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FIXED POINT THEOREMS IN EXTENDED CONE METRIC SPACE

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Abstract

This paper introduces the concept of an extended cone metric and discusses several of its properties. Within the context of extended cone metric spaces, we establish fixed point results for mappings that satisfy specific contractive conditions. Our results not only expand upon but also complement the well-established results of the existing literature. We enhance the theoretical foundation and potential applications of these mathematical structures by providing a broader understanding and new insights into the behavior of mappings in extended cone metric spaces through this work.

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1. Introduction

The examination of fixed point theorems has been a central theme in mathematical analysis, with applications that extend across a variety of fields. The pioneering works of L.G. Huang and X. Zhang [1], which established foundational results for contractive mappings in cone metric spaces, have provided extensive exploration of traditional fixed point theorems in metric and cone metric spaces. S. Czerwik [3] broadened the concept to b-metric spaces, while B.E. Rhoades [2] compared various definitions of contractive mappings, thereby enriching the understanding of fixed point theory.

The concept of extended cone metric spaces, which is a generalization of cone metric spaces, opens up new possibilities for fixed point results. Huang and Xu [4] investigated the practical applications of fixed point theorems in cone b-metric spaces. This paper expands upon these seminal works and delves deeper into the topic of fixed point theorems in the context of extended cone metric spaces.

Authors such as Abhishikta Das and T. Bag [13] have made substantial contributions to the field by investigating mappings in extended cone metric spaces, which are among the main developments in this field. Similarly, Hemavathy and Maheswari [14] expanded upon these concepts within the extended cone S_b-metric space framework. Jerolina Fernandez [15]. have recently conducted research on extended Nb-cone metric spaces, which underscores the concepts' extensive applicability and adaptability.

Additionally, advanced research conducted by Kushal Royy et al. [16] and Wasfi Shatanawi and Taqi A.M. Shatnawi [17] has further generalized fixed point theorems in extended cone b-metric spaces, integrating new forms of contractions. By demonstrating its extensive applicability and potential for further research, this body of work collectively enriches the fixed point theory. Readers may refer more results of cone metric space in [5–12, 18, 19].

This paper endeavors to add to the rapidly developing field of extended cone metric spaces by investigating the properties and fixed point results of mappings, thereby broadening the applicability of well-established mathematical principles and providing novel insights.

2. Preliminaries

Definition 2.1. [1] Let E be a real Banach space, θ the zero of E and P a subset of E. Then P is called a cone if and only if

(1) P is closed, nonempty and $P \neq \{\theta\}$,
(2) If $a, b \in \mathbb{R}$, $a, b \geq 0$ and $u, v \in P$, then $au + bv \in P$,
(3) If both $u \in P$ and $-u \in P$, then $u = \theta$.

Definition 2.2. [1] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies the following conditions:

(1) $0 < d(u, v)$ and $d(u, v) = 0$ iff $u = v, \forall u, v \in X$,
(2) $d(u, v) = d(v, u)$ for all $u, v \in X$
(3) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.3. Let X be a nonempty set and $\theta : X \times X \to [1, \infty)$. Suppose the mapping $d_\theta : X \times X \to E$ satisfies the following conditions:

(1) $0 < d_\theta(u, v)$ and $d_\theta(u, v) = 0$ if and only if $u = v, \forall u, v \in X$
(2) $d_\theta(u, v) = d_\theta(v, u)$ for all $u, v \in X$
(3) $d_\theta(u, v) \leq \theta(u, w)[d_\theta(u, w) + d_\theta(w, v)]$ for all $u, v, w \in X$

Then the pair (X, d_θ) is called an extended cone metric space.

Example 2.4. *Let $X = \{1, 2, 3\}$. We define*

$$\theta : X \times X \to \mathbb{R}^+$$

and

$$d_\theta : X \times X \to \mathbb{R}^+$$

as $\theta(u, v) = 1 + u + v$, for all $u, v \in X$ and $d_\theta(1, 1) = d_\theta(2, 2) = d_\theta(3, 3) = 0, d_\theta(1, 2) = d_\theta(2, 1) = 80, d_\theta(1, 3) = d_\theta(3, 1) = 1000, d_\theta(2, 3) = d_\theta(3, 2) = 600$. Then the pair (X, d_θ) is an extended cone metric space.

Definition 2.5. Let (X, d_θ) be an extended cone metric space.

(1) A sequence $\{x_n\}$ in X is said to converge to $x \in X$, if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d_\theta(x_n, x) < \varepsilon$ for all $n > N$. We write $\lim_{n\to\infty} x_n = x$.
(2) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d_\theta(x_m, x_n) < \varepsilon$ for all $m, n > N$.

Definition 2.6. If every Cauchy sequence in X is convergent, then (X, d_θ) is a complete extended cone metric space.

Lemma 2.7. *Consider the extended cone metric space (X, d_θ). Assuming that d_θ is a continuous function, it follows that every convergent sequence possesses a unique limit point.*

3. Fixed Point Theorems

Here, we shall establish fixed-point theorems for extended cone metric space.

Theorem 3.1. *Let (X, d_θ) be a complete extended cone metric with the property that d_θ is continuous functional. Let $f : X \to X$ satisfy the following conditions:*

$$d_\theta(fu, fv) \le k d_\theta(u, v) \tag{3.1}$$

for all $u, v \in X$ where $k \in [0, 1)$ be such that for each $u_0 \in X, \lim_{n,m} \to \theta(u_n, u_m) < \frac{1}{k}$, here $u_n = f^n x_0, n = 1, 2, 3, \ldots$. Then f has precisely one fixed point x. Moreover for each $v \in X, f^n v \to x$.

Proof. We choose an arbitrary point $u_0 \in X$. We define the iterative sequence $\{u_n\}$ by

$$u_0, Tu_0 = u_1, u_2 = Tu_1 = T(Tu_0) = T^2(u_0) \ldots, u_n = T^n u_0 \ldots \tag{3.2}$$

Then by successively applying inequality 3.1 we obtain

$$d_\theta(u_n, u_{n+1}) \le k^n d_\theta(u_0, u_1) \tag{3.3}$$

So long as $m > n$, we may deduce from triangle inequality and equation 3.2 that

$$\begin{aligned} d_\theta(u_n, u_m) &\le \theta(u_n, u_m) + \theta(u_n, u_m)\theta(u_{n+1}, u_m)k^{n+1} d_\theta(u_0, u_1) + \ldots \\ &+ \theta(u_n, u_m)\theta(u_{n+1}, u_m)\theta(u_{n+2}, u_m)\ldots\theta(u_{m-2}, u_m)\theta(u_{n-1}, u_m)k^{m+1} d_\theta(u_0, u_1) \\ &\le d_\theta(u_0, u_1)[\theta(u_1, u_m)\theta(u_2, u_m)\ldots\theta(u_{n-1}, u_m)\theta(u_n, u_m k^n + \ldots \\ &+ \theta(u_1, u_m)\theta(u_2, u_m)\ldots\theta(u_n, u_m)\theta(u_{n+1}, u_m + k^{n+1} + \ldots \\ &+ \theta(u_n, u_m)\theta(u_2, u_m)\ldots\theta(u_n, u_m)\theta(u_{n+1}, u_m)\ldots\theta(u_{m-2}, u_m)\theta(u_{m-1}, u_m)k^{m-1}] \end{aligned}$$

Since $lim_{n,m\to\infty}\theta(u_{n-1}, u_m)k < 1$ so that the series $\sum_{m=1}^{\infty} k^n \prod_{i=1}^{n} \theta(u_i, u_m)$ converges by ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{m=1}^{\infty} k^n \prod_{i=1}^{n} \theta(u_i, u_m), S_n = \sum_{j=1}^{n} k^j \prod_{i=1}^{j} \theta(u_i, x_m)$$

Thus for $m > n$ above inequality implies:

$$d_\theta(u_n, u_m) \leq d_\theta(u_0, u_1)[S_{m-1} - S_n]$$

Letting $n \to \infty$ we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete let $u_n \to x \in X$:

$$\begin{aligned} d_\theta(fx, x) &= \theta(fx, x)[d_\theta(fx, u_n) + d_\theta(u_n, x)] \\ &\leq \theta(fx, x)[kd_\theta(x, u_{n-1}) + d_\theta(u_n, x)] \\ d_\theta(fx, x) &= 0, n \to \infty \\ d_\theta(fx, x) &= 0 \end{aligned}$$

Hence, there is a fixed point x in f. The fact that $k < 1$ makes it easy to use inequality 3.1 to induce uniqueness. □

Definition 3.2. Consider a mapping $T : X \to X$ and let $u_0 \in X$ be a fixed point. The orbit $\mathcal{O}(u_0)$ of u_0 is defined as the set $\{u_0, Tx_0, T^2x_0, \ldots\}$. A function G from X to the set of real numbers is considered to be $T-$orbitally lower semi-continuous at $t \in X$ if, for each sequence $\{u_n\} \subset \mathcal{O}(u_0)$ that converges to t, the inequality

$$G(t) \leq \lim_{n\to\infty} inf.\ G(u_n)$$

holds.

Theorem 3.3. *Let (X, d_θ) be a complete extended cone metric where d_θ is continuous functional. Let $f : X \to X$, and there exists $u_0 \in X$ that satisfy*

$$d_\theta(fv, f^2v) \leq kd_\theta(v, Tv) \tag{3.4}$$

for each $v \in \mathcal{O}(u_0)$ where $k \in [0, 1)$ be such that for $u_0 \in X, \lim_{n,m\to\infty} < \frac{1}{k}$, here $u_n = f^nu_0, n = 0, 1, 2, \ldots$. Then

$$f^nx_0 \to x \in X(n \to \infty).$$

Moreover x is a fixed point of f if and only if $G(u) = d(u, Tu)$ is $f-$orbitally lower semi continuous at x.

Proof. We choose an arbitrary point $u_0 \in X$. We define the iterative sequence $\{u_n\}$ by

$$u_0, fu_0 = u_1, u_2 = fu_1 = f(fu_0) = f^2(u_0)\ldots, u_n = f^nu_0\ldots \tag{3.5}$$

Now for $v = fu_0$ by successively applying 3.3 we obtain:

$$d_\theta(f^nu_0, f^{n+1}u_0) = d_\theta(u_n, u_{n+1}) \leq k^nd_\theta(u_0, u_1) \tag{3.6}$$

Using the same steps, we used in Theorem 3.1, We may assert that the sequence $\{x_n\}$ is a Cauchy sequence. Given that X is a complete space, we may conclude that the sequence $u_n = T^n u_0$ converges to $x \in X$. If G is orbitally lower semi-continuous at $x \in X$, then

$$\begin{aligned} d_\theta(x, fx) &\leq lim_{n\to\infty} inf.\ d_\theta f^n u_0, f^{n+1} u_0 \\ &\leq lim_{n\to\infty} inf.\ k^n d_\theta(u_0, u_1) = 0 \end{aligned}$$

Conversely, let $x = fx$ and $\{u_n\} \in O(u)$ with $\{u_n\} \to x$. Then

$$G(x) = d(x, fx) = 0 \leq lim_{n\to\infty} inf.\ G(u_n) = d(f^n u_0, f^{n-1} u_0)$$

□

EXAMPLE 3.4. *Let $X = [0, \infty)$. Define $d_\theta : X \times X \to \mathbb{R}^+$ and $\theta : X \times X \to [1, \infty)$ as*

$$d_\theta = (u - v)^2, \theta(u, v) = u + v + 2$$

for all $u, v \in X$. Then d_θ is a complete extended cone metric on X. Define $f : X \to X$ by $fu = \frac{u}{2}$. Then we can show that all conditions of the above the theorem are satisfied. Hence f has unique fixed point.

4. Conclusion and Future Work

We have introduced the concept of extended cone metric spaces and have investigated their fundamental properties in this study. Within these spaces, we established numerous fixed point theorems for mappings that satisfy particular contractive conditions. Our findings enhance the current corpus of knowledge by not only extending but also complementing well-established fixed point theorems in traditional metric and cone metric spaces. The results emphasize the adaptability and durability of extended cone metric spaces in fixed point theory, thereby facilitating a more profound comprehension of their structure and applications. In light of the findings of this investigation, there are numerous potential directions for future research. A potential approach is to examine the applicability of extended cone metric spaces in a variety of disciplines, including dynamic systems, differential equations, and optimization.

Conflicts of Interest: The authors declare no conflict of interest.

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