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ON CONTACT *CR*-LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE SASAKIAN STATISTICAL MANIFOLD

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Abstract

This study aims to investigate the geometry of contact *CR*-lightlike submanifolds of an indefinite Sasakian statistical manifold. It characterizes the integrability of distributions of these submanifolds and formulates findings regarding the subbundles within the tangent bundle inherent in the indefinite Sasakian statistical structure. A condition for a maximal integrable manifold of the radical distribution to be totally geodesic has also been provided.

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1. Introduction

The inspection of submanifolds of a contact metric manifold has been an active area of interest for the geometers from time to time but it had essentially been limited to differentiable manifolds with non-degenerate metric. So, an important class of submanifolds known as lightlike submanifolds, initiated by Duggal [8], having wide applications in various branches of mathematics where the metric is degenerate, gathered immense popularity. It led to the development of many significant results for lightlike geometry of several contact metric manifolds. In this direction, the contact *CR*-lightlike submanifolds were introduced by [9] and further analyzed by [7], [12], [20] and others.

Through the analysis of geometric structures on a set of certain probability distributions, a significant branch of study known as statistical manifolds, have come into existence. The statistical structure was first introduced by Rao in [18]. Thereafter, many geometers [1], [13], [6], [4] et.al worked in this direction and studied the properties of Riemannian manifold of a statistical model. Later, Amari [2], [3] introduced statistical manifold based on the information geometry which is the study of probability and information from the view point of differential geometry. His work was subsequently researched by [15], [21], [19] et.al.

By combining the notion of statistical manifold with the contact metric manifold, Furuhata et.al [11] introduced the notion of Sasakian statistical manifold and developed some conditions for a real hypersurface in a holomorphic statistical manifold. This work was further investigated by [10], [22], [16], [17] et.al. Oguzhan Bahadir [5] related the concept of lightlike geometry to the Sasakian statistical structure and developed some relations among induced geometrical objects with respect to dual connections in a lightlike hypersurface of an indefinite statistical manifold.

Thus inspired, we study the geometry of contact CR-lightlike submanifolds in the indefinite Sasakian statistical manifold and exemplify its structure. Various results for the integrability of distributions and subbundles of the tangent bundle have also been derived.

2. Preliminaries

The basic theory of lightlike submanifolds following [8], [9] and [14] is as below:

A submanifold M^m immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called r-lightlike submanifold if it admits a degenerate metric g induced from $\bar{g}$ whose radical distribution $Rad(TM)$ of rank r, where $m, n \geq 1$, $1 \leq r \leq m$, is defined by

$$Rad(T_xM) = \{\xi \in T_xM : g_x(\xi, X) = 0,\ X \in \Gamma(TM),\ x \in M\}.$$

Also,

$$Rad(TM) = TM \cap TM^{\perp},$$

where

$$TM^{\perp} = \bigcup_{x \in M} \{u \in T_x\bar{M} : \bar{g}(u, v) = 0, \forall\ v \in T_xM\}.$$

Consider $S(TM)$, known as Screen distribution, as a complementary distribution of radical distribution in TM, i.e.,

$$TM = Rad(TM) \perp S(TM),$$

and $S(TM^{\perp})$, called screen transversal vector bundle as a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$, i.e.,

$$TM^{\perp} = Rad(TM) \perp S(TM^{\perp}).$$

As $S(TM)$ is non degenerate vector subbundle of $T\bar{M}|_M$, we have

$$T\bar{M}|_M = S(TM) \perp S(TM)^{\perp},$$

where $S(TM)^{\perp}$ is the complementary orthogonal vector subbundle of $S(TM)$ in $T\bar{M}|_M$.

If $tr(TM)$ and $ltr(TM)$ denote the complementary vector bundles to TM in $T\bar{M}|_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$, then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}),$$

$$T\bar{M}\ |_M = TM \oplus tr(TM) = \{Rad(TM) \oplus ltr(TM)\} \perp S(TM) \perp S(TM^{\perp}). \quad (2.1)$$

For any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $(S(TM))^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$. Therefore, there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ where $i = 1, 2, ...r$.

Let $\tilde{\nabla}$ be the Levi-Civita connection on $\bar{M}$, then the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \qquad \forall\ X, Y \in \Gamma(TM), \quad (2.2)$$

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \qquad \forall\ X \in \Gamma(TM), V \in \Gamma(tr(TM)), \quad (2.3)$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^{\perp}_X V\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$ respectively. Here, ∇ and $\nabla^{\perp}$ are linear connections on M and the vector bundle $tr(TM)$ respectively.

Considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively, (2.2) and (2.3) become

$$\tilde{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y),$$

$$\tilde{\nabla}_X V = -A_V X + D^l_X V + D^s_X V.$$

In particular, we have

$$\tilde{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad \tilde{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W),$$

where $X, Y \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$.

Here $h^l(X, Y) = Lh(X, Y)$, $h^s(X, Y) = Sh(X, Y)$, $D^l_X V = L(\nabla^{\perp}_X V)$, $D^s_X V = S(\nabla^{\perp}_X V)$, $\nabla^l_X N$, $D^l(X, W) \in \Gamma(ltr(TM))$, $\nabla^s_X W, D^s(X, N) \in \Gamma(S(TM^{\perp}))$ and $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$; wherein h^l and h^s are respectively called the lightlike second fundamental form and the screen second fundamental form on M .

Let P be the projection morphism of TM on $S(TM)$. We consider the following decomposition:

$$\nabla_X PY = \nabla'_X PY + h'(X, PY), \qquad \nabla_X \xi = -A'_\xi X + \nabla'^{\perp}_X \xi,$$

where $\{\nabla'_X Y, A'_\xi X\} \in \Gamma(S(TM))$ and $\{h'(X, PY), \nabla'^{\perp}_X \xi\} \in \Gamma(Rad(TM))$; ∇' and $\nabla'^{\perp}$ are linear connections on $S(TM)$ and $Rad(TM)$ respectively.

Using the above equations, we obtain

$$\bar{g}(h^l(X,PY),\xi) = g(A'_\xi X, PY), \qquad \bar{g}(h'(X,PY),N) = g(A_N X, PY),$$
$$\bar{g}(h^l(X,\xi),\xi) = 0, \qquad g(A'_\xi PX, PY) = g(PX, A'_\xi PY),$$
$$A'_\xi \xi = 0,$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

Definition 2.1. A pair $(\bar{\nabla}, \bar{g})$ is called an indefinite statistical structure on a semi-Riemannian manifold $\bar{M}$ where $\bar{g}$ is a semi-Riemannian metric of constant index $q \geq 1$ on $\bar{M}$, if $\bar{\nabla}$ is torsion free and the Codazzi equation

$$(\bar{\nabla}_X \bar{g})(Y,Z) = (\bar{\nabla}_Y \bar{g})(X,Z),$$

holds for any $X, Y, Z \in \Gamma(T\bar{M})$.

Moreover, there exists $\bar{\nabla}^*$ which is a dual connection of $\bar{\nabla}$ with respect to $\bar{g}$, satisfying

$$X\bar{g}(Y,Z) = \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}^*_X Z) \qquad \forall\ X, Y, Z \in \Gamma(T\bar{M}). \tag{2.4}$$

If $(\bar{M}, \bar{g}, \bar{\nabla})$ is an indefinite statistical manifold, then so is $(\bar{M}, \bar{g}, \bar{\nabla}^*)$. Hence the indefinite statistical manifold is denoted by $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$.

The Gauss and Weingarten formulae in the context of lightlike submanifold (M, g) of an indefinite statistical manifold $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$ are as follows:

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h^l(X,Y) + h^s(X,Y), \\ \bar{\nabla}^*_X Y &= \nabla^*_X Y + h^{*l}(X,Y) + h^{*s}(X,Y), \end{aligned} \tag{2.5}$$

$$\begin{aligned} \bar{\nabla}_X V &= -A_V X + D^l_X V + D^s_X V, \\ \bar{\nabla}^*_X V &= -A^*_V X + D^{*l}_X V + D^{*s}_X V, \\ \bar{\nabla}_X N &= -A_N X + \nabla^l_X N + D^s(X,N), \\ \bar{\nabla}^*_X N &= -A^*_N X + \nabla^{*l}_X N + D^{*s}(X,N), \\ \bar{\nabla}_X W &= -A_W X + \nabla^s_X W + D^l(X,W), \\ \bar{\nabla}^*_X W &= -A^*_W X + \nabla^{*s}_X W + D^{*l}(X,W), \end{aligned} \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(tr(TM))$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(STM^\perp)$.

Considering the corresponding projection morphism P of tangent bundle TM to the screen distribution, we have the following decomposition with respect to ∇ and ∇^*:

$$\nabla_X PY = \nabla'_X PY + h'(X,PY), \quad \nabla^*_X PY = \nabla^{*\prime}_X PY + h^{*\prime}(X,PY), \tag{2.7}$$

$$\nabla_X \xi = -A'_\xi X + \nabla'^{\perp}_X \xi, \quad \nabla^*_X \xi = -A^{*\prime}_\xi X + \nabla^{*\prime\perp}_X \xi, \tag{2.8}$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$.

Then the following holds:

$$\begin{aligned} \bar{g}(h^l(X, PY), \xi) &= g(A^{*\prime}_\xi X, PY), \\ \bar{g}(h^{*l}(X, PY), \xi) &= g(A'_\xi X, PY), \end{aligned} \tag{2.9}$$

$$\begin{aligned} \bar{g}(h'(X, PY), N) &= g(A^*_N X, PY), \\ \bar{g}(h^{*\prime}(X, PY), N) &= g(A_N X, PY). \end{aligned} \tag{2.10}$$

3. Indefinite Sasakian statistical manifold

Following [11], we consider a Levi-Civita connection $\widehat{\bar{\nabla}}$ with respect to $\bar{g}$ such that $\widehat{\bar{\nabla}} = \frac{1}{2}(\bar{\nabla} + \bar{\nabla}^*)$.

For a statistical manifold $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$, the difference $(1, 2)$ tensor K of a torsion free affine connection $\bar{\nabla}$ and Levi-Civita connection $\widehat{\bar{\nabla}}$ is defined as

$$K(X, Y) = K_X Y = \bar{\nabla}_X Y - \widehat{\bar{\nabla}}_X Y.$$

Since $\bar{\nabla}$ and $\widehat{\bar{\nabla}}$ are torsion free, we have

$$K_X Y = K_Y X, \qquad \bar{g}(K_X Y, Z) = \bar{g}(Y, K_X Z), \tag{3.1}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$.

Also

$$K(X, Y) = \widehat{\bar{\nabla}}_X Y - \bar{\nabla}^*_X Y.$$

From the above equations, we get

$$K(X, Y) = \frac{1}{2}(\bar{\nabla}_X Y - \bar{\nabla}^*_X Y).$$

Definition 3.1. [9] An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called contact metric manifold if there are a $(1, 1)$ tensor field ϕ, a vector field ν called characteristic vector field and a 1-form η such that

$$\begin{gathered} \bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y), \qquad \bar{g}(\nu, \nu) = 1, \\ \phi^2(X) = -X + \eta(X)\nu, \quad \bar{g}(X, \nu) = \eta(X), \quad \bar{g}(\phi X, Y) + \bar{g}(X, \phi Y) = 0, \\ d\eta(X, Y) = \bar{g}(X, \phi Y) \quad \forall\ X, Y \in \Gamma(T\bar{M}). \end{gathered} \tag{3.2}$$

It follows that $\phi\nu = 0$, $\eta o \phi = 0$, $\eta(\nu) = 1$. Then $(\phi, \nu, \eta, \bar{g})$ is called contact metric structure of $\bar{M}$.

Also, $\bar{M}$ has a normal contact structure if $N_\phi + d\eta \otimes \nu = 0$, where N_ϕ is the Nijenhuis tensor field.

A normal contact metric manifold $\bar{M}$ is called an indefinite Sasakian manifold if

$$\widehat{\bar{\nabla}}_X \nu = -\phi X,$$

$$(\widehat{\bar{\nabla}}_X \phi)Y = \bar{g}(X, Y)\nu - \eta(Y)X,$$

holds for any $X, Y \in \Gamma(T\bar{M})$, where $\widehat{\bar{\nabla}}$ is Levi-Civita Connection.

Definition 3.2. [5] A quadruplet $(\bar{\nabla} = \widehat{\bar{\nabla}} + K, \bar{g}, \phi, \nu)$ is called an indefinite Sasakian statistical structure on $\bar{M}$ if

1. $(\bar{g}, \phi, \nu)$ is an indefinite Sasakian structure on $\bar{M}$,
2. $(\bar{\nabla}, \bar{g})$ is a statistical structure on $\bar{M}$ and the condition

$$K(X, \phi Y) = -\phi K(X, Y), \tag{3.3}$$

holds for any $X, Y \in \Gamma(T\bar{M})$.

Then $(\bar{M}, \bar{\nabla}, \bar{g}, \phi, \nu)$ is called an indefinite Sasakian statistical manifold. If $(\bar{M}, \bar{\nabla}, \bar{g}, \phi, \nu)$ is an indefinite Sasakian statistical manifold, then so is $(\bar{M}, \bar{\nabla}^*, \bar{g}, \phi, \nu)$.

Definition 3.3. [5] Let $(\bar{M}, \bar{\nabla}, \bar{g})$ be an indefinite statistical manifold and $(\bar{g}, \phi, \nu)$ an almost contact metric structure on $\bar{M}$. Then $(\bar{\nabla}, \bar{g}, \phi, \nu)$ is an indefinite Sasakian statistical struture if and only if the following conditions hold:

$$\bar{\nabla}_X \phi Y - \phi \bar{\nabla}^*_X Y = \bar{g}(Y, X)\nu - \bar{g}(Y, \nu)X, \tag{3.4}$$

$$\bar{\nabla}_X \nu = -\phi X + \bar{g}(\bar{\nabla}_X \nu, \nu)\nu,$$

for all the vector fields X, Y on $\bar{M}$.

4. Contact *CR*-lightlike submanifold of an indefinite Sasakian statistical manifold

Definition 4.1. A submanifold M of an indefinite Sasakian statistical manifold $(\bar{M}, \bar{g})$ is said to be a contact CR-lightlike submanifold if the following conditions hold:

1. $Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \phi(Rad(TM)) = \{0\}$,
2. there exist vector bundles D_o and D' over M such that

$$S(TM) = \{\phi(Rad(TM)) \oplus D'\} \perp D_o \perp \{\nu\}, \tag{4.1}$$

$$\phi D_o = D_o, \qquad \phi(D') = L_1 \perp ltr(TM),$$

where D_o is non-degenerate and L_1 is a vector subbundle of $S(TM^\perp)$.

Thus, the following decomposition:

$$TM = D \oplus \{\nu\} \oplus D', \qquad D = Rad(TM) \perp \phi(Rad(TM)) \perp D_o.$$

Denote the orthogonal complement subbundle to the vector subbundle L_1 in $S(TM^{\perp})$ by $L_1^{\perp}$. For a contact CR-lightlike submanifold M, we put

$$\phi X = fX + wX \quad \forall\, X \in \Gamma(TM),$$

where $fX \in \Gamma(D)$ and $wX \in \Gamma(L_1 \perp ltr(TM))$. Similarly, we have

$$\phi W = BW + CW \quad \forall\, W \in \Gamma(S(TM^{\perp})),$$

where $BW \in \Gamma(\phi L_1)$ and $CW \in \Gamma(L_1^{\perp})$.

Following [11], we have dealt with the basic structure of a contact CR-lightlike submanifold of an indefinite Sasakian statistical manifold and elaborated it with an example.

Remark 4.2. *Let $(\bar{g}, \phi, \nu)$ be an indefinite Sasakian structure on $\bar{M}$. By setting*

$$K(X, Y) = \bar{g}(X, \nu)\bar{g}(Y, \nu)\nu,$$

for any $X, Y \in \Gamma(T\bar{M})$ such that K satisfies (3.1) and (3.3), we obtain an indefinite Sasakian statistical structure $(\bar{\nabla}^{\lambda} = \widehat{\bar{\nabla}} + \lambda K, \bar{g}, \phi, \nu)$ on $\bar{M}$ for $\lambda \in C^{\infty}(\bar{M})$.

Inspired by [9], we have the following example:

Example 4.3. *Let $\bar{M} = (\mathbb{R}^9_2, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-, +, +, +, -, +, +, +, +)$ with respect to canonical basis*

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}.$$

Here $(\mathbb{R}^9_2, \phi_o, \nu, \eta, \bar{g})$ denotes the indefinite Sasakian manifold with its degenerate structure given as:

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{4} y^i dx^i), \qquad \nu = 2\partial z,$$

$$\bar{g} = \eta \otimes \eta + \frac{1}{4}(-dx^1 \otimes dx^1 - dy^1 \otimes dy^1 + dx^3 \otimes dx^3 + dy^3 \otimes dy^3 + dx^4 \otimes dx^4 + dy^4 \otimes dy^4),$$

$$\phi_o(\sum_{i=1}^{4}(X_i\partial x^i + Y_i\partial y^i) + Z\partial z) = \sum_{i=1}^{4}(Y_i\partial x^i - X_i\partial y_i) + \sum_{i=1}^{4} Y_i y^i \partial z,$$

for the cartesian coordinates $(x^i; y^i; z)$.

As per Remark (4.2), if we set $K(X, Y) = \bar{g}(X, \nu)\bar{g}(Y, \nu)\nu$ and take $\lambda = 1$, then $(\bar{\nabla} = \widehat{\bar{\nabla}} + K, \bar{g}, \phi, \nu)$ defines an indefinite Sasakian statistical structure on $\bar{M}$.

Consider a submanifold M of $\mathbb{R}^9_2$ defined by

$$x^1 = y^4, \quad x^2 = \sqrt{1 - (y^2)^2}, \quad y^2 \neq \pm 1.$$

A local frame of TM is given by

$$e_1 = 2(\partial x_1 + \partial y_4 + y^1\partial z), \quad e_2 = 2(\partial x_4 - \partial y_1 + y^4\partial z),$$

$$e_3 = \partial x_3 + y^3\partial z, \quad e_4 = \partial y_3, \quad e_5 = -\frac{y^2}{x^2}\partial x_2 + \partial y_2 - \frac{(y^2)^2}{x_2}\partial z,$$

$$e_6 = \partial x_4 + \partial y_1 + y^4\partial z, \quad e_7 = \nu = 2\partial z.$$

We see that,

$Rad(TM) = span\{e_1\}$, $\phi_o Rad(TM) = span\{e_2\}$ and $Rad(TM) \cup \phi_o Rad(TM) = \{0\}$.

Further, $\phi_o(e_3) = -e_4$ which shows that $D_o = \{e_3, e_4\}$ is invariant with respect to ϕ_o.

Hence, we have

$$S(TM^\perp) = span\{W = \partial x_2 + \frac{y^2}{x^2}\partial y_2 + y^2\partial z\}$$

such that $\phi_o(W) = -e_5$. Also, $ltr(TM) = span\ \{N = -\partial x_1 + \partial y_4 - y^1\partial z\}$ where $\phi_o(N) = e_6$.

Therefore, M becomes a contact CR-lightlike submanifold of the indefinite Sasakian statistical manifold.

4.1. Results on subbundles of the tangent bundle

Definition 4.4. Let $\bar{M}$ be an indefinite Sasakian statistical manifold and M be a contact CR-lightlike submanifold of $\bar{M}$. Then, distribution D_o is integrable if and only if $[X, Y] \in \Gamma(D_o)$ for any $X, Y \in \Gamma(D_o)$.

Theorem 4.5. *For a contact CR-lightlike submanifold M of an indefinite Sasakian statistical manifold $\bar{M}$, $Rad(TM)$ is integrable if and only if*

(i) $\bar{g}(h^{*l}(\xi', \nu), \xi'') + \bar{g}(h^l(\xi', \nu), \xi'') = \bar{g}(h^{*l}(\xi'', \nu), \xi') + \bar{g}(h^l(\xi'', \nu), \xi')$,
(ii) $\bar{g}(A'_{\xi''}\xi', Z) + \bar{g}(A^{*\prime}_{\xi''}\xi', Z) = \bar{g}(A'_{\xi'}\xi'', Z) + \bar{g}(A^{*\prime}_{\xi'}\xi'', Z)$,
(iii) $\bar{g}(A'_{\xi''}\xi', \phi\xi) + \bar{g}(A^{*\prime}_{\xi''}\xi', \phi\xi) = \bar{g}(A'_{\xi'}\xi'', \phi\xi) + \bar{g}(A^{*\prime}_{\xi'}\xi'', \phi\xi)$,
(iv) $\bar{g}(h^{*s}(\xi', \phi\xi''), W) + \bar{g}(h^s(\xi', \phi\xi''), W) = \bar{g}(h^{*s}(\xi'', \phi\xi'), W) + \bar{g}(h^s(\xi'', \phi\xi'), W)$,
(v) $\bar{g}(h^{*\prime}(\xi', \phi\xi''), N) + \bar{g}(h'(\xi', \phi\xi''), N) = \bar{g}(h^{*\prime}(\xi'', \phi\xi'), N) + \bar{g}(h'(\xi'', \phi\xi'), N)$,

for all $\xi, \xi', \xi'' \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$, $W \in \Gamma(L_1)$ and $Z \in \Gamma(D_o)$.

Proof. As per the concept of contact CR-lightlike submanifold of an indefinite Sasakian statistical manifold $\bar{M}$, $Rad(TM)$ is integrable if and only if

for all $\xi, \xi', \xi'' \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$, $W \in \Gamma(L_1)$ and $Z \in \Gamma(D_0)$, we have

$$\bar{g}([\xi', \xi''], \nu) = \bar{g}([\xi', \xi''], Z) = \bar{g}([\xi', \xi''], \phi\xi) = \bar{g}([\xi', \xi''], \phi W) = \bar{g}([\xi', \xi''], \phi N) = 0. \tag{4.2}$$

Using the statistical character of $\bar{M}$ and the equations (2.4), (2.5) and (2.1), we have for $\{\nu\} \in \Gamma(S(TM))$

$$\begin{aligned}
&\bar{g}([\xi', \xi''], \nu) = \bar{g}(\widehat{\bar{\nabla}}_{\xi'}\xi'', \nu) - \bar{g}(\widehat{\bar{\nabla}}_{\xi''}\xi', \nu) \\
&= \tfrac{1}{2}[\bar{g}(\bar{\nabla}_{\xi'}\xi'', \nu) + \bar{g}(\bar{\nabla}^*_{\xi'}\xi'', \nu) - \bar{g}(\bar{\nabla}_{\xi''}\xi', \nu) - \bar{g}(\bar{\nabla}^*_{\xi''}\xi', \nu)] \\
&= \tfrac{1}{2}[-\bar{g}(\xi'', \bar{\nabla}^*_{\xi'}\nu) - \bar{g}(\xi'', \bar{\nabla}_{\xi'}\nu) + \bar{g}(\xi', \bar{\nabla}^*_{\xi''}\nu) + \bar{g}(\xi', \bar{\nabla}_{\xi''}\nu)] \\
&= \tfrac{1}{2}[-\bar{g}(\xi'', h^{*l}(\xi', \nu)) - \bar{g}(\xi'', h^{l}(\xi', \nu)) + \bar{g}(\xi', h^{*l}(\xi'', \nu)) + \bar{g}(\xi', h^{l}(\xi'', \nu))].
\end{aligned}$$

Therefore, (i) follows using the hypothesis.

Further, (2.4), (2.5) and (2.9) implies

$$\begin{aligned}
&\bar{g}([\xi', \xi''], Z) = \tfrac{1}{2}\bar{g}(\bar{\nabla}_{\xi'}\xi'' + \bar{\nabla}^*_{\xi'}\xi'', Z) - \tfrac{1}{2}\bar{g}(\bar{\nabla}_{\xi''}\xi' + \bar{\nabla}^*_{\xi''}\xi', Z) \\
&= \tfrac{1}{2}[-\bar{g}(\xi'', \bar{\nabla}^*_{\xi'}Z) - \bar{g}(\xi'', \bar{\nabla}_{\xi'}Z) + \bar{g}(\xi', \bar{\nabla}^*_{\xi'}Z) + \bar{g}(\xi', \bar{\nabla}_{\xi''}Z)] \\
&= \tfrac{1}{2}[-\bar{g}(A'_{\xi''}\xi', Z) - \bar{g}(A^{*\prime}_{\xi''}\xi', Z) + \bar{g}(A'_{\xi'}\xi'', Z) + \bar{g}(A^{*\prime}_{\xi'}\xi'', Z)],
\end{aligned}$$

and $\bar{g}([\xi', \xi''], \phi\xi) = \frac{1}{2}[-\bar{g}(\xi'', \bar{\nabla}^*_{\xi'}\phi\xi) - \bar{g}(\xi'', \bar{\nabla}_{\xi'}\phi\xi) + \bar{g}(\xi', \bar{\nabla}^*_{\xi''}\phi\xi) + \bar{g}(\xi', \bar{\nabla}_{\xi''}\phi\xi)]$
$= \frac{1}{2}[-\bar{g}(A'_{\xi''}\xi', \phi\xi) - \bar{g}(A^{*\prime}_{\xi''}\xi', \phi\xi) + \bar{g}(A'_{\xi'}\xi'', \phi\xi) + \bar{g}(A^{*\prime}_{\xi'}\xi'', \phi\xi)]$.

Since $\bar{M}$ is an indefinite Sasakian statistical manifold, therefore

$$\begin{aligned}
&\bar{g}([\xi', \xi''], \phi W) = \tfrac{1}{2}\bar{g}(\bar{\nabla}_{\xi'}\xi'' + \bar{\nabla}^*_{\xi'}\xi'', \phi W) - \tfrac{1}{2}\bar{g}(\bar{\nabla}_{\xi''}\xi' + \bar{\nabla}^*_{\xi''}\xi', \phi W) \\
&= \tfrac{1}{2}[-\bar{g}(\phi\bar{\nabla}_{\xi'}\xi'', W) - \bar{g}(\phi\bar{\nabla}^*_{\xi'}\xi'', W) + \bar{g}(\phi\bar{\nabla}_{\xi''}\xi', W) + \bar{g}(\phi\bar{\nabla}^*_{\xi''}\xi', W)] \\
&= \tfrac{1}{2}[-\bar{g}(\bar{\nabla}^*_{\xi'}\phi\xi'' - \bar{g}(\xi'', \xi')\nu + \bar{g}(\xi'', \nu)\xi', W) - \bar{g}(\bar{\nabla}_{\xi'}\phi\xi'' - \bar{g}(\xi'', \xi')\nu + \bar{g}(\xi'', \nu)\xi', W) \\
&\quad +\bar{g}(\bar{\nabla}^*_{\xi''}\phi\xi' - \bar{g}(\xi', \xi'')\nu + \bar{g}(\xi', \nu)\xi'', W) + \bar{g}(\bar{\nabla}_{\xi''}\phi\xi' - \bar{g}(\xi', \xi'')\nu + \bar{g}(\xi', \nu)\xi'', W)] \\
&= \tfrac{1}{2}[-\bar{g}(h^{*s}(\xi', \phi\xi''), W) - \bar{g}(h^{s}(\xi', \phi\xi''), W) + \bar{g}(h^{*s}(\xi'', \phi\xi'), W) + \bar{g}(h^{s}(\xi'', \phi\xi'), W)],
\end{aligned}$$

and $\bar{g}([\xi', \xi''], \phi N) = \bar{g}(\widehat{\bar{\nabla}}_{\xi'}\xi'', \phi N) - \bar{g}(\widehat{\bar{\nabla}}_{\xi''}\xi', \phi N)$

$$\begin{aligned}
&= \tfrac{1}{2}[-\bar{g}(\bar{\nabla}^*_{\xi'}\phi\xi'' - \bar{g}(\xi'', \xi')\nu + \bar{g}(\xi'', \nu)\xi', N) - \bar{g}(\bar{\nabla}_{\xi'}\phi\xi'' - \bar{g}(\xi'', \xi')\nu + \bar{g}(\xi'', \nu)\xi', N) \\
&\quad +\bar{g}(\bar{\nabla}^*_{\xi''}\phi\xi' - \bar{g}(\xi', \xi'')\nu + \bar{g}(\xi', \nu)\xi'', N) + \bar{g}(\bar{\nabla}_{\xi''}\phi\xi' - \bar{g}(\xi', \xi'')\nu + \bar{g}(\xi', \nu)\xi'', N)] \\
&= \tfrac{1}{2}[-\bar{g}(\phi\xi'', A_N\xi') - \bar{g}(\phi\xi'', A^*_N\xi') + \bar{g}(\phi\xi', A_N\xi'') + \bar{g}(\phi\xi', A^*_N\xi'')].
\end{aligned}$$

Hence (ii), (iii), (iv) and (v) follows from (4.2).

□

THEOREM 4.6. *Let $(\bar{M}, \bar{\nabla}, \bar{g}, \phi, \nu)$ be an indefinite Sasakian statistical manifold and M be a contact CR-lightlike submanifold of $\bar{M}$. Then $\phi Rad(TM)$ is integrable if and only if*

(i) $\bar{g}(h^{l}(\phi X, \phi Z), Y) + \bar{g}(h^{*l}(\phi X, \phi Z), Y) = \bar{g}(h^{l}(\phi Y, \phi Z), X) + \bar{g}(h^{*l}(\phi Y, \phi Z), X)$,
(ii) $\bar{g}(A^{*\prime}_{\xi}\phi X, Y) + \bar{g}(A'_{\xi}\phi X, Y) = \bar{g}(A^{*\prime}_{\xi}\phi Y, X) + \bar{g}(A'_{\xi}\phi Y, X)$,
(iii) $\bar{g}(h^{*s}(\phi X, Y), W) + \bar{g}(h^{s}(\phi X, Y), W) = \bar{g}(h^{*s}(\phi Y, X), W) + \bar{g}(h^{s}(\phi Y, X), W)$,

(iv) $\bar{g}(A^{*\prime}_{Y}\phi X, \phi N) + \bar{g}(A'_{Y}\phi X, \phi N) = \bar{g}(A^{*\prime}_{X}\phi Y, \phi N) + \bar{g}(A'_{X}\phi Y, \phi N)$,
for all $X, Y \in \Gamma(Rad(TM))$, $Z \in \Gamma(D_0)$, $\xi \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$
and $W \in \Gamma(S(TM^{\perp}))$.

Proof. Using (2.8), (2.9) and (3.4), we derive

$$\begin{aligned}
\bar{g}([\phi X, \phi Y], Z) &= \tfrac{1}{2}[\bar{g}(\phi\bar{\nabla}^{*}_{\phi X}Y + \bar{g}(Y, \phi X)\nu, Z) + \bar{g}(\phi\bar{\nabla}_{\phi X}Y + \bar{g}(Y, \phi X)\nu, Z) \\
&\quad - \bar{g}(\phi\bar{\nabla}^{*}_{\phi Y}X + \bar{g}(X, \phi Y)\nu, Z) - \bar{g}(\phi\bar{\nabla}_{\phi Y}X + \bar{g}(X, \phi Y)\nu, Z)] \\
&= \tfrac{1}{2}[-\bar{g}(\bar{\nabla}^{*}_{\phi X}Y, \phi Z) - \bar{g}(\bar{\nabla}_{\phi X}Y, \phi Z) + \bar{g}(\bar{\nabla}^{*}_{\phi Y}X, \phi Z) + \bar{g}(\bar{\nabla}_{\phi Y}X, \phi Z)] \\
&= \tfrac{1}{2}[\bar{g}(h^{l}(\phi X, \phi Z), Y) + \bar{g}(h^{*l}(\phi X, \phi Z), Y) - \bar{g}(h^{l}(\phi Y, \phi Z), X) - \bar{g}(h^{*l}(\phi Y, \phi Z), X)],
\end{aligned}$$

and

$$\begin{aligned}
\bar{g}([\phi X, \phi Y], \phi\xi) &= \tfrac{1}{2}[\bar{g}(\bar{\nabla}_{\phi X}\phi Y, \phi\xi) + \bar{g}(\bar{\nabla}^{*}_{\phi X}\phi Y, \phi\xi) - \bar{g}(\bar{\nabla}_{\phi Y}\phi X, \phi\xi) - \bar{g}(\bar{\nabla}^{*}_{\phi Y}\phi X, \phi\xi)] \\
&= \tfrac{1}{2}[\bar{g}(\phi\bar{\nabla}^{*}_{\phi X}Y + \bar{g}(Y, \phi X)\nu, \phi\xi) + \bar{g}(\phi\bar{\nabla}_{\phi X}Y + \bar{g}(Y, \phi X)\nu, \phi\xi) - \bar{g}(\phi\bar{\nabla}^{*}_{\phi Y}X + \bar{g}(X, \phi Y)\nu, \phi\xi) \\
&\quad - \bar{g}(\phi\bar{\nabla}_{\phi Y}X + \bar{g}(X, \phi Y)\nu, \phi\xi)] \\
&= \tfrac{1}{2}[\bar{g}(\bar{\nabla}^{*}_{\phi X}Y, \xi) + \bar{g}(\bar{\nabla}_{\phi X}Y, \xi) - \bar{g}(\bar{\nabla}^{*}_{\phi Y}X, \xi) - \bar{g}(\bar{\nabla}_{\phi Y}X, \xi)] \\
&= \tfrac{1}{2}[\bar{g}(A^{*\prime}_{\xi}\phi X, Y) + \bar{g}(A'_{\xi}\phi X, Y) - \bar{g}(A^{*\prime}_{\xi}\phi Y, X) - \bar{g}(A'_{\xi}\phi Y, X)].
\end{aligned}$$

Further, $\bar{g}([\phi X, \phi Y], \phi W) = \bar{g}(\widehat{\bar{\nabla}}_{\phi X}\phi Y, \phi W) - \bar{g}(\widehat{\bar{\nabla}}_{\phi Y}\phi X, \phi W)$

$$\begin{aligned}
&= \tfrac{1}{2}[\bar{g}(\phi\bar{\nabla}^{*}_{\phi X}Y + \bar{g}(Y, \phi X)\nu, \phi W) + \bar{g}(\phi\bar{\nabla}_{\phi X}Y + \bar{g}(Y, \phi X)\nu, \phi W) - \bar{g}(\phi\bar{\nabla}^{*}_{\phi Y}X + \bar{g}(X, \phi Y)\nu, \phi W) \\
&\quad - \bar{g}(\phi\bar{\nabla}_{\phi Y}X + \bar{g}(X, \phi Y)\nu, \phi W)] \\
&= \tfrac{1}{2}[-\bar{g}(\bar{\nabla}^{*}_{\phi X}Y, \phi^{2}W) - \bar{g}(\bar{\nabla}_{\phi X}Y, \phi^{2}W) + \bar{g}(\bar{\nabla}^{*}_{\phi Y}X, \phi^{2}W) + \bar{g}(\bar{\nabla}_{\phi Y}X, \phi^{2}W)] \\
&= \tfrac{1}{2}[\bar{g}(h^{*s}(\phi X, Y), W) + \bar{g}(h^{s}(\phi X, Y), W) - \bar{g}(h^{*s}(\phi Y, X), W) - \bar{g}(h^{s}(\phi Y, X), W)].
\end{aligned}$$

Now the CR-lightlike structure of M and the indefinite Sasakian statistical character of $\bar{M}$ gives

$$\begin{aligned}
\bar{g}([\phi X, \phi Y], N) &= \bar{g}(\widehat{\bar{\nabla}}_{\phi X}\phi Y, N) - \bar{g}(\widehat{\bar{\nabla}}_{\phi Y}\phi X, N) \\
&= \tfrac{1}{2}[\bar{g}(\phi\bar{\nabla}^{*}_{\phi X}Y + \bar{g}(Y, \phi X)\nu, N) + \bar{g}(\phi\bar{\nabla}_{\phi X}Y + \bar{g}(Y, \phi X)\nu, N) - \bar{g}(\phi\bar{\nabla}^{*}_{\phi Y}X + \bar{g}(X, \phi Y)\nu, N) \\
&\quad - \bar{g}(\phi\bar{\nabla}_{\phi Y}X + \bar{g}(X, \phi Y)\nu, N)] \\
&= \tfrac{1}{2}[\bar{g}(A^{*\prime}_{Y}\phi X, \phi N) + \bar{g}(A'_{Y}\phi X, \phi N) - \bar{g}(A^{*\prime}_{X}\phi Y, \phi N) - \bar{g}(A'_{X}\phi Y, \phi N)].
\end{aligned}$$

M being a contact CR-lightlike submanifold implies that $\phi(Rad(TM))$ is integrable if and only if

$$\bar{g}([\phi X, \phi Y], Z) = \bar{g}([\phi X, \phi Y], \nu) = \bar{g}([\phi X, \phi Y], \phi W) = \bar{g}([\phi X, \phi Y], \phi\xi) = \bar{g}([\phi X, \phi Y], N) = 0, \tag{4.3}$$

for $X, Y \in \Gamma(Rad(TM))$, $Z \in \Gamma(D_o)$, $W \in \Gamma(L_1)$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$. Hence from the above assertion (4.3), we get the desired result.

□

4.2. Characterization of distributions

Now, we characterize the integrability of distributions for the contact CR-lightlike submanifolds as follows:

Theorem 4.7. *If M is a contact CR-lightlike submanifold of an indefinite Sasakian statistical manifold $\bar{M}$, then each maximal integrable manifold of the radical distribution is totally geodesic in M if and only if*

(i) $\bar{g}(\phi\xi'', A'_{\xi}\xi') + \bar{g}(\phi\xi'', A^{*\prime}_{\xi}\xi') = 0,$

(ii) $\bar{g}(h^s(\xi', \phi\xi''), W) + \bar{g}(h^{s*}(\xi', \phi\xi''), W) = 0,$

(iii) $\bar{g}(h'(\xi', \phi\xi''), N) + \bar{g}(h^{*\prime}(\xi', \phi\xi''), N) = 0,$

$\forall\ \xi, \xi', \xi'' \in \Gamma(Rad(TM)),\ N \in \Gamma(ltr(TM))$ *and* $W \in \Gamma(S(TM^{\perp}))$.

Proof. The theory of contact CR-lightlike submanifold of an indefinite Sasakian statistical manifold gives that each leaf of $Rad(TM)$ defines totally geodesic foliation in M if and only if

$$\bar{g}(\widehat{\nabla}_{\xi'}\xi'', \nu) = \bar{g}(\widehat{\nabla}_{\xi'}\xi'', Z) = \bar{g}(\widehat{\nabla}_{\xi'}\xi'', \phi\xi) = \bar{g}(\widehat{\nabla}_{\xi'}\xi'', \phi W) = \bar{g}(\widehat{\nabla}_{\xi'}\xi'', \phi N) = 0, \quad (4.4)$$

for all $\xi, \xi', \xi'' \in \Gamma(Rad(TM)),\ N \in \Gamma(ltr(TM)),\ W \in \Gamma(S(TM^{\perp}))$ and $Z \in \Gamma(D_o)$.

Since $\bar{M}$ is an indefinite Sasakian statistical manifold and (2.8) holds, therefore

$$\begin{aligned}
&\bar{g}(\widehat{\nabla}_{\xi'}\xi'', \phi\xi) = \bar{g}(\widehat{\bar{\nabla}}_{\xi'}\xi'', \phi\xi)\\
&= \tfrac{1}{2}[\bar{g}(\bar{\nabla}_{\xi'}\xi'', \phi\xi) + \bar{g}(\bar{\nabla}^{*}_{\xi'}\xi'', \phi\xi)]\\
&= \tfrac{1}{2}[-\bar{g}(\bar{\nabla}^{*}_{\xi'}\phi\xi'' - \bar{g}(\xi'', \xi')\nu + \bar{g}(\xi'', \nu)\xi', \xi) - \bar{g}(\bar{\nabla}_{\xi'}\phi\xi'' - \bar{g}(\xi'', \xi')\nu + \bar{g}(\xi'', \nu)\xi', \xi)]\\
&= \tfrac{1}{2}[\bar{g}(\phi\xi'', \bar{\nabla}_{\xi'}\xi) + \bar{g}(\phi\xi'', \bar{\nabla}^{*}_{\xi'}\xi) + 2\bar{g}(\bar{g}(\xi'', \xi')\nu, \xi)]\\
&= \tfrac{1}{2}[-\bar{g}(\phi\xi'', A'_{\xi}\xi') - \bar{g}(\phi\xi'', A^{*\prime}_{\xi}\xi')],
\end{aligned}$$

and $\bar{g}(\widehat{\nabla}_{\xi'}\xi'', \phi W) = \bar{g}(\widehat{\bar{\nabla}}_{\xi'}\xi'', \phi W)$

$$\begin{aligned}
&= -\tfrac{1}{2}\bar{g}(\phi\bar{\nabla}_{\xi'}\xi'', W) - \tfrac{1}{2}\bar{g}(\phi\bar{\nabla}^{*}_{\xi'}\xi'', W)\\
&= \tfrac{1}{2}[-\bar{g}(\bar{\nabla}^{*}_{\xi'}\phi\xi'' - \bar{g}(\xi'', \xi')\nu + \bar{g}(\xi'', \nu)\xi', W) - \bar{g}(\bar{\nabla}_{\xi'}\phi\xi'' - \bar{g}(\xi'', \xi')\nu + \bar{g}(\xi'', \nu)\xi', W)]\\
&= \tfrac{1}{2}[-\bar{g}(h^{s*}(\xi', \phi\xi''), W) - \bar{g}(h^{s}(\xi', \phi\xi''), W)].
\end{aligned}$$

Also, from (2.6) and (2.10), we get

$$\begin{aligned}
&\bar{g}(\widehat{\nabla}_{\xi'}\xi'', \phi N) = \tfrac{1}{2}\bar{g}(\bar{\nabla}_{\xi'}\xi'' + \bar{\nabla}^{*}_{\xi'}\xi'', \phi N)\\
&= \tfrac{1}{2}[-\bar{g}(\bar{\nabla}^{*}_{\xi'}\phi\xi'' - \bar{g}(\xi'', \xi')\nu + \bar{g}(\xi'', \nu)\xi', N) - \bar{g}(\bar{\nabla}_{\xi'}\phi\xi'' - \bar{g}(\xi'', \xi')\nu + \bar{g}(\xi'', \nu)\xi', N)]\\
&= \tfrac{1}{2}[-\bar{g}(h'(\xi', \phi\xi''), N) - \bar{g}(h^{*\prime}(\xi', \phi\xi''), N)].
\end{aligned}$$

Hence the required result follows from (4.4).

□

Theorem 4.8. *For a contact CR-lightlike submanifold M of an indefinite Sasakian statistical manifold $(\bar{M}, \bar{\nabla}, \bar{g}, \phi, \nu)$, D' is integrable if and only if*

$$\nabla_X \phi Y + \nabla^*_X \phi Y = \nabla_Y \phi X + \nabla^*_Y \phi X,$$

for any $X, Y \in \Gamma(D')$.

Proof. Consider

$$h(X, \phi Y) + h^*(X, \phi Y) - h(Y, \phi X) - h^*(Y, \phi X) = \bar{\nabla}_X \phi Y - \nabla_X \phi Y + \bar{\nabla}^*_X \phi Y - \nabla^*_X \phi Y$$
$$-\bar{\nabla}_Y \phi X + \nabla_Y \phi X - \bar{\nabla}^*_Y \phi X + \nabla^*_Y \phi X,$$

for any $X, Y \in \Gamma(D')$.

The concept of indefinite Sasakian statistical manifold and the equation (3.2) gives

$$h(X, \phi Y) + h^*(X, \phi Y) - h(Y, \phi X) - h^*(Y, \phi X) = \phi(2\widehat{\bar{\nabla}}_X Y) - \eta(Y)X - \nabla_X \phi Y - \eta(Y)X$$
$$-\nabla^*_X \phi Y - \phi(2\widehat{\bar{\nabla}}_Y X) + \eta(X)Y + \nabla_Y \phi X$$
$$+\eta(X)Y + \nabla^*_Y \phi X.$$

Since M is a contact CR-lightlike submanifold of $\bar{M}$, therefore, using (4.1), we get

$$h(X, \phi Y) + h^*(X, \phi Y) - h(Y, \phi X) - h^*(Y, \phi X) = 2\phi[X, Y] - \nabla_X \phi Y - \nabla^*_X \phi Y + \nabla_Y \phi X + \nabla^*_Y \phi X$$
$$= 2f[X, Y] + 2w[X, Y] - \nabla_X \phi Y - \nabla^*_X \phi Y + \nabla_Y \phi X + \nabla^*_Y \phi X.$$

On comparing the tangential parts of the above equation, we obtain

$$2f[X, Y] = \nabla_X \phi Y + \nabla^*_X \phi Y - \nabla_Y \phi X - \nabla^*_Y \phi X.$$

The result follows from the integrability of D'.

□

Theorem 4.9. *Let $\bar{M}$ be an indefinite Sasakian statistical manifold and M be a contact CR-lightlike submanifold of $\bar{M}$. Then the distribution D_o is integrable if and only if*

(i) $\bar{g}(h'(X, Y), N) + \bar{g}(h^{*\prime}(X, Y), N) = \bar{g}(h'(Y, X), N) + \bar{g}(h^{*\prime}(Y, X), N)$,
(ii) $\bar{g}(\nabla'_X Y, \nu) + \bar{g}(\nabla^{*\prime}_X Y, \nu) = \bar{g}(\nabla'_Y X, \nu) + \bar{g}(\nabla^{*\prime}_Y X, \nu)$,
(iii) $\bar{g}(h^{*\prime}(X, \phi Y), N) + \bar{g}(h'(X, \phi Y), N) = \bar{g}(h^{*\prime}(Y, \phi X), N) + \bar{g}(h'(Y, \phi X), N)$,
(iv) $\bar{g}(h^{*s}(X, \phi Y), W) + \bar{g}(h^s(X, \phi Y), W) = \bar{g}(h^{*s}(Y, \phi X), W) + \bar{g}(h^s(Y, \phi X), W)$,
(v) $\bar{g}(\nabla'_X Y, \phi\xi) + \bar{g}(\nabla^{*\prime}_X Y, \phi\xi) = \bar{g}(\nabla'_Y X, \phi\xi) + \bar{g}(\nabla^{*\prime}_Y X, \phi\xi)$,

$\forall\ X, Y \in \Gamma(D_o), N \in \Gamma(ltr(TM))$, *$W \in \Gamma(S(TM^\perp))$ and $\xi \in \Gamma(Rad(TM))$.*

Proof. For a contact CR-lightlike submanifold, D_0 is integrable if and only if

$$\bar{g}([X,Y],\phi W)=\bar{g}([X,Y],\phi\xi)=\bar{g}([X,Y],\nu)=\bar{g}([X,Y],\phi N)=\bar{g}([X,Y],N)=0, \quad (4.5)$$

$\forall\ X,Y\in\Gamma(D_o), N\in\Gamma(ltr(TM)), W\in\Gamma(S(TM^{\perp}))$ and $\xi\in\Gamma(Rad(TM))$.

Using (2.10) for a statistical manifold with connections $\bar{\nabla}$ and $\bar{\nabla}^*$, we have

$$\begin{aligned}
&\bar{g}([X,Y],N)=\bar{g}(\widehat{\bar{\nabla}}_XY-\widehat{\bar{\nabla}}_YX,N)\\
&=\tfrac{1}{2}[\bar{g}(\bar{\nabla}_XY,N)+\bar{g}(\bar{\nabla}^*_XY,N)-\bar{g}(\bar{\nabla}_YX,N)-\bar{g}(\bar{\nabla}^*_YX,N)]\\
&=\tfrac{1}{2}[\bar{g}(Y,A^*_NX)+\bar{g}(Y,A_NX)-\bar{g}(X,A^*_NY)-\bar{g}(X,A_NY)]\\
&=\tfrac{1}{2}[\bar{g}(h'(X,Y),N)+\bar{g}(h^{*\prime}(X,Y),N)-\bar{g}(h'(Y,X),N)-\bar{g}(h^{*\prime}(Y,X),N)],
\end{aligned}$$

and from (2.7), we obtain

$$\begin{aligned}
&\bar{g}([X,Y],\nu)=\bar{g}(\widehat{\bar{\nabla}}_XY,\nu)-\bar{g}(\widehat{\bar{\nabla}}_YX,\nu)\\
&=\tfrac{1}{2}[\bar{g}(\bar{\nabla}_XY,\nu)+\bar{g}(\bar{\nabla}^*_XY,\nu)-\bar{g}(\bar{\nabla}_YX,\nu)-\bar{g}(\bar{\nabla}^*_YX,\nu)]\\
&=\tfrac{1}{2}[\bar{g}(\nabla_XY,\nu)+\bar{g}(\nabla^*_XY,\nu)-\bar{g}(\nabla_YX,\nu)-\bar{g}(\nabla^*_YX,\nu)]\\
&=\tfrac{1}{2}[\bar{g}(\nabla'_XY,\nu)+\bar{g}({\nabla^*}'_XY,\nu)-\bar{g}(\nabla'_YX,\nu)-\bar{g}({\nabla^*}'_YX,\nu)].
\end{aligned}$$

Since $S(TM^{\perp})$ is orthogonal to $ltr(TM)$ and $ltr(TM)$ is self-orthogonal, therefore

$$\begin{aligned}
&\bar{g}([X,Y],\phi N)=\tfrac{1}{2}\bar{g}(\bar{\nabla}_XY+\bar{\nabla}^*_XY,\phi N)-\tfrac{1}{2}\bar{g}(\bar{\nabla}_YX+\bar{\nabla}^*_YX,\phi N)\\
&=\tfrac{1}{2}[-\bar{g}(\bar{\nabla}^*_X\phi Y-\bar{g}(Y,X)\nu+\bar{g}(Y,\nu)X,N)-\bar{g}(\bar{\nabla}_X\phi Y-\bar{g}(Y,X)\nu+\bar{g}(Y,\nu)X,N)\\
&+\bar{g}(\bar{\nabla}^*_Y\phi X-\bar{g}(X,Y)\nu+\bar{g}(X,\nu)Y,N)+\bar{g}(\bar{\nabla}_Y\phi X-\bar{g}(X,Y)\nu+\bar{g}(X,\nu)Y,N)]\\
&=\tfrac{1}{2}[-\bar{g}(\nabla^*_X\phi Y,N)-\bar{g}(\nabla_X\phi Y,N)+\bar{g}(\nabla^*_Y\phi X,N)+\bar{g}(\nabla_Y\phi X,N)]\\
&=\tfrac{1}{2}[-\bar{g}(h^{*\prime}(X,\phi Y),N)-\bar{g}(h'(X,\phi Y),N)+\bar{g}(h^{*\prime}(Y,\phi X),N)+\bar{g}(h'(Y,\phi X),N)],
\end{aligned}$$

and

$$\begin{aligned}
&\bar{g}([X,Y],\phi W)=\tfrac{1}{2}[\bar{g}(\bar{\nabla}_XY,\phi W)+\bar{g}(\bar{\nabla}^*_XY,\phi W)-\bar{g}(\bar{\nabla}_YX,\phi W)-\bar{g}(\bar{\nabla}^*_YX,\phi W)]\\
&=\tfrac{1}{2}[-\bar{g}(\bar{\nabla}^*_X\phi Y-\bar{g}(Y,X)\nu+\bar{g}(Y,\nu)X,W)-\bar{g}(\bar{\nabla}_X\phi Y-\bar{g}(Y,X)\nu+\bar{g}(Y,\nu)X,W)\\
&+\bar{g}(\bar{\nabla}^*_Y\phi X-\bar{g}(X,Y)\nu+\bar{g}(X,\nu)Y,W)+\bar{g}(\bar{\nabla}_Y\phi X-\bar{g}(X,Y)\nu+\bar{g}(X,\nu)Y,W)]\\
&=\tfrac{1}{2}[-\bar{g}(h^{*s}(X,\phi Y),W)-\bar{g}(h^s(X,\phi Y),W)+\bar{g}(h^{*s}(Y,\phi X),W)+\bar{g}(h^s(Y,\phi X),W)].
\end{aligned}$$

Also, $\bar{g}([X,Y],\phi\xi)=\bar{g}(\widehat{\bar{\nabla}}_XY,\phi\xi)-\bar{g}(\widehat{\bar{\nabla}}_YX,\phi\xi)$
$=\tfrac{1}{2}[\bar{g}(\nabla'_XY,\phi\xi)+\bar{g}({\nabla^*}'_XY,\phi\xi)-\bar{g}(\nabla'_YX,\phi\xi)-\bar{g}({\nabla^*}'_YX,\phi\xi)]$.

Thus the proof follows using (4.5).

□

Corollary 4.10. *For the shape operators A_N and A^*_N in contact CR-lightlike submanifold M of an indefinite Sasakian statistical manifold $\bar{M}$, the necessary and sufficient conditions for the integrability of D_o are*

(i) $\bar{g}(A_N^* X, Y) + \bar{g}(A_N X, Y) = \bar{g}(A_N^* Y, X) + \bar{g}(A_N Y, X)$,
(ii) $\bar{g}(\nabla'_X Y, \nu) + \bar{g}(\nabla^{*\prime}_X Y, \nu) = \bar{g}(\nabla'_Y X, \nu) + \bar{g}(\nabla^{*\prime}_Y X, \nu)$,
(iii) $\bar{g}(A_N^* X, \phi Y) + \bar{g}(A_N X, \phi Y) = \bar{g}(A_N^* Y, \phi X) + \bar{g}(A_N Y, \phi X)$,
(iv) $\bar{g}(h^{*s}(X, \phi Y), W) + \bar{g}(h^s(X, \phi Y), W) = \bar{g}(h^{*s}(Y, \phi X), W) + \bar{g}(h^s(Y, \phi X), W)$,
(v) $\bar{g}(h^{*l}(X, \phi Y), \xi) + \bar{g}(h^l(X, \phi Y), \xi) = \bar{g}(h^{*l}(Y, \phi X), \xi) + \bar{g}(h^l(Y, \phi X), \xi)$,

for any $X, Y \in \Gamma(D_0), N \in \Gamma(ltr(TM)), W \in \Gamma(L_1)$ *and* $\xi \in \Gamma(Rad(TM))$.

Proof. Using (2.10) in the given hypothesis, we have

$$\bar{g}(A_N^* X, Y) + \bar{g}(A_N X, Y) = \bar{g}(A_N^* Y, X) + \bar{g}(A_N Y, X),$$

$$\text{and } \bar{g}(A_N^* X, \phi Y) + \bar{g}(A_N X, \phi Y) = \bar{g}(A_N^* Y, \phi X) + \bar{g}(A_N Y, \phi X),$$

for any $X, Y \in \Gamma(D_0), N \in \Gamma(ltr(TM))$.

Further, $\bar{M}$ being an indefinite Sasakian statistical manifold, implies

$$\begin{aligned}\bar{g}([X, Y], \phi\xi) = \frac{1}{2}[&-\bar{g}(\bar{\nabla}^*_X \phi Y, \xi) + \bar{g}(\bar{g}(Y, X)\nu, \xi) - \bar{g}(\bar{\nabla}_X \phi Y, \xi) + \bar{g}(\bar{g}(Y, X)\nu, \xi) + \bar{g}(\bar{\nabla}^*_Y \phi X, \xi) \\ &-\bar{g}(\bar{g}(X, Y)\nu, \xi) + \bar{g}(\bar{\nabla}_Y \phi X, \xi) - \bar{g}(\bar{g}(X, Y)\nu, \xi)] \\ = \frac{1}{2}[-\bar{g}(h^{*l}(X, \phi Y), \xi) &- \bar{g}(h^l(X, \phi Y), \xi) + \bar{g}(h^{*l}(Y, \phi X), \xi) + \bar{g}(h^l(Y, \phi X), \xi)].\end{aligned}$$

Hence, the result. □

Conflicts of Interest: The authors declare no conflict of interest.

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