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AN ANALOGUE OF MILLOUX'S THEOREM FOR COMPOSITE FUNCTIONS

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Abstract

The main purpose of this paper is to prove analogous of Milloux's theorem for composite functions.

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1. Introduction and Definitions

In the value distribution theory, Milloux theorem plays a cardinal role to correlate the characteristic function of a given meromorphic function with its derivatives. In 2016, Dyavanal and Rathod [2] proved the same for the domain annuli and established some interesting results as an application. Now it is natural to ask whether can we establish the analogous of Milloux inequality for composite meromorphic and entire functions.

In the following, we introduce the definitions and notaions of [3] which will be used in this paper.

Definition 1.1. The positive logarithmic function $\log^+ x$ for $x \geq 0$ is defined as follows

$$\log^+ x = max(\log x, 0).$$

For all $x > 0$, it is evident that

$$\log x = \log^+ x - \log^+ \frac{1}{x}.$$

For a non-constant meromorphic function $f(z)$ in the disc $|z| \leq R$ $(0 < R < \infty)$, Nevanlinna defined the following functions.

Definition 1.2. $m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$
which is nothing but the average of the positive logarithm of $|f(z)|$ on the circle $|z| = r$.

Definition 1.3. The counting function of poles of $f(z)$, denoted by $N(r, f)$ is defined as

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

where $n(t, f)$ is the number of poles of $f(z)$ in the disc $|z| \leq t$, multiple poles are counted according to their multiplicities and $n(0, f)$ denotes the multiplicity of poles of $f(z)$ at the origin.

Definition 1.4. The characteristic function of $f(z)$, denoted by $T(r, f)$ is defined as

$$T(r, f) = m(r, f) + N(r, f)\ .$$

Definition 1.5. A meromorphic function $a(z)$ is called a small function with respect to a meromorphic function $f(z)$ if

$$T(r, a) = S(r, f)$$

i.e.,

$$T(r, a) = o(T(r, f))$$

i.e.,

$$\frac{T(r, a)}{T(r, f)} \to 0 \ \ as \ \ r \to \infty$$

possibly outside a set of finite linear measure.

In the value distribution theory, Milloux [3] proved the following remarkable theorems to study the characteristic function of the derivative of a given function as follows.

Theorem 1.1. *Let $f(z)$ be a non-constant meromorphic function in the complex plane. If*

$$\Psi(z) = \sum_{i=0}^{k} a_i(z) f^{(i)}(z)\ ,$$

where k is a positive integer and $a_0(z), a_1(z), a_2(z), ..., a_k(z)$ are small functions of $f(z)$, then

$$m(r, \frac{\Psi}{f}) = S(r, f) \tag{1.1}$$

and

$$T(r, \Psi) \leq T(r, f) + k\bar{N}(r, f) + S(r, f) \leq (k + 1)T(r, f) + S(r, f)\ , \tag{1.2}$$

$f^{(i)}(z)$ being the i-th order derivative of $f(z)$.

Theorem 1.2. *Let $f(z)$ be a non-constant meromorphic function in the complex plane. If*

$$\Psi(z) = \sum_{i=0}^{k} a_i(z) f^{(i)}(z)\ ,$$

where k is a positive integer and $a_0(z), a_1(z), a_2(z), ..., a_k(z)$ are small functions of $f(z)$, is not a constant, then

$$T(r, f) < \bar{N}(r, f) + N(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{\Psi - 1}) - N_0(r, \frac{1}{\Psi'}) + S(r, f)\ , \tag{1.3}$$

where in $N_0(r, \frac{1}{\Psi'})$ only zeros of Ψ' not corresponding to the repeated roots of $\Psi = 1$ are to be considered.

2. Preliminary Results

To reach the main results of this paper we need the following lemmas.

Lemma 2.1. *[1] If $f(z)$ is meromorphic and $g(z)$ is entire then for all large values of r*

$$T(r, f \circ g) \le (1 + o(1))\frac{T(r, g)}{\log M(r, g)} T(M(r, g), f) \ . \tag{2.1}$$

Lemma 2.2. *[3] Let $f(z)$ be a non-constant meromorphic function in the complex plane. Then*

$$m(r, \frac{f'}{f}) = S(r, f) \ . \tag{2.2}$$

Lemma 2.3. *For a non-constant meromorphic function f*

$$N(r, f^{(k)}) \le N(r, f) + k\bar{N}(r, f) \ . \tag{2.3}$$

Proof. Since $N(r, f') \le N(r, f) + \bar{N}(r, f)$, we can write

$$\begin{aligned} N(r, f^{(k)}) &\le N(r, f^{(k-1)}) + \bar{N}(r, f^{(k-1)}) \\ &\le N(r, f^{(k-2)}) + \bar{N}(r, f^{(k-2)}) + \bar{N}(r, f^{(k-1)}) \\ &\le N(r, f^{(k-3)}) + \bar{N}(r, f^{(k-3)}) + \bar{N}(r, f^{(k-2)}) + \bar{N}(r, f^{(k-1)}) \\ &\quad \dots \quad \dots \quad \dots \\ &\le N(r, f) + \bar{N}(r, f) + \bar{N}(r, f') + \bar{N}(r, f'') + \dots + \bar{N}(r, f^{(k-1)}) \\ &= N(r, f) + k\bar{N}(r, f), \quad \textit{since} \ \ \bar{N}(r, f) = \bar{N}(r, f') = \bar{N}(r, f'') = \dots = \bar{N}(r, f^{(k-1)}) \ . \end{aligned}$$

□

Lemma 2.4. *[1] Let $f(z)$ be a non-constant meromorphic function in the complex plane and $g(z)$ be a non-constant entire function other than a linear polynomial. Then for sufficiently large r*

$$\bar{N}(r, f \circ g) \le \sum_{|b_i| \le M(r,g)} \bar{N}(r, \frac{1}{g(z) - b_i}) \ , \tag{2.4}$$

where $b_i (i = 1, 2, ...)$ are poles of f.

3. Main Results

In this section we prove the following theorems, which are analogous of Milloux results for composite functions.

Theorem 3.1. *Let $f(z)$ be a non-constant meromorphic function in the complex plane and $g(z)$ be a non-constant entire function other than a linear polynomial. If*

$$\Psi(z) = \sum_{i=0}^{k} a_i(z)(f \circ g)^{(i)}(z) \ ,$$

where k is a positive integer and $a_0(z), a_1(z), a_2(z), ..., a_k(z)$ are small functions of $f(z)$, then

$$m(r, \frac{\Psi}{f \circ g}) = S(r_1, f) + S(r_1, g) \tag{3.1}$$

and

$$T(r, \Psi) \leq (k+2)[T(r_1, f) + T(r_1, g)] + S(r_1, f) + S(r_1, g) \tag{3.2}$$

where $r_1 = M(r, g)$.

Proof. First we consider the case when $\Psi(z) = (f \circ g)^{(k)}(z)$ and use induction on the number k to prove the conclusion of the theorem.

By Theorem 1.1 for non-constant meromorphic function $f \circ g$ and using Lemma 2.1 we have for all large values of r

$$\begin{aligned} m(r, \frac{(f \circ g)'}{f \circ g}) &= S(r, f \circ g) \\ &= o(T(r, f \circ g)) \\ &= T(r, f \circ g)\, o(1) \\ &\leq [(1 + o(1))\frac{T(r, g)}{\log M(r, g)} T(M(r, g), f)]\, o(1) \\ &\leq [(1 + o(1))T(M(r, g), f)]\, o(1) \\ &= o(1)\, T(r_1, f)\,, \quad as \quad r_1 = M(r, g) \\ &\leq o(1)\, [T(r_1, f) + T(r_1, g)] \\ &= o(T(r_1, f)) + o(T(r_1, g)) \\ &= S(r_1, f) + S(r_1, g)\,. \end{aligned} \tag{3.3}$$

Again by using Lemma 2.1 and the assumption that $g(z)$ is a non-constant entire function other than a linear polynomial, we have for all large values of r

$$\begin{aligned} T(r, (f \circ g)') &= T(r, (f' \circ g)\, g') \\ &\leq T(r, f' \circ g) + T(r, g') \\ &\leq [(1 + o(1))\frac{T(r, g)}{\log M(r, g)} T(M(r, g), f')] + T(r, g') \\ &\leq [(1 + o(1))T(M(r, g), f')] + T(r, g') \\ &= (1 + o(1))T(r_1, f') + T(r, g')\,, \quad as \quad r_1 = M(r, g) \\ &\leq (1 + o(1))T(r_1, f') + T(r_1, g') \\ &\leq \frac{3}{2}[T(r_1, f') + T(r_1, g')]. \end{aligned} \tag{3.4}$$

Again, using Lemma 2.2 and Lemma 2.3, we have

$$
\begin{aligned}
T(r, f^{'}) &= m(r, f^{'}) + N(r, f^{'}) \\
&\le m(r, f) + m(r, \frac{f^{'}}{f}) + N(r, f) + \bar{N}(r, f) \\
&= T(r, f) + m(r, \frac{f^{'}}{f}) + \bar{N}(r, f) \\
&= T(r, f) + \bar{N}(r, f) + S(r, f) \qquad (3.5)\\
&\le 2T(r, f) + S(r, f). \qquad (3.6)
\end{aligned}
$$

Using (3.6) in (3.4) we get

$$
\begin{aligned}
T(r, (f \circ g)^{'}) &\le \frac{3}{2}[2T(r_1, f) + S(r_1, f) + 2T(r_1, g) + S(r_1, g)] \\
&= 3[T(r_1, f) + T(r_1, g)] + S(r_1, f) + S(r_1, g) \\
&= (1 + 2)[T(r_1, f) + T(r_1, g)] + S(r_1, f) + S(r_1, g). \qquad (3.7)
\end{aligned}
$$

Hence in view of (3.3) and (3.7), we can say that the theorem is true for $k = 1$.

Now we suppose that the theorem is true for $k = n$

$$
i.e., \quad m(r, \frac{(f \circ g)^{(n)}}{f \circ g}) = S(r_1, f) + S(r_1, g) \qquad (3.8)
$$

and

$$
T(r, (f \circ g)^{(n)}) \le (n + 2)[T(r_1, f) + T(r_1, g)] + S(r_1, f) + S(r_1, g)\,. \qquad (3.9)
$$

Now we have by using Lemma 2.1, Lemma 2.2 and (3.8)

$$
\begin{aligned}
m(r, \frac{(f \circ g)^{(n+1)}}{f \circ g}) &\le m(r, \frac{(f \circ g)^{(n+1)}}{(f \circ g)^{(n)}}) + m(r, \frac{(f \circ g)^{(n)}}{f \circ g}) \\
&\le S(r, (f \circ g)^{(n)}) + S(r_1, f) + S(r_1, g) \\
&= S(r, f \circ g) + S(r_1, f) + S(r_1, g) \\
&= o(T(r, f \circ g)) + S(r_1, f) + S(r_1, g) \\
&= T(r, f \circ g)\, o(1) + S(r_1, f) + S(r_1, g) \\
&\le [(1 + o(1))\frac{T(r, g)}{\log M(r, g)} T(M(r, g), f)]\, o(1) + S(r_1, f) + S(r_1, g) \\
&\le (1 + o(1))T(M(r, g), f)\, o(1) + S(r_1, f) + S(r_1, g) \\
&= o(1)\, T(r_1, f) + S(r_1, f) + S(r_1, g)\,, \quad as \quad r_1 = M(r, g) \\
&= o(T(r_1, f)) + S(r_1, f) + S(r_1, g) \\
&= S(r_1, f) + S(r_1, f) + S(r_1, g) \\
&= S(r_1, f) + S(r_1, g). \qquad (3.10)
\end{aligned}
$$

Now using Lemma 2.1, (3.5) and (1.2) for the function $(f \circ g)^{(n)}$, we have

$$\begin{aligned}
T(r,(f\circ g)^{(n+1)}) &\le T(r,(f\circ g)^{(n)}) + \bar{N}(r,(f\circ g)^{(n)}) + S(r,(f\circ g)^{(n)})\\
&\le [T(r,(f\circ g)) + n\bar{N}(r,f\circ g) + S(r,f\circ g)] + \bar{N}(r,(f\circ g)^{(n)}) + S(r,(f\circ g)^{(n)})\\
&= T(r,(f\circ g)) + (n+1)\bar{N}(r,f\circ g) + S(r,f\circ g) + S(r,(f\circ g)^{(n)}),\\
&\qquad\qquad \textit{since } \bar{N}(r,(f\circ g)^{(n)}) = \bar{N}(r,f\circ g)\\
&\le (n+2)T(r,f\circ g) + S(r,f\circ g)\\
&\le (n+2)[(1+o(1))\frac{T(r,g)}{\log M(r,g)}T(M(r,g),f)] + o(T(r,f\circ g))\\
&\le (n+2)[(1+o(1))T(r_1,f)] + T(r,f\circ g)\,o(1)\\
&\le (n+2)(1+o(1))T(r_1,f) + [(1+o(1))T(r_1,f)]\,o(1)\\
&\le (n+3)T(r_1,f) + o(1)\,T(r_1,f)\\
&\le (n+3)[T(r_1,f)+T(r_1,g)] + o(1)\,[T(r_1,f)+T(r_1,g)]\\
&= (n+3)[T(r_1,f)+T(r_1,g)] + o(T(r_1,f)) + o(T(r_1,g))\\
&= (n+3)[T(r_1,f)+T(r_1,g)] + S(r_1,f) + S(r_1,g). \qquad (3.11)
\end{aligned}$$

In view of (3.10) and (3.11) we can say that the theorem is true for $k = n+1$ as well.

Now we consider the general case.

Clearly

$$\begin{aligned}
m(r,\frac{\Psi}{f\circ g}) &= m(r,\frac{\sum_{i=0}^{k} a_i(z)(f\circ g)^{(i)}(z)}{f\circ g})\\
&\le \sum_{i=0}^{k} m(r,\frac{a_i(f\circ g)^{(i)}}{f\circ g}) + \log(k+1)\\
&\le \sum_{i=0}^{k} [m(r,a_i) + m(r,\frac{(f\circ g)^{(i)}}{f\circ g})] + \log(k+1)\\
&\le S(r_1,f) + S(r_1,g), \qquad (3.12)
\end{aligned}$$

since $a_i(i = 0,1,2,...,k)$ are small functions of f and $m(r,\frac{(f\circ g)^{(i)}}{f\circ g}) = S(r_1,f) + S(r_1,g)$ for all i which is already proved by induction.

Also using (3.12), we have

$$\begin{aligned}
m(r,\Psi) &\le m(r,f\circ g) + m(r,\frac{\Psi}{f\circ g})\\
&\le m(r,f\circ g) + S(r_1,f) + S(r_1,g). \qquad (3.13)
\end{aligned}$$

Again, by using Lemma 2.3 and the fact that $a_i(i = 0,1,2,...,k)$ are small functions of

f, we have

$$\begin{aligned} N(r, \Psi) &= N(r, \sum_{i=0}^{k} a_i(z)(f \circ g)^{(i)}(z)) \\ &\leq N(r, \sum_{i=0}^{k} a_i (f \circ g)^{(k)}) \\ &= N(r, (f \circ g)^{(k)} \sum_{i=0}^{k} a_i) \\ &\leq N(r, (f \circ g)^{(k)}) + \sum_{i=0}^{k} N(r, a_i) \\ &\leq N(r, f \circ g) + k\bar{N}(r, f \circ g) + S(r_1, f). \end{aligned} \tag{3.14}$$

Thus by using (3.13), (3.14) and the Lemma 2.1, we get

$$\begin{aligned} T(r, \Psi) &= m(r, \Psi) + N(r, \Psi) \\ &\leq [m(r, f \circ g) + S(r_1, f) + S(r_1, g)] + [N(r, f \circ g) + k\bar{N}(r, f \circ g) + S(r_1, f)] \\ &= m(r, f \circ g) + N(r, f \circ g) + k\bar{N}(r, f \circ g) + S(r_1, f) + S(r_1, g) \\ &= T(r, f \circ g) + k\bar{N}(r, f \circ g) + S(r_1, f) + S(r_1, g) \\ &\leq (k+1)T(r, f \circ g) + S(r_1, f) + S(r_1, g) \\ &\leq (k+1)[(1+o(1))\frac{T(r, g)}{\log M(r, g)} T(M(r, g), f)] + S(r_1, f) + S(r_1, g) \\ &\leq (k+1)[(1+o(1))T(M(r, g), f)] + S(r_1, f) + S(r_1, g) \\ &\leq (k+1)(1+o(1))T(r_1, f) + S(r_1, f) + S(r_1, g) \\ &\leq (k+2)T(r_1, f) + S(r_1, f) + S(r_1, g) \\ &\leq (k+2)[T(r_1, f) + T(r_1, g)] + S(r_1, f) + S(r_1, g), \end{aligned}$$

which completes the proof of the theorem. □

Note 3.1. : It can be noted that when $g(z) = z$ then we have

$$T(r, \Psi) \leq (k+2)T(r, f) + S(r, f)$$

instead of

$$T(r, \Psi) \leq (k+1)T(r, f) + S(r, f)$$

of the result of Theorem 1.1

Theorem 3.2. *Let $f(z)$ be a non-constant meromorphic function in the complex plane with n distinct poles and $g(z)$ be a non-constant entire function except a linear polynomial such that $f \circ g$ does not have any zero inside $|z| \leq r$. Also let*

$$\Psi(z) = \sum_{i=0}^{k} a_i(z)(f \circ g)^{(i)}(z) \,,$$

where k is a positive integer and $a_0(z), a_1(z), a_2(z), ..., a_k(z)$ are small functions of $f(z)$. If $\Psi(z)$ is not a constant, then

$$T(r, f\circ g) \le nT(r,g) + \bar{N}(r, \frac{1}{\Psi-1}) - N_0(r, \frac{1}{\Psi'}) + S(r_1, f) + S(r_1, g),$$

where $r_1 = M(r,g)$ and in $N_0(r, \frac{1}{\Psi'})$ only zeros of Ψ' not corresponding to the repeated roots of $\Psi = 1$ are to be considered.

Proof. First we apply Nevanlinna Second Fundamental Theorem to the non-constant meromorphic function $\Psi(z)$ where $0, 1$ being the distinct finite complex numbers, to get

$$m(r,\Psi) + m(r, \frac{1}{\Psi}) + m(r, \frac{1}{\Psi-1}) \le 2T(r,\Psi) - N_1(r) + S(r,\Psi) \tag{3.15}$$

where

$$N_1(r) = 2N(r,\Psi) - N(r,\Psi') + N(r, \frac{1}{\Psi'}). \tag{3.16}$$

Also using (3.16) and Nevanlinna First Fundamental Theorem, we have

$$\begin{aligned}
2T(r,\Psi) - N_1(r) &= T(r,\Psi) + T(r,\Psi) - [2N(r,\Psi) - N(r,\Psi') + N(r, \frac{1}{\Psi'})] \\
&= [m(r,\Psi) + N(r,\Psi)] + [T(r, \frac{1}{\Psi-1}) + O(1)] - 2N(r,\Psi) + N(r,\Psi') \\
&\quad - N(r, \frac{1}{\Psi'}) \\
&= m(r,\Psi) + [m(r, \frac{1}{\Psi-1}) + N(r, \frac{1}{\Psi-1})] + [N(r,\Psi') - N(r,\Psi)] \\
&\quad - N(r, \frac{1}{\Psi'}) + O(1) \\
&= m(r,\Psi) + m(r, \frac{1}{\Psi-1}) + [N(r,\Psi') - N(r,\Psi)] + [N(r, \frac{1}{\Psi-1}) \\
&\quad - N(r, \frac{1}{\Psi'})] + O(1).
\end{aligned} \tag{3.17}$$

Now by using Lemma 2.4, Nevanlinna First Fundamental Theorem, the fact that $a_i (i = 0, 1, 2, ..., k)$ are small functions of f and assuming $b_j (j = 1, 2, ..., n)$ are poles of f, we have

$$\begin{aligned}
N(r,\Psi') - N(r,\Psi) &= \bar{N}(r,\Psi) \\
&= \bar{N}(r, \sum_{i=0}^{k} a_i (f\circ g)^{(i)}) \\
&= \bar{N}(r, \sum_{i=0}^{k} a_i (f\circ g)) \\
&= \bar{N}(r, (f\circ g) \sum_{i=0}^{k} a_i)
\end{aligned}$$

$$
\begin{aligned}
&\leq \bar{N}(r, f \circ g) + \bar{N}(r, \sum_{i=0}^{k} a_i) \\
&\leq \sum_{|b_j| \leq M(r,g)} \bar{N}(r, \frac{1}{g - b_j}) + \sum_{i=0}^{k} \bar{N}(r, a_i) \\
&\leq \sum_{|b_j| \leq M(r,g)} T(r, \frac{1}{g - b_j}) + S(r, f) \\
&\leq n[T(r, g) + O(1)] + S(r, f) \\
&= nT(r, g) + S(r, f). \qquad (3.18)
\end{aligned}
$$

And since in $N_0(r, \frac{1}{\Psi'})$ only zeros of Ψ' not corresponding to the repeated roots of $\Psi - 1 = 0$ are to be considered, we have

$$
\begin{aligned}
N(r, \frac{1}{\Psi - 1}) - N(r, \frac{1}{\Psi'}) &= N(r, \frac{1}{\Psi - 1}) - [N(r, \frac{1}{(\Psi - 1)'}) + N_0(r, \frac{1}{\Psi'})] \\
&= [N(r, \frac{1}{\Psi - 1}) - N(r, \frac{1}{(\Psi - 1)'})] - N_0(r, \frac{1}{\Psi'}) \\
&= \bar{N}(r, \frac{1}{\Psi - 1}) - N_0(r, \frac{1}{\Psi'}). \qquad (3.19)
\end{aligned}
$$

Using (3.18) and (3.19) in (3.17) we get

$$
2T(r, \Psi) - N_1(r) \leq m(r, \Psi) + m(r, \frac{1}{\Psi - 1}) + nT(r, g) + \bar{N}(r, \frac{1}{\Psi - 1}) - N_0(r, \frac{1}{\Psi'}) + S(r, f). \qquad (3.20)
$$

Again since $r_1 = M(r, g)$, from (3.2) we get

$$
o(T(r, \Psi)) \leq (k + 2)[o(T(r_1, f)) + o(T(r_1, g))] + S(r_1, f) + S(r_1, g).
$$

$$
\begin{aligned}
\textit{i.e.,} \quad S(r, \Psi) &\leq (k + 2)[S(r_1, f) + S(r_1, g)] + S(r_1, f) + S(r_1, g) \\
&= S(r_1, f) + S(r_1, g). \qquad (3.21)
\end{aligned}
$$

Now we use (3.20) in (3.15) to get

$$
\begin{aligned}
m(r, \Psi) + m(r, \frac{1}{\Psi}) + m(r, \frac{1}{\Psi - 1}) &\leq m(r, \Psi) + m(r, \frac{1}{\Psi - 1}) + nT(r, g) + \bar{N}(r, \frac{1}{\Psi - 1}) \\
&\quad - N_0(r, \frac{1}{\Psi'}) + S(r, f) + S(r, \Psi)
\end{aligned}
$$

$$
\textit{i.e.,} \quad m(r, \frac{1}{\Psi}) \leq nT(r, g) + \bar{N}(r, \frac{1}{\Psi - 1}) - N_0(r, \frac{1}{\Psi'}) + S(r, f) + S(r_1, f) + S(r_1, g)
$$

using (3.21)

$$
\leq nT(r, g) + \bar{N}(r, \frac{1}{\Psi - 1}) - N_0(r, \frac{1}{\Psi'}) + S(r_1, f) + S(r_1, g), \qquad (3.22)
$$

$$
\textit{since} \;\; S(r, f) \leq S(r_1, f) \;\; \textit{as} \;\; r \leq r_1 = M(r, g).
$$

Now by using Jensen-Nevanlinna formula, (3.1), (3.22) and the fact that $f \circ g$ does not have any zero inside $|z| \leq r$ implying $N(r, \frac{1}{f \circ g}) = 0$, we get

$$
\begin{aligned}
T(r, f \circ g) &= T(r, \frac{1}{f \circ g}) + O(1) \\
&= m(r, \frac{1}{f \circ g}) + N(r, \frac{1}{f \circ g}) + O(1) \\
&\leq m(r, \frac{\Psi}{f \circ g}) + m(r, \frac{1}{\Psi}) + O(1) \\
&\leq [S(r_1, f) + S(r_1, g)] + nT(r, g) + \bar{N}(r, \frac{1}{\Psi - 1}) - N_0(r, \frac{1}{\Psi'}) + S(r_1, f) + S(r_1, g) \\
&\leq nT(r, g) + \bar{N}(r, \frac{1}{\Psi - 1}) - N_0(r, \frac{1}{\Psi'}) + S(r_1, f) + S(r_1, g),
\end{aligned}
$$

which completes the proof of the theorem. □

4. Concluding Remark

Here the composition of functions f and g are so taken that f is a non-constant meromorphic function and g is restricted to be an entire function other than a linear polynomial. So, the scope for studying the analogy when g is not restricted to be an entire function only remains open. Also, here the result corresponding to Milloux's theorem is proved for composition of only two functions. But the scope for finding the same for composition of n such functions remains open.

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