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A STUDY ON SOLUTION OF TWO NON-LINEAR EXPONENTIAL DIOPHANTINE EQUATIONS $11^x + 9^y = z^2$ AND $7^x + 17^y = z^2$ IN NON-NEGATIVE INTEGER

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Abstract

In this paper, we consider the polynomial Diophantine equation whose results are confined to integers of the form $a^x + b^y = z^2$, numerous researchers are interested and studied similar types of Diophantine equations. These equations have consummate involvement when a problem require a global comprehensive result. So, we're interested to turn up the result (if any) of the nonlinear exponential Diophantine equations $11^x + 9^y = z^2$ and $7^x + 17^y = z^2$. This study profoundly affects distinct methodologies to make out the establishment of multitudinous precise challenges: by exploring the going evolution in the accurate supposition, the command juncture of this check is to purport for per users with acquiring the refinement of pinpoint calculation and its unborn aspect in modern engineering era.

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1. Introduction

Number theory accounted as an advanced computation, the oldest branches of mathematics which studies the common properties of positive integers and their abecedarian property. A veritably wide and cornucopian exploration field in mathematics is the supposition of number theory, which has such a fascinating and the absorbing connection with the addition and multiplication of structures of integers. Fermat last theorem was established in 1995 by Wiles, A. abutted with collaborative effort of numerous other mathematicians which confirms that no non-trivial result of the equation $x^n + y^n = z^n$ for whole numbers, n>2. Number Theory used the basic properties of arithmetic computation and set up intriguing and unanticipated relationships between different numbers to prove that validity of these relationships. Number theoretic approach is applied in several fields, some of which are cryptography, material science, science, acoustics and also in devices sciences. In this paper we substantially concentrated on necessary improvements and some operation of number theory that can be useful to us, keeping the idea of the overview, according to the early history of numbers and explore the involvement, productive and practical situation dealing in our daily life. A group of mathematicians has done a lot of work in this direction and their contribution suggested studying distinct properties of the integers with different work frames. Number theory formed a unified inflow for advanced core system, that has precipitously improved. Generally, in number theory we study the nature of positive integers and prime numbers in extending way. Some numerical problems can be crack using a by unique factor decomposition approach. The development of mathematical tool theory of numbers, polynomial rigorous questions of supposition were proposed and further worked out, attracting more people to the null proposition. Former history shows that, the methodologies and arrangements to dive alike class of problems have arisen, and some conjectures have been created. Acu showed that only two solutions are possible, when variable takes the values (x = 3, y = 0, z = 3) and (x = 2, y = 1, z = 3) of the nonlinear Diophantine equation $2^x + 5^y = z^2$, in 2007. Sroysang (2012) obtained the single result of non-linear exponential Diophantine equation $3^x + 5^y = z^2$ is (1,0,2). Latterly, Sroysang (2013) derived the complete proof of three solutions (0,1,2), (3,0,3) and (4,2,5) for non-linear exponential $2^x + 3^y = z^2$. Tanakan (2014) calculated the result at (0,3,3) belongs to positive integer for the non-linear diophantine equation $19^x + 2^y = z^2$. Sroysang (2014) proved that non-linear Diophantine equation $5^x + 43^y = z^2$ is not solvable for non-negative integers. Fergy and Rabago (2016) studies five solutions in non-negative integers for Diophantine equation, $2^x + 17^y = z^2$. Sugandha, et al. (2018) discussed non-linear Diophantine equation $11^x + 13^y = z^2$ and find no solution in non negative integers. Li. (2019) suggested the Diophantine equation with prime numbers. Some Diophantine equations and inequalities with primes has been analysed by Baker (2021). Porto et. al. (2023) find the solution of exponential Diophantine Equation $p.3^x + p^y = z^2$ with p a Prime Number they also find unique solution (x,y,z) = (0,1,2) if p=2.

The paper divided into five sections, in next section 2 discussed the methodology such as some lemmas and theorems after that we considered two cases for solving exponential Diophantine $11^x + 9^y = z^2$ are given in section 3. In Section 4 of this paper is to represent solution procedure for the another nonlinear Diophantine equation $7^x + 17^y = z^2$ which can be helpful to make strategies and techniques to tackle issues have arisen in future research work. Section 5, illustrate the concluding remark and at the rest of the paper listed bibliography and arranged in alphabetically order for the convenience of the reader.

2. Preliminaries

In this section, we will study some definitions/lemma that will be used in this paper.

LEMMA 2.1. *The conjecture was computed by Catalan where $\{a, b, x, y\} = \{3, 2, 2, 3\} \in$ Z (set of integers), for the Diophantine equation $a^x - b^y = 1$, has unique solution with condition min {a, b, x, y} >1. (See. Ref. 5).*

Lemma 2.1, can be understand by this Diophantine equation $1 + p^x = z^2$, where p is an odd prime, x and z be whole number (W) has a unique solution (p, x, z) = (3, 1, 2).

Solution: For z is even, z = 2k. Since p is an odd prime number, so $p \equiv 1(mod4)$ or $p \equiv 3(mod4)$. So, we have two cases

Case 2.1.1:

Let $p \equiv 1(mod4)$,
$\Rightarrow \quad p^x \equiv 1(mod4)$
also $\quad 1 + p^x = z^2$
and $\quad z^2 \equiv 0(mod4)$
It is contradiction.

Case 2.1.2:

we consider $p \equiv 3(mod4)$,
if x is even: $p^x \equiv 1(mod4)$, but it is not possible.
Now, x is odd,
So, we consider the value of x = (2t +1).
$1 + p^x = z^2$ or equivalently
$1 + p^{2t+1} = (2k)^2$
So that $p^{2t+1} = (2k)^2 - 1$
$\Rightarrow \quad p^{2t+1} = (2k - 1)(2k + 1)$
Let u+v = 2t+1 and u > v,
so, $p^u p^v = (2k - 1)(2k + 1)$
$\Rightarrow \quad p^v(p^{u-v} - 1) = 2$
It tells us the value v = 0 and $2 = p^{2t+1} - 1$
Thus, t = 0, p = 3, x = 1 and z = 2.

So Diophantine equation $1 + p^x = z^2$ in non-negative integer solution is (p, x, z) = (3,1,2).

LEMMA 2.2. *If x, y, z is a primitive Pythagorean triple then x and y are of opposite parity.*

PROOF. Let x, y, z is a primitive Pythagorean triple then $x^2 + y^2 = z^2$, gcd (x, y, z) = 1.

If x, y both even, then 2|x, 2|y
$\Rightarrow \quad 2|x^2, 2|y^2$
$\Rightarrow \quad 2|x^2 + y^2$
$\Rightarrow \quad 2|z^2$
$\Rightarrow \quad 2|z$
$\Rightarrow \quad 2|gcd(x, y, z)$
$\Rightarrow \quad gcd(x, y, z) \gtrsim 2$ which is not true
and we also show that, if x, y both are odd then $x^2 \equiv 1(mod4)$
and $y^2 \equiv 1(mod4)$ $\quad$ [$\because a^2 \equiv 1(mod4)$, if a is odd integer]
$\Rightarrow \quad x^2 + y^2 \equiv 2(mod4)$
$\Rightarrow \quad z^2 \equiv 2(mod4)$ which is not possible $\quad$ [$\because a^2 \equiv 0(mod4)\ \forall$ integer a]

hence one of x and y is odd and other is even or x and y are of opposite parity. □

3. Main work for Equation 1

THEOREM 3.1. *Determine the solutions (if any) for the Non-linear exponential Diophantine equation* $11^x + 9^y = z^2$

PROOF. Let x, y and z be non-negative integers such that $11^x + 9^y = z^2$, x, y, z>0. . . (i) and consider one of them variable may be zero. If we take z = 0 is obviously not possible

if x = 0
$1 + 9^y = z^2$
$z^2 - 9^y = 1$
$z^2 - (3y)^2 = 1$ which is not possible by lemma 2.1
if y = 0
$11^x + 1 = z^2$ . . . (ii)
Now, we will check equation (ii) is solvable or not

Method 3.1.1: we have

$z^2 - 11^x = 1$
if x is even
x = 2t for some integer t
$z^2 - (11^t)^2 = 1$
which is not possible by Catalan conjecture Lemma 2.1
if x is odd
x = 2t + 1
x = u + v, u is greater than v
$z^2 - 1 = 11^x$
$(z - 1)(z + 1) = 11^x$
It implies that $11^u - 11^v = 2$
$11^u - 11^v \equiv 2(mod 10)$
$1 - 1 \equiv 2(mod 10)$
$0 \equiv 2(mod 10)$ which is a contradiction.

Method 3.1.2:

Let $z^2 - 11^x = 1$
$z^2 \equiv 11^x + 1(mod 10)$
$z^2 \equiv 1 + 1 \equiv 2(mod 10)$
which implies $z^2 \equiv 2(mod 5)$
but Legendre symbol [2/5] = -1. So, it is not solvable.

We consider the equation (i) under mod 10, for further use these results,
$11^x + 9^y \equiv z^2(mod 10)$
$1 + (-1)^y \equiv z^2(mod 10)$
If y is even then $z^2 \equiv 2(mod 10)$ which is not possible

y is odd
also 10|z
now z is even as it is divisible by 10, so we let z = 2k for some integer k
also, y is odd, so we let y = 2m + 1, Now, we consider all cases for finding parity of x.

Case 3.1.3: x is even

we have, x = 2t, then equation (i) becomes, $11^{2t}+9^y = z^2$ or $(11^t)^2+(3^y)^2 = z^2$ there exist $11^t, 3^y, z$ is a Pythagorean triple, Since $gcd(11^t, 3^y) = 1$ and also, $gcd(11^t, 3^y, z) = 1$, [if gcd (a, b) =1 then $gcd(a^n, b) = 1$]. It is observed that $11^t, 3^y, z$ is a primitive Pythagorean triple then by lemma 2, 11^t and 3^y are of opposite parity. which is obviously a contradiction.

Case 3.1.4: x is odd

x = 2t + 1
$11^{2t+1} + 9^{2m+1} = (2k)^2$
Thus, $11.11^{2t} + 9.9^{2m} = 4k^2$
or $11.11^{2t} + 11.9^{2m} = 4k^2 + 2.9^{2m}$
Therefore $11(11^{2t} + 9^{2m}) = 2(2k^2 + 9^{2m})$
Implying that $11 = 2k^2 + 9^{2m}$ and $2 = 11^{2t} + 9^{2m}$
The above equation gives t = 0, m = 0
Which implies $2k^2 + 1 = 11$ or $k^2 = 5$, this is impossible for $k \in Z$.

Hence Proved the given non-linear exponential Diophantine equation $11^x+9^y = z^2$ has no solution in non-negative whole numbers. □

4. Main work for Equation 2

THEOREM 4.1. *The Diophantine equation $7^x + 17^y = z^2$ has no solution in positive integers x, y, z*

PROOF. Let x, y, z be a solution of given exponential equation $7^x + 17^y = z^2 \ldots$ (i)
Firstly, we consider the one unknown variable i.e. x, y, z is 0. If z = 0, is obviously not possible because if z = 0 the equation will not balance as $7^x + 17^y > 0$.

We know that (see ref. 9), $1 + p^x = z^2$, p is prime has no solutions for p > 3, x, y cannot be equal to 0. We continue to go with, min (x, y, z) > 0 and, 7^x is odd, 17^y is odd. So $7^x + 17^y = z^2$ is even also, z is even. now, we use modulo 4 to find the parity of x.

$\Rightarrow \quad z^2 \equiv 0(mod4)$ also, $17^y \equiv 1^y \equiv 1(mod4)$
$\Rightarrow \quad 7^x \equiv 0 - 1 \equiv 3(mod4)$
$\Rightarrow$ x is odd

Now we let, z = 2k (for even) and x = 2t+1 (for odd), here k and t are different variable because if t = k then x = z+1 which is not always possible. We take two cases for y.

Case 4.1.1: y is even,
then y = 2m, so equation (i) becomes $7^{2t+1} + 17^{2m} = (2k)^2$

$\Rightarrow \quad 7^{2t+1} = (2k)^2 - 17^{2m} = (2k - 17^m)(2k + 17^m)$
Now again let x = u + v, u > v; then from lemma 1,

$2.17^m = 7^u - 7^v = 7^v(7^{u-v} - 1)$, v = 0 then u = x, and $2.17^m = 7^x - 1$.

Observe that $7^x - 1 \equiv 6(mod 8)$

$\Rightarrow \quad 2.17^m \equiv 6(mod 8)$

$\Rightarrow \quad 17^m \equiv 3(mod 4)$ which is contradict $17 \equiv 1(mod 4)$, go to next case.

Case 4.1.2: y is odd

Then, y = 2m+1
$7^{2t+1} + 17^{2m+1} = (2k)^2$
$7.7^{2t} + 17.17^{2m} = 4k^2$
$7(7^{2t} + 3.17^{2m}) = 4(k^2 + 17^{2m})$
$7 = k^2 + 17^{2m}$
$4 = 7^{2t} + 3.17^{2m}$

Let $t \neq 0$, t > 0. Such that 2t > 1, $7^{2t} > 7$ also, $3.17^{2m} > 3$, where $m \gtrsim 0$. Thus, $7^{2t} + 3.17^{2m} > 10$, which contradicts equation (ii). Then, t = 0, also, if m > 0., $3.17^{2m} > 51$, again contradicts equation (ii). So that, t = m = 0, gives x = y = 1. but then $z^2 = 7 + 17 = 24$ which is not a perfect square. Hence the Diophantine Equation $7^x + 17^y = z^2$ has no solution in non-negative integers. □

5. Conclusion

This paper seeks to find the truth and answers through explanations, in order to learn about, we have been computed two non-linear Diophantine equation $11^x + 9^y = z^2$ & $7^x + 17^y = z^2$, it has been shown that both Diophantine equations have no solutions for any x, y, z ∈ W.

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