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SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY A CONVOLUTION OPERATOR

S. A. JADHAV, P. G. JADHAV, S. B. GAIKWAD and P. THIRUPATHI REDDY ✉

Abstract

In this paper, a new complex convolution operator defined on analytic functions related to Lambda function and (p, k) extended Kummer hypergeometric function is considered. Also, properties like coefficient bounds, convex linear combinations, integral means inequalities, partial sums, growth and distortion theorems, partial sums are studied and radius of close to convexity, starlikeness and convexity.

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1. Introduction

Let A denote the class of analytic functions l defined on the unit disc $U = \{z : z \in \mathbb{C}, |z| < 1\}$ with normalization $l(0) = 0$, $l'(0) = 1$.Such a function has a Taylor series expansion about the origin in the form

$$l(z) = z + \sum_{m=2}^{\infty} a_m z^m. \tag{1.1}$$

We denote by T subclass of A consisting of functions of the form

$$l(z) = z - \sum_{m=2}^{\infty} a_m z^m, \quad (a_m \geq 0). \tag{1.2}$$

This subclass was introduced and studied by Silverman [9].
A function $l \in A$ is called starlike of order α $(0 \leq \alpha < 1)$ if and only if

$$\Re\left(\frac{zl'(z)}{l(z)}\right) > \alpha, \quad (z \in U).$$

A function $l \in A$ is called convex of order α $(0 \leq \alpha < 1)$ if and only if

$$\Re\left(1 + \frac{zl''(z)}{l'(z)}\right) > \alpha, \quad (z \in U).$$

We denote the class of all starlike functions of order α by $S^*(\alpha)$ and the class convex functions of order α by $K(\alpha)$.

The class $\beta - UCV$ was introduced by Kanas and Wisniowska [4], β-uniformly convex function is a natural extension of the classical convexity.
For $-1 < \alpha \leq 1$ and $\beta \geq 0$ a function $l \in A$ is said to be in the class
(i) β-uniformly starlike functions of order α is denoted by $S_p(\alpha,\beta)$ if it satisfies the condition

$$\Re\left(\frac{zl'(z)}{l(z)} - \alpha\right) > \beta\left|\frac{zl'(z)}{l(z)} - 1\right|, \quad z \in U,$$

and

(ii)β -uniformly convex functions of order α is denoted by $UCV(\alpha,\beta)$ if it satisfies the condition

$$\Re\left(1 + \frac{zl''(z)}{l'(z)} - \alpha\right) \geq \beta\left|\frac{zl''(z)}{l'(z)}\right|, \quad z \in U.$$

It is of interest to state that $UCV(\alpha, 0) = K(\alpha)$ and $S_p(\alpha, 0) = S^*(\alpha)$.
Moreover, for $\beta = 1$ corresponds to the class of uniformly convex functions UCV introduced by Goodman [2, 3] and studied by Ronning [7, 8].
The class $\beta - S_p$ is related to the class $\beta - UCV$ by means of the well-known Alexander equivalence between the usual classes of convex K and starlike S^* functions.

Murugusundaramoorthy and Magesh [6] introduced the following subclasses of A.

For $0 \leq \lambda < 1$, $0 \leq \alpha < 1$ and $\beta \geq 0$, we let $S_p(\lambda, \alpha, \beta)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\Re\left\{\frac{zl'(z)}{(1-\lambda)l(z) + \lambda zl'(z)} - \alpha\right\} > \beta\left|\frac{zl'(z)}{(1-\lambda)l(z) + \lambda zl'(z)} - 1\right|, \quad z \in U$$

and also, let $UCV(\lambda, \alpha, \beta)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\Re\left\{\frac{l'(z) + zl''(z)}{l'(z) + \lambda zl''(z)} - \alpha\right\} > \beta\left|\frac{l'(z) + zl''(z)}{l'(z) + \lambda zl''(z)} - 1\right|, \quad z \in U.$$

For $l \in A$ given by (1.1) and $w(z)$ given by

$$w(z) = z + \sum_{m=2}^{\infty} b_m z^m.$$

Convolution of l and w denoted by $l * w$, is defined as

$$(l * w)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m. \tag{1.3}$$

In [1] Mohamed Abdalla introduced and discussed the (p,k)-extended Gauss hypergeometric function $\mathfrak{W}(p,k;z)$ and (p,k)-extended Kummer hypergeometric function $\mathfrak{Y}(p,k;z)$ as follows

$$\mathfrak{W}(p,k;z)=\sum_{m=0}^{\infty}\frac{(\zeta_1)_{m,k}(\zeta_2)_{m,k}}{(\zeta_3)_{m,k}}\frac{z^m}{(pm)!} \tag{1.4}$$

$$\mathfrak{Y}(p,k;z)=\sum_{m=0}^{\infty}\frac{(\zeta_1)_{m,k}}{(\zeta_3)_{m,k}}\frac{z^m}{(pm)!} \tag{1.5}$$

where $k\in\mathbb{R}^+$ and $\zeta_1,\zeta_2,z\in\mathbb{C}$ and $\zeta_3\in\mathbb{C}\backslash\mathbb{Z}_0^-$, p is positive integer, $(\zeta)_{m,k}$ is the k-Pochhammer symbol given in the form

$$(\zeta)_{m,k}=\frac{\Gamma^k(\zeta+mk)}{\Gamma^k(\zeta)} \tag{1.6}$$
$$=\begin{cases}\zeta(\zeta+k)\ldots(\zeta+(m-1)k), & m\in\mathbb{N},\zeta\in\mathbb{C}\\ 1 & m=0,k\in\mathbb{R}^+,u\in\mathbb{C}\backslash\{0\}\end{cases}$$

$$z\mathfrak{Y}(p,k;z)=z+\sum_{m=2}^{\infty}\frac{(\zeta_1)_{m-1,k}}{(\zeta_3)_{m-1,k}}\frac{z^m}{(p(m-1))!}. \tag{1.7}$$

Let us recall lambda function [14] defined by

$$\Lambda(z,\varrho)=\sum_{m=0}^{\infty}\frac{z^m}{(2m+1)^{\varrho}}$$

where $z\in U$, $\varrho\in\mathbb{C}$, when $|z|<1$, $\mathfrak{R}(\varrho)>1$, when $|z|=1$ and let

$$\Lambda^{(-1)}(z,\varrho)=1+\sum_{m=2}^{\infty}\frac{(\mu+1)_{m-1}(2m-1)^{\varrho}}{(m-1)!}z^{m-1}$$

be defined such that

$$\Lambda(z,\varrho)*\Lambda^{(-1)}(z,\varrho)=\frac{1}{(1-z)^{\mu+1}},\quad \mu>(-1).$$

Also, $z\Lambda^{(-1)}(z,\varrho)$ be defined as the following

$$(z\Lambda(z,\varrho))*(z\Lambda^{(-1)}(z,\varrho))=\frac{z}{(1-z)^{\mu+1}}$$
$$=z+\sum_{m=2}^{\infty}\frac{(\mu+1)_{m-1}}{(m-1)!}z^m,\quad \mu>(-1).$$

where

$$(z\Lambda^{(-1)}(z,\varrho)) = z + \sum_{m=2}^{\infty} \frac{(\mu+1)_{m-1}(2m-1)^{\varrho}}{(m-1)!} z^m. \tag{1.8}$$

Based on the convolution tool motivated by, we define the following new complex convolution operator for $l \in A$

$$\begin{aligned}\Im_{\mu,\varrho}(\zeta_1,\zeta_3)l(z) &= (z\mathfrak{Y}(p,k;z)) * (z\Lambda^{(-1)}(z,\varrho)) * l(z)\\ &= z + \sum_{m=2}^{\infty} \frac{(\mu+1)_{m-1}(2m-1)^{\varrho}(\zeta_1)_{m-1,k}}{(m-1)!(p(m-1))!(\zeta_3)_{m-1,k}} a_m z^m\end{aligned}$$

$$\Im_{\mu,\varrho}(\zeta_1,\zeta_3)l(z) = z + \sum_{m=2}^{\infty} \phi_m(\mu,\varrho,\zeta_1,\zeta_3) a_m z^m \tag{1.9}$$

where,

$$\phi_m(\mu,\varrho,\zeta_1,\zeta_3) = \frac{(\mu+1)_{m-1}(2m-1)^{\varrho}(\zeta_1)_{m-1,k}}{(m-1)!(p(m-1))!(\zeta_3)_{m-1,k}}. \tag{1.10}$$

Now, we define new subclass $S_{\mu}^{\varrho}(\lambda,\gamma)$ of functions belonging to class A motivated by Venkateswarlu and Thirupathi Reddy [15], G. Murugusundaramoorthy and Magesh[6].

Definition 1.1. The function $l(z)$ of the form (1.1) is in the class $S_{\mu}^{\varrho}(\lambda,\gamma)$ if it satisfies the inequality

$$\Re\left\{\frac{z(\Im_{\mu,\varrho}(\zeta_1,,\zeta_3)l(z))'}{(1-\lambda)z+\lambda\Im_{\mu,\varrho}(\zeta_1,\zeta_3)l(z)} - \gamma\right\} > \left|\frac{z(\Im_{\mu,\varrho}(\zeta_1,,\zeta_3)l(z))'}{(1-\lambda)z+\lambda\Im_{\mu,\varrho}(\zeta_1,\zeta_3)l(z)} - 1\right|$$

.
Further, we define $TS_{\mu}^{\varrho}(\lambda,\gamma) = S_{\mu}^{\varrho}(\lambda,\gamma) \bigcap T$.

2. Coefficient Estimates

Theorem 2.1. *The function $l(z)$ of the form (1.1) is in $S_{\mu}^{\varrho}(\lambda,\gamma)$ if*

$$\sum_{m=2}^{\infty}[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m| \le (1-\gamma), \tag{2.1}$$

where $0 \le \lambda \le 1, 0 \le \gamma < 1$ and $\phi_m(\mu,\varrho,\zeta_1,\zeta_3)$ is given by (1.10).

Proof. It is enough to show that

$$\left|\frac{z(\Im_{\mu,\varrho}(\zeta_1,\zeta_3)l(z))'}{(1-\lambda)z+\lambda\Im_{\mu,\varrho}(\zeta_1,\zeta_3)l(z)} - 1\right| - \Re\left\{\frac{z(\Im_{\mu,\varrho}(\zeta_1,\zeta_3)l(z))'}{(1-\lambda)z+\lambda\Im_{\mu,\varrho}(\zeta_1,\zeta_3)l(z)} - \gamma\right\} \le 1-\gamma.$$

We have

$$\left|\frac{z(\Im_{\mu,\varrho}(\zeta_1,\zeta_3)l(z))'}{(1-\lambda)z+\lambda\Im_{\mu,\varrho}(\zeta_1,\zeta_3)l(z)}-1\right|-\Re\left\{\frac{z(\Im_{\mu,\varrho}(\zeta_1,\zeta_3)l(z))'}{(1-\lambda)z+\lambda\Im_{\mu,\varrho}(\zeta_1,\zeta_3)l(z)}-\gamma\right\}$$

$$\leq 2\left|\frac{z(\Im_{\mu,\varrho}(\zeta_1,\zeta_3)f(z))'}{(1-\lambda)z+\lambda\Im_{\mu,\varrho}(\zeta_1,\zeta_3)l(z)}-1\right|$$

$$\leq \frac{2\sum_{m=2}^{\infty}(m-\lambda)\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m||z|^{m-1}}{1-\sum_{m=2}^{\infty}\lambda\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m||z|^{m-1}}$$

$$\leq \frac{2\sum_{m=2}^{\infty}(m-\lambda)\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m|}{1-\sum_{m=2}^{\infty}\lambda\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m|}.$$

The last expression is bounded above by $(1-\gamma)$, if

$$2\sum_{m=2}^{\infty}(m-\lambda)\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m| \leq (1-\gamma)\left(1-\sum_{m=2}^{\infty}\lambda\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m|\right),$$

which is equivalent to

$$\sum_{m=2}^{\infty}[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m| \leq (1-\gamma).$$

This is true by hypothesis and the proof is complete. □

Theorem 2.2. *Let $0\leq\lambda\leq 1$, $0\leq\gamma<1$ then l is of the form* (1.2) *to be in the class $TS_{\mu}^{\varrho}(\lambda,\gamma)$ if and only if*

$$\sum_{m=2}^{\infty}[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m| \leq (1-\gamma) \tag{2.2}$$

where $0\leq\lambda\leq 1, 0\leq\gamma<1$ and $\phi_m(\mu,\varrho,\zeta_1,\zeta_3)$ is given by (1.10).

Proof. In the view of above theorem (2.1) we need only to prove necessity
If $f\in TS_{\mu}^{\varrho}(\lambda,\gamma)$ and z real then

$$\Re\left\{\frac{1-\sum_{m=2}^{\infty}m\phi_m(\mu,\varrho,\zeta_1,\zeta_3)a_m z^{m-1}}{1-\sum_{m=2}^{\infty}\lambda\phi_m(\mu,\varrho,\zeta_1,\zeta_3)a_m z^{m-1}}-\gamma\right\} > \left|\frac{\sum_{m=2}^{\infty}(m-\lambda)\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m||z|^{m-1}}{1-\sum_{m=2}^{\infty}\lambda\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m||z|^{m-1}}\right|$$

$z \to 1$ along real axis, we obtain the desired inequality

$$1-\gamma-\sum_{m=2}^{\infty}(m-\lambda\gamma)\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m| \geq \sum_{m=2}^{\infty}(m-\lambda)\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m|$$

$$\sum_{m=2}^{\infty}[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m| \leq (1-\gamma)$$

where $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$ and $\phi_m(\mu,\varrho,\zeta_1,\zeta_3)$ is given by (1.10). □

Corollary 2.3. *If $l(z) \in TS_{\mu}^{\varrho}(\lambda,\gamma)$, then*

$$|a_m| \leq \frac{1-\gamma}{[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)} \tag{2.3}$$

where $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$ and $\phi_m(\mu,\varrho,\zeta_1,\zeta_3)$ is given by (1.10). Equality holds for the function

$$l(z) = z - \frac{1-\gamma}{[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)}z^m. \tag{2.4}$$

3. Convex Linear Combinations

Theorem 3.1. *Let $l_1(z) = z$ and*

$$l_m(z) = z - \frac{1-\gamma}{[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)}z^m, \quad (m \geq 2) \tag{3.1}$$

then $l(z) \in TS_{\mu}^{\varrho}(\lambda,\gamma)$ if and only if it can be expressed in the form

$$l(z) = \sum_{m=1}^{\infty} w_m l_m(z), \quad w_m \geq 0, \quad \sum_{m=1}^{\infty} w_m = 1. \tag{3.2}$$

Proof. Suppose $l(z)$ can be written as in (3.2), then

$$l(z) = z - \sum_{m=2}^{\infty} w_m \frac{1-\gamma}{[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)}z^m.$$

Now,

$$\begin{aligned}\sum_{m=2}^{\infty} w_m \frac{(1-\gamma)[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)}{(1-\gamma)[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)} &= \sum_{m=2}^{\infty} w_m \\ &= 1 - w_1 \leq 1.\end{aligned}$$

Thus, $l(z) \in TS_{\mu}^{\varrho}(\lambda, \gamma)$.
Conversely,
let, $l(z) \in TS_{\mu}^{\varrho}(\lambda, \gamma)$ then by using (2.3), we get

$$w_m = \frac{[2m - \lambda(\gamma + 1)]\phi_m(\mu, \varrho, \zeta_1, \zeta_3)}{1 - \gamma} a_m, \quad (m \geq 2)$$

and $w_1 = 1 - \sum_{m=2}^{\infty} w_m$, then we have $l(z) = \sum_{m=1}^{\infty} w_m l_m(z)$ and the proof is complete. □

Theorem 3.2. *The class $TS_{\mu}^{\varrho}(\lambda, \gamma)$ is a convex set.*

Proof. Let the function

$$l_j(z) = z - \sum_{m=2}^{\infty} a_{m,j} z^m, \quad a_{m,j} \geq 0, \ j = 1, 2 \tag{3.3}$$

be in the class $TS_{\mu}^{\varrho}(\lambda, \gamma)$. We have to show that the function $k(z)$ defined by
$k(z) = \eta l_1(z) + (1 - \eta) l_2(z), \ 0 \leq \eta < 1$ in the class $TS_{\mu}^{\varrho}(\lambda, \gamma)$.

$$k(z) = z - \sum_{m=2}^{\infty} [\eta a_{m,1} + (1 - \eta) a_{m,2}] z^m.$$

Using Theorem 2.2, we get

$$\begin{aligned} \sum_{m=2}^{\infty} [2m - \lambda(\gamma + 1)] \eta \phi_m(\mu, \varrho, \zeta_1, \zeta_3) a_{m,1} + \sum_{m=2}^{\infty} [2m - \lambda(\gamma + 1)](1 - \eta) \phi_m(\mu, \varrho, \zeta_1, \zeta_3) a_{m,2} \\ \leq \eta(1 - \gamma) + (1 - \eta)(1 - \gamma) \\ \leq (1 - \gamma). \end{aligned}$$

Which implies $k(z) \in TS_{\mu}^{\varrho}(\lambda, \gamma)$. Hence $TS_{\mu}^{\varrho}(\lambda, \gamma)$ is convex. □

4. Integral Means Inequalities

Two functions l and g which are analytic in U,the function l is said to be subordinate to g in U if there exists a function w analytic in U with $w(0) = 0$, $|w(z)| < 1, (z \in U)$ such that $l(z) = g(w(z)), (z \in U)$.
We denote subordination by $l(z) \prec g(z)$.

Lemma 4.1. *If the functions l and g are analytic in U with $l(z) \prec g(z)$, then for $\tau > 0$ and $z = re^{i\varphi}, \ 0 < r < 1$,*

$$\int_0^{2\pi} |g(re^{i\varphi})|^{\tau} d\varphi \leq \int_0^{2\pi} |l(re^{i\varphi})|^{\tau} d\varphi.$$

Now, we discuss the integral means inequalities for the function l in $TS_{\mu}^{\varrho}(\lambda, \gamma)$.

Silverman found that the function $l_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality conjectured in[10] settled in [11], that

$$\int_0^{2\pi} |l(re^{i\varphi})|^\tau d\varphi \leq \int_0^{2\pi} |l_2(re^{i\varphi})|^\tau d\varphi$$

for all $l \in T$, $\tau > 0$ and $0 < r < 1$.
Now, we prove Silverman's conjecture for the functions $l(z) \in TS_\mu^\varrho(\lambda, \gamma)$ by using the concept of subordination between analytic function and a subordination theorem of Littlewood [5].

Theorem 4.2. *Suppose* $l \in TS_\mu^\varrho(\lambda, \gamma),\ \tau > 0,\ 0 \leq \lambda \leq 1,\ 0 \leq \gamma < 1$ *and* $l_2(z)$ *is defined by*

$$l_2(z) = z - \frac{1-\gamma}{\varphi_2(\lambda, \gamma)} z^2, \tag{4.1}$$

where $\varphi_2 = [4 - \lambda(\gamma + 1)]\phi_m(\mu, \varrho, \zeta_1, \zeta_3)$ *and* $\phi_m(\mu, \varrho, \zeta_1, \zeta_3)$ *is given by* (1.10) *then* $z = re^{i\varphi}, 0 < r < 1$

$$\int_0^{2\pi} |l(z)|^\tau d\varphi \leq \int_0^{2\pi} |l_2(z)|^\tau d\varphi. \tag{4.2}$$

Proof. For $l(z) = z - \sum\limits_{m=2}^{\infty} a_m z^m$ Lemma (4.1) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{m=2}^{\infty} a_m z^{m-1} \right|^\tau d\varphi \leq \int_0^{2\pi} \left| 1 - \frac{1-\gamma}{\varphi_2(\lambda, \gamma)} z \right|^\tau d\varphi.$$

By Lemma (4.1) it is enough to show that

$$1 - \sum_{m=2}^{\infty} a_m z^{m-1} \prec 1 - \frac{1-\gamma}{\varphi_2(\lambda, \gamma)} z.$$

Assuming

$$1 - \sum_{m=2}^{\infty} a_m z^{m-1} \prec 1 - \frac{1-\gamma}{\varphi_2(\lambda, \gamma)} w(z),$$

and using (2.2), we obtain

$$|w(z)| = \left| \sum_{m=2}^{\infty} \frac{\varphi_2(\lambda, \gamma)}{1-\gamma} a_m z^{m-1} \right| \leq |z| \sum_{m=2}^{\infty} \frac{\varphi_2(\lambda, \gamma)}{1-\gamma} |a_m| \leq |z|.$$

Where

$$\varphi_m(\lambda,\gamma) = [2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3).$$

This completes the proof. □

5. Growth and Distortion Theorems

THEOREM 5.1. *Let the function l defined by (1.2) be in the class $TS_{\mu}^{\varrho}(\lambda,\gamma)$. Then for* $|z| = r < 1$

$$r - \frac{1-\gamma}{[4-\lambda(1+\gamma)]\phi_2(\mu,\varrho,\zeta_1,\zeta_3)}r^2 \le |l(z)| \le r + \frac{1-\gamma}{[4-\lambda(1+\gamma)]\phi_2(\mu,\varrho,\zeta_1,\zeta_3)}r^2 \quad (5.1)$$

equality holds for the function

$$l(z) = z - \frac{1-\gamma}{[4-\lambda(1+\gamma)]\phi_2(\mu,\varrho,\zeta_1,\zeta_3)}z^2. \quad (5.2)$$

PROOF. By theorem 2.2, note that

$$[4-\lambda(\gamma+1)]\phi_2(\mu,\varrho,\zeta_1,\zeta_3)\sum_{m=2}^{\infty} a_m \le \sum_{m=2}^{\infty}[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)|a_m| \le (1-\gamma)$$

. That is

$$\sum_{m=2}^{\infty} a_m \le \frac{1-\gamma}{[4-\lambda(\gamma+1)]\phi_2(\mu,\varrho,\zeta_1,\zeta_3)}$$

We have,

$$l(z) = z - \sum_{m=2}^{\infty} a_m z^m.$$

$$\begin{aligned}
|l(z)| &= \left| z - \sum_{m=2}^{\infty} a_m z^m \right| \\
&\le r + \sum_{m=2}^{\infty} |a_m| r^m \\
&\le r + r^2 \sum_{m=2}^{\infty} |a_m| \\
&\le r + r^2 \sum_{m=2}^{\infty} \frac{1-\gamma}{[4-\lambda(1+\gamma)]\phi_2(\mu,\varrho,\zeta_1,\zeta_3)}.
\end{aligned}$$

Which gives right hand side inequality of (5.1).
Other inequality can be used by using similar argument. □

Theorem 5.2. *Let the function l defined by (1.2) be in the class $TS_{\mu}^{\varrho}(\lambda,\gamma)$. Then for* $|z| = r < 1$

$$1 - \frac{2(1-\gamma)}{[4-\lambda(1+\gamma)\phi_2](\mu,\varrho,\zeta_1,\zeta_3)} r \le |l'(z)| \le 1 + \frac{2(1-\gamma)}{[4-\lambda(1+\gamma)]\phi_2(\mu,\varrho,\zeta_1,\zeta_3)} r \quad (5.3)$$

equality holds for the function given by (5.2).

6. Partial Sums

Using earlier methods of Silverman [12] and Silvia[13] on partial sums of analytic functions, we consider the ratio of the real part of function $l(z)$ which is defined by

$$l(z) = z - \sum_{m=2}^{\infty} a_m z^m$$

to its sequence of partial sums $l_1(z) = z$

$$l_n(z) = z - \sum_{m=2}^{n} a_m z^m.$$

Theorem 6.1. *Let $l(z) \in TS_{\mu}^{\varrho}(\lambda,\gamma)$ given by* (1.2) *and define $l_1(z)$ and $l_n(z)$ by $l_1(z) = z$,*

$$l_n(z) = z - \sum_{m=2}^{n} a_m z^m$$

then

$$\Re\left\{\frac{l(z)}{l_n(z)}\right\} > 1 - \frac{1}{B_{n+1}}, \quad z \in U \quad (6.1)$$

and

$$\Re\left\{\frac{l_n(z)}{l(z)}\right\} > \frac{B_{n+1}}{1+B_{n+1}}, \quad z \in U \quad (6.2)$$

where

$$B_n = \frac{[2n-\lambda(\gamma+1)]\phi_n(\mu,\varrho,\zeta_1,\zeta_3)}{1-\gamma}.$$

Proof. We use same technique used by Silverman [12]. The function $l(z) \in TS_{\mu}^{\varrho}(\lambda,\gamma)$ if and only if $\sum_{m=2}^{\infty} B_m a_m \le 1$.
Where, $B_{n+1} > B_n > 1$ therefore we have

$$\sum_{m=2}^{n} |a_m| + B_{n+1} \sum_{m=n+1}^{\infty} |a_m| \le \sum_{m=2}^{\infty} B_m |a_m| \le 1.$$

Let

$$\frac{1+p(z)}{1-p(z)} = B_{n+1}\left\{\frac{l(z)}{l_n(z)} - \left(1 - \frac{1}{B_{n+1}}\right)\right\}$$
$$= \frac{1 - B_{n+1}\sum\limits_{m=n+1}^{\infty} a_m z^{m-1} - \sum\limits_{m=2}^{n} a_m z^{m-1}}{1 - \sum\limits_{m=2}^{n} a_m z^{m-1}},$$

where

$$p(z) = \frac{-B_{n+1}\sum\limits_{m=n+1}^{\infty} a_m z^{m-1}}{2 - 2\sum\limits_{m=2}^{n} a_m z^{m-1} - B_{n+1}\sum\limits_{m=n+1}^{\infty} a_m z^{m-1}}.$$

Now,

$$|p(z)| \le \frac{B_{n+1}\sum\limits_{m=n+1}^{\infty} a_m}{2 - 2\sum\limits_{m=2}^{n} a_m - B_{n+1}\sum\limits_{m=n+1}^{\infty} a_m}.$$

$|p(z)| \le 1$ if and only if

$$2B_{n+1}\sum_{m=n+1}^{\infty} a_m \le 2 - 2\sum_{m=2}^{n} a_m,$$

which is equivalent to

$$\sum_{m=2}^{n} a_m + B_{n+1}\sum_{m=n+1}^{\infty} a_m \le 1,$$

which gives (6.1).
Similarly, take

$$\frac{1+w(z)}{1-w(z)} = (1+B_{n+1})\left\{\frac{l_n(z)}{l(z)} - \left(1 - \frac{B_{n+1}}{1+B_{n+1}}\right)\right\}$$
$$= \frac{(1+B_{n+1})\sum\limits_{m=n+1}^{\infty} a_m z^{m-1} - \sum\limits_{m=2}^{n} a_m z^{m-1}}{1 - \sum\limits_{m=2}^{\infty} a_m z^{m-1}},$$

where

$$w(z) = \frac{(1+B_{n+1})\sum\limits_{m=n+1}^{\infty} a_m z^{m-1}}{2 - 2\sum\limits_{m=2}^{n} a_m z^{m-1} + (B_{n+1}-1)\sum\limits_{m=n+1}^{\infty} a_m z^{m-1}}.$$

Now,

$$|w(z)| \leq \frac{(1+B_{n+1})\sum\limits_{m=n+1}^{\infty} a_m}{2-2\sum\limits_{m=2}^{n} a_m z^{m-1} - (B_{n+1}-1)\sum\limits_{m=n+1}^{\infty} a_m z^{m-1}}.$$

$|w(z)| \leq 1$ if and only if

$$2B_{n+1}\sum_{m=n+1}^{\infty} a_m \leq 2-2\sum_{m=2}^{n} a_m,$$

which is equivalent to

$$\sum_{m=2}^{n} a_m + B_{n+1}\sum_{m=n+1}^{\infty} a_m \leq 1,$$

which gives (6.2). □

7. Radii of Close-to-convexity, Starlikeness and Convexity

Now, we obtain the radii of convexity, close-to-convexity and starlikeness for the class $TS_{\mu}^{\varrho}(\lambda,\gamma)$.
The following Theorems 7.1, 7.2, and 7.3 can be proved the similer techniques of Silverman's [9] and hence omitted.

THEOREM 7.1. *Let the function $l(z)$ defined by (1.2) belong to the class $TS_{\mu}^{\varrho}(\lambda,\gamma)$, then $l(z)$ is close-to-convex of order ξ $(0 \leq \xi < 1)$ in the disc $|z| < r_1$, where*

$$r_1 = \inf_{m\geq 2}\left[\frac{(1-\xi)[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)}{m(1-\gamma)}\right]^{\frac{1}{m-1}}, \ (m \geq 2) \tag{7.1}$$

The result is sharp for the function $l(z)$ given in (2.4).

THEOREM 7.2. *Let the function $l(z)$ defined by (1.2) belong to the class $TS_{\mu}^{\varrho}(\lambda,\gamma)$ then $l(z)$ is starlike of order ξ $(0 \leq \xi < 1)$ in the disc $|z| < r_2$, where*

$$r_2 = \inf_{m\geq 2}\left[\frac{(1-\xi)[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)}{(m-\xi)(1-\gamma)}\right]^{\frac{1}{m-1}}, \ (m \geq 2) \tag{7.2}$$

The result is sharp for the function $l(z)$ given in (2.4).

THEOREM 7.3. *Let the function $l(z)$ defined by (1.2) belong to the class $TS^{\varrho}_{\mu}(\lambda, \gamma)$, then $l(z)$ is convex of order ξ $(0 \leq \xi < 1)$ in the disc $|z| < r_3$, where*

$$r_3 = \inf_{m\geq 2}\left[\frac{(1-\xi)[2m-\lambda(\gamma+1)]\phi_m(\mu,\varrho,\zeta_1,\zeta_3)}{m(m-\xi)(1-\gamma)}\right]^{\frac{1}{m-1}}, \ (m \geq 2) \tag{7.3}$$

The result is sharp for the function $l(z)$ given in (2.4).

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References

[1] M. Abdalla, M. Hidan, S. M. Boulaaras and B. B. Cherif, *Investigation of extended k-hypergeometric functions and associated fractional integrals*, Mathematical Problems in Engineering **2021** (2021) 1–11.

[2] A. W. Goodman, *On uniformly starlike functions*, Journal of Mathematical Analysis and Applications **155** (1991) 364–370.

[3] A. W. Goodman, *On uniformly convex functions* , Ann. Polon. Math. **56** (1991) 87–92.

[4] S. Kanas and A. Wisniowska, *Conic regions and k-uniform convexity*, J. Compu. Appl. Math., **105**(1999) 327–336.

[5] J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc., **23** (1925) 481–519.

[6] G. Murugusundaramoorthy and N. Magesh, *On certain subclasses of analytic functions associated with hypergeometric functions*, Applied Mathematics Letters, **24** (2011) 494–500.

[7] F. Ronning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc, **118** (1993) 189–196.

[8] F. Ronning, *Integral representation of bounded starlike functions*, Ann. Polon. Math., **60** (1995) 289–297.

[9] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math soc, **51** (1975) 109–116.

[10] H. Silverman, *A survey with open problems on univalent functions whose coefficients are negative*, Rocky. Mountain J. Math., **21** (1995) 1099–1125.

[11] H. Silverman, *Integral means for univalent functions with negative coefficients*, Houston J. Math., **23** (1997) 169–174.

[12] H. Silverman, *Partial sums of starlike and convex functions*, J. Anal. Appl., **209** (1997) 221–227.

[13] E. M. Silvia, *Partial sums of convex functions of order α*, Houston J. Math., **11** (1985) 397–404.

[14] J. Spanier and K. B. Oldham, *The zeta number and related functions*, Chapter 3 in Atlas of functions, Washington, Dc: Hemisphere, 25–33, 1987.

[15] B. Venkateswarlu and P. Thirupathi Reddy, *A certain subclass of uniformly convex functions defined by bessel functions*, Proyecciones Journal of Mathematics, **38** (2019) 719–731.

S. A. Jadhav, Department of Mathematics, New Arts, Commerce and Science College, Ahmednagar-414 001, Maharashtra, India
e-mail: jadhavsonali108@gmail.com

P. G. Jadhav, Department of Mathematics Balasaheb Jadhav Arts, Commerce and Science College, Pune-412 411, Maharashtra, India.
e-mail: pgjmaths1@gmail.com

S. B. Gaikwad, Department of Mathematics, New Arts, Commerce and Science College,
Ahmednagar-414 001, Maharashtra, India
e-mail: sbgmathsnagar@gmail.com

P. Thirupathi Reddy, (Corresponding author)
Department of Mathematics, DRK Institute of Science and Technology, Bowrampet, Hyderabad- 500 043, Telangana, India.
e-mail: reddypt2@gmail.com