text{ is not incident with } v_i \text{ in } P_n\right\}$.
Hence

$$\begin{aligned} l(v_i) &= \frac{1}{n+m-1-2deg_{P_n}(v_i)} \sum_{a_j \in N(v_i)} \left(\frac{n+m-1-2deg_{P_n}(v_i) - deg_{P_n^{--}}(a_j)}{n+m-1-2deg_{P_n}(v_i) + deg_{P_n^{--}}(a_j)} \right) \\ &= \frac{1}{2n-6} \left(\frac{2n-6-deg_{P_n^{--}}(e_j)}{2n-6+deg_{P_n^{--}}(e_j)} + \frac{2n-6-deg_{P_n^{--}}(v_j)}{2n-6+deg_{P_n^{--}}(v_j)} \right) \\ &= \frac{1}{2n-6} \left(\left[\frac{2n-6-(n-2)}{2n-6+n-2} \right](n-3) + \left[\frac{2n-6-(2n-4)}{2n-6+2n-4} \right] 2 \right) \\ &= \frac{1}{2(n-3)} \left(\frac{(n-3)(n-4)}{3n-8} - \frac{2}{2n-5} \right) \end{aligned}$$

which completes the proof.

Corollary 4.13. *The leverage centers of P_n^{--} and P_n are different with same leverage type.*

5. Conclusion

We have established the exact formulae for the leverage centrality of nodes for the generalized transformation path graphs P_n^{++}, P_n^{+-}, P_n^{-+}, and P_n^{--} for $n \geq 5$. In each case, the leverage center and the leverage type are compared with the original graph P_n. This offers fresh insights into molecular topology as a variety of different structural properties of the underlying molecules could be modeled by this knowledge. Our journey continues as we explore more permutations and extend the study to other connected graphs. The potential applications span diverse fields, inviting further research in molecular science and beyond.

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Conceptualisation: S. Sinumol, R. Sunil Kumar ; *Software:* S. Sinumol ; *Writing-Original Draft:* S. Sinumol

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