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FIXED POINT RESULTS ON DUALISTIC PARTIAL METRIC SPACES

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Abstract

In this paper, we obtain a fixed point result utilizing F-functions in the context of dualistic partial metric space. Our result generalizes recent results in [5], [7] and many others. An illustrative example is included. Additionally, we highlight mathematical bugs that appear in some recent papers in the context of dualistic partial metric space.

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1. Introduction

Matthews [1] introduced the partial metric space by observing that the self-distance of a point need not be zero. He also obtained Banach fixed point theorem in the context of partial metric space. Neill [10] extended the range set of partial metric space to the set of real numbers, and introduced dualistic partial metric space. Further, Oltra et. al. [11] investigate Banach fixed point theorem in the dualistic partial metric space.

Theorem 1.1 ([11]). *Let f be a mapping of a complete dualistic partial metric space (X, p) into itself such that there is a real number c with $0 \leq c < 1$, satisfying*

$$|p(f(x), f(y))| \leq c|p(x, y)|,$$

for all $x, y \in X$. Then f has a unique fixed point.

Afterthat many fixed point theorems in dualistic partial metric space, have been obtained by various researchers. See, [2–9, 12], and references therein.

In 2012, Wardowski [13] obtained a fixed point theorem using $F-$contraction in the complete metric space. Inspired by this, Nazam et. al. [7] in 2021 studied a class of function $\mathcal{F} = \{F | F : (0, \infty) \to \mathbb{R}\}$ satisfying the following properties:

(i) F is strictly increasing,

(ii) For any sequence of positive terms $\{a_n\}$, $\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} F(a_n) = -\infty$,

(iii) There is k in $(0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Nazam et. al. [7] also obtained a fixed point result on the dualistic partial metric space by utilizing the above F-functions.

Theorem 1.2 ([7]). *Let (X, d) be a complete dualistic partial metric space, $F \in \mathcal{F}$, and $T : X \to X$ be a continuous mapping for which there exist $\tau > 0$ such that, for all $x, y \in X$, the following implication holds:*

$$d(Tx, Ty) \neq 0 \Rightarrow \tau + F(|d(Tx, Ty)|) \leq F(|d(x, y)|). \tag{1.1}$$

Then T possesses a unique fixed point.

In this paper, we obtain a fixed point result using F-contraction in the dualistic partial metric space. Our result generalizes recent results in [7], [5] and many others. An illustrative example is also included.

2. Preliminaries

Now, we recall some important definitions, remarks, and lemmas needed for this work.

Definition 2.1 ([1]). Let X be a non-empty set. A partial metric on X is a mapping $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,
(i) $x = y \iff p(x, x) = p(x, y) = p(y, y)$
(ii) $p(x, x) \leq p(x, y)$
(iii) $p(x, y) = p(y, x)$
(iv) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.
The pair (X, p) is said to be a partial metric space.

Definition 2.2 ([10]). Let X be a non-empty set. A dualistic partial metric on X is a mapping $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,
(i) $x = y \iff d(x, x) = d(x, y) = d(y, y)$
(ii) $d(x, x) \leq d(x, y)$
(iii) $d(x, y) = d(y, x)$
(iv) $d(x, y) \leq d(x, z) + d(z, y) - d(z, z)$.
The pair (X, d) is said to be a dualistic partial metric space.

Remark 2.3 ([7, 11]). *Each partial metric space is dualistic partial metric space. But the converse is not true in general.*

Remark 2.4 ([11]). *Let (X, d) be a dualistic partial metric space. Then, the open ball centered at $x_0 \in X$ and radius $r > 0$ is denoted by $B(x_0, r)$, and defined as $B(x_0, r) = \{x \in X : d(x, x_0) < r + d(x_0, x_0)\}$. The collection of all open balls form a base for the topology τ_d in X.*

Remark 2.5 ([11]). *If (X, d) is a dualistic partial metric space, then the function $d^* : X \times X \to \mathbb{R}^+$ such that*

$$d^*(x, y) = d(x, y) - d(x, x), \tag{2.1}$$

is a quasi metric on X; and,

$$D_d^*(x,y) = \max\{d^*(x,y), d^*(y,x)\} \tag{2.2}$$

is a metric on X. It is said to be an induced metric on (X,d).

Definition 2.6 ([11]). Let (X,d) be a dualistic partial metric space. Then,

(i) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if and only if $\lim_{n\to\infty} d(x_n, x) = d(x,x)$,

(ii) a sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim_{m,n\to\infty} d(x_m, x_n)$ exists (and finite),

(iii) X is said to be complete if every Cauchy sequence in it converges to a point $x \in X$ with respect to τ_d. Furthermore,

$$\lim_{m,n\to\infty} d(x_m, x_n) = d(x,x).$$

Lemma 2.7 ([11]). *Let (X,d) be a dualistic partial metric space. Then,*

(i) every Cauchy sequence $\{x_n\}$ in (X, D_d^*) is also a Cauchy sequence in (X,d);

(ii) (X,d) is complete if and only if the induced metric space (X, D_d^*) is complete;

(iii) a sequence $\{x_n\}$ in X converges to an element $x \in X$ with respect D_d^* if and only if

$$\lim_{n\to\infty} d(x, x_n) = d(x,x) = \lim_{m,n\to\infty} d(x_m, x_n).$$

Recently, Nazam et. al. [5] introduce convergence comparison property as follows:

Definition 2.8 ([5]). Let (X,d) be a dualistic partial metric space and $T : X \to X$ be a mapping. A mapping T has a convergence comparison property (CCP) if for every $\{x_n\}$ in X such that $x_n \to x$, T satisfies the following condition:

$$d(x,x) \le d(Tx, Tx).$$

3. Main Result

First, we prove a fixed point result using F-functions in the dualistic partial metric space.

Theorem 3.1. *Let (X,d) be a complete dualistic partial metric space. Let $T : X \to X$ be a mapping and $F \in \mathcal{F}$. Suppose, there exist $\tau > 0$ such that*

$$d(Tx,Ty) \neq 0 \implies \tau + F(|d(Tx,Ty)|) \le F(\mathcal{M}(x,y)), \quad \forall\ x,y \in X \tag{3.1}$$

where

$$\mathcal{M}(x,y) = \max\{|d(x,y)|, |d(x,Tx)|, |d(y,Ty)|\},$$

If T is continuous or T has a convergence comparison property (CCP), then T possesses a unique fixed point.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Clearly, if there is n_0 such that $x_{n_0+1} = x_{n_0}$, then the proof is complete. So, assume that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$. Using equation (3.1), we have

$$F(|d(x_n, x_{n+1})|) \leq F(\mathcal{M}(x_{n-1}, x_n)) - \tau, \tag{3.2}$$

where

$$\begin{aligned}\mathcal{M}(x_{n-1}, x_n) = \max\{&|d(x_{n-1}, x_n)|, |d(x_{n-1}, Tx_{n-1})|, |d(x_n, x_{n+1})|\} \\ &= \max\{|d(x_{n-1}, x_n|, |d(x_n, x_{n+1})|\}\end{aligned}$$

If $\mathcal{M}(x_{n-1}, x_n) = |d(x_n, x_{n+1})|$, then equation (3.2) becomes

$$F(|d(x_n, x_{n+1})|) \quad \leq \quad F(\mathcal{M}(x_{n-1}, x_n)) - \tau = F(|d(x_n, x_{n+1})|) - \tau,$$

which is a contradiction. Hence, $\mathcal{M}(x_{n-1}, x_n) = |d(x_{n-1}, x_n)|$. So, from (3.2), we have

$$F(|d(x_n, x_{n+1})|) \leq F(\mathcal{M}(x_{n-1}, x_n)) - \tau = F(|d(x_{n-1}, x_n)|) - \tau,$$

Thus, we get

$$\begin{aligned} F(|d(x_n, x_{n+1})|) &\leq F(|d(x_{n-1}, x_n)|) - \tau \\ &\leq F(|d(x_{n-2}, x_{n-1})|) - 2\tau \\ &\vdots \\ &\leq F(|d(x_0, x_1)|) - n\tau. \end{aligned} \tag{3.3}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} F(|d(x_n, x_{n+1})|) = -\infty. \tag{3.4}$$

By using (F_2), we have

$$\lim_{n\to\infty} |d(x_n, x_{n+1})| = 0. \tag{3.5}$$

Now, consider the self distances, for $n \in \mathbb{N}$,

$$F(|d(x_n, x_n)|) \leq F(\mathcal{M}(x_{n-1}, x_{n-1})) - \tau, \tag{3.6}$$

where

$$\begin{aligned} \mathcal{M}(x_{n-1}, x_{n-1}) &= \max\{|d(x_{n-1}, x_{n-1})|, |d(x_{n-1}, x_n)|, |d(x_{n-1}, x_n)|\} \\ &= \max\{|d(x_{n-1}, x_{n-1})|, |d(x_{n-1}, x_n)|\} \end{aligned}$$

Case 1: If $\mathcal{M}(x_{n-1}, x_{n-1}) = |d(x_{n-1}, x_{n-1})|$, then from (3.6), we have

$$\begin{aligned} F(|d(x_n, x_n)|) &\leq F(|d(x_{n-1}, x_{n-1})|) - \tau, \\ &\leq F(|d(x_0, x_0)|) - n\tau. \end{aligned}$$

Case 2: If $\mathcal{M}(x_{n-1}, x_{n-1}) = |d(x_{n-1}, x_n)|$, then from (3.6), we have

$$\begin{aligned} F(|d(x_n, x_n)|) &\leq F(|d(x_{n-1}, x_n)|) - \tau, \\ &\leq F(|d(x_0, x_1)|) - n\tau. \end{aligned}$$

Letting $n \to \infty$ in both cases, we have

$$\lim_{n\to\infty} |d(x_n, x_n)| = 0.$$

Continuing from (3.5), using property (iii) of F-functions, there is $h \in (0, 1)$ such that

$$\lim_{n\to\infty} |d(x_n, x_{n+1})|^h F(|d(x_n, x_{n+1})|) = 0.$$

From (3.3),

$$|d(x_n, x_{n+1})|^h F(|d(x_n, x_{n+1})|) \leq |d(x_n, x_{n+1})|^h [F(|d(x_0, x_1)|) - n\tau],$$
$$|d(x_n, x_{n+1})|^h [F(|d(x_n, x_{n+1})|) - F(|d(x_0, x_1)|)] \leq |d(x_n, x_{n+1})|^h n\tau \leq 0.$$

Letting $n \to \infty$ and taking advantage of the properties of the function F, we get that $n|d(x_n, x_{n+1})|^h \to 0$ as $n \to \infty$. There is $N_1 \in \mathbb{N}$ such that

$$|d(x_n, x_{n+1})| \leq \frac{1}{n^{\frac{1}{h}}}, \quad n \geq N_1. \tag{3.7}$$

Similarly, there is $N_2 \in \mathbb{N}$ such that, for any $n \geq N_2$,

$$|d(x_n, x_n)| \leq \frac{1}{n^{\frac{1}{h}}}, \quad n \geq N_2. \tag{3.8}$$

From (3.7) and (3.8), consider $m > n \geq \max\{N_1, N_2\}$,

$$\begin{aligned} d^*(x_n, x_m) &\leq \sum_{i=0}^{m-n-1} d^*(x_{n+i}, x_{n+i+1}) \\ &\leq \sum_{i=0}^{m-n-1} (|d(x_{n+i}, x_{n+i+1})| + |d(x_{n+i}, x_{n+i})|) \\ &\leq 2\sum_{i=0}^{m-n-1} \frac{1}{i^{\frac{1}{h}}}. \end{aligned}$$

Taking the limit to ∞, it follows that $d^*(x_n, x_m)$ converges to 0. Applying an analogous procedure, we get that $d^*(x_m, x_n) \to 0$, hence, $D_d^*(x_n, x_m) \to 0$, so $\{x_n\}$ is a Cauchy sequence in the complete metric space (M, D_d^*). Let x be its limit. Then, by lemma 2.7

$$\lim_{n\to\infty} d(x_n, x) = d(x, x) = \lim_{m,n\to\infty} d(x_n, x_m). \tag{3.9}$$

Also, observe that

$$\begin{aligned} 0 = \lim_{n,m\to\infty} d^*(x_n, x_m) &= \lim_{n,m\to\infty} [d(x_n, x_m) - d(x_n, x_n)] \\ \implies \lim_{n,m\to\infty} d(x_n, x_m) &= \lim_{n\to\infty} d(x_n, x_n) = 0. \end{aligned}$$

Consequently, $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ and so $\{x_n\}$ is a Cauchy sequence in (X, d). From (3.9), we obtain

$$\lim_{n\to\infty} d(x_n, x) = d(x, x) = 0.$$

Now, we show that x is a fixed point of T.

From equation (3.1), we have

$$F(|d(x_n, Tx)|) \leq F(\mathcal{M}(x_{n-1}, x)) - \tau \tag{3.10}$$

where $\mathcal{M}(x_{n-1}, x) = \max\{|d(x_{n-1}, x)|, |d(x_{n-1}, Tx_{n-1})|, |d(x, Tx)|\}$. As $n \to \infty$ in (3.10), we have

$$F(|d(x, Tx)|) \leq F(|d(x, Tx)|) - \tau,$$

which is a contradiction. So, $d(x, Tx) = 0$.

If T is a continuous mapping, then $\{Tx_n\}$ converges to Tx. This implies that $d(Tx_n, Tx) \to d(Tx, Tx)$ as $n \to \infty$. So, $d(x_{n+1}, Tx) \to d(Tx, Tx)$ as $n \to \infty$. Also,

$$d(x, Tx) \leq d(x, x_{n+1}) + d(x_{n+1}, Tx) - d(x_{n+1}, x_{n+1}),$$
$$\text{and} \quad d(x_{n+1}, Tx) \leq d(x_{n+1}, x) + d(x, Tx) - d(x, x);$$

by considering $n \to \infty$, we have $d(x, Tx) \leq d(Tx, Tx)$ and $d(Tx, Tx) \leq d(x, Tx)$. Thus, $d(x, Tx) = d(Tx, Tx)$. Thus, $d(x, x) = d(x, Tx) = d(Tx, Tx) = 0$. So, $Tx = x$.

If T has CCP, then $0 = d(x, x) \leq d(Tx, Tx)$. Also, $d(Tx, Tx) \leq d(x, Tx) = 0$. Thus, $d(x, x) = d(x, Tx) = d(Tx, Tx) = 0$. So, $Tx = x$.

Now, we prove the uniqueness of the fixed point of T. Assume that x and y are two distinct fixed points of T. If $d(x, y) \neq 0$, then the following relations hold true:

$$F(|d(x, y)|) = F(|d(Tx, Ty)|) \leq F(|d(x, y)|) - \tau,$$

which is contradiction. Therefore, $d(x, y) = 0$. Similarly, it can be proved that $d(x, x) = 0$ and $d(y, y) = 0$. It follows that $x = y$, and so the fixed point is unique. □

Now, we illustrate our result through an example.

EXAMPLE 3.2. *Let* $X = \{0, -2, -0.1\}$ *and* $d : X \times X \to \mathbb{R}$*; where,*

$$d(x, y) = \begin{cases} |x - y|, & x \neq y \\ \max\{x, y\}, & x = y. \end{cases}$$

Then (X, d) *is a complete dualistic partial metric space. Define a mapping* $T : X \to X$ *by*

$$T(x) = \begin{cases} 0, & x \in \{0, -0.1\} \\ -0.1, & x = -2 \end{cases}$$

For cases $x = y = 0$; $x = 0, y = -0.1$; $x = -0.1, y = 0$; *and* $x = y = -0.1$, *we have* $d(Tx, Ty) = 0$. *So, condition* (3.1) *of our result is trivially true. Rest of the cases are*

as follow:
Case 1: *If* $(x, y) \in \{(0, -2), (-2, 0)\}$, *then*

$$|d(T0, T(-2))| = |d(T(-2), T0)| = 0.1; \quad \mathcal{M}(0, -2) = \mathcal{M}(-2, 0) = 2.$$

Case 2: *If* $(x, y) \in \{(-0.1, -2), (-2, -0.1)\}$, *then*

$$|d(T(-0.1), T(-2))| = |d(T(-2), T(-0.1))| = 0.1; \quad \mathcal{M}(-0.1, -2) = \mathcal{M}(-2, -0.1) = 1.9.$$

Case 3: *If* $(x, y) = (-2, -2)$, *then*

$$|d(T(-2), T(-2))| = 0.1; \quad \mathcal{M}(-2, -2) = 2.$$

Clearly, T has CCP. Hence, all conditions of Theorem 3.1 are satisfied and T has a unique fixed point 0.

Remark 3.3. *Clearly, our theorem 3.1 generalizes the results due to Nazam et. al. [7], Nazam et. al. [5], Oltra and Valero [11], and Valero [12] in the context of dualistic partial metric space.*

In the following remark, we highlight mathematical bugs that appear in some recent papers ([3], [5], [4], and [9]) in the context of dualistic partial metric space.

Remark 3.4. *Nazam et. al. [3] obtain a fixed point theorem using Dass-Gupta contraction on the dualistic partial metric space. However, the following contractive definition used in [3],*

$$|d(Tx, Ty)| \leq |\frac{\alpha d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)}| + \beta|d(x, y)| \text{ for all } x, y \in X;$$

is not valid in the case of $d(x, y) = -1$. Also, the contractive definition used in Theorem 3 of [5] is not well defined in case of $d(x, y) = 0$.

$$|d(Tx, Ty)| \leq |\frac{a\, d(y, Ty) d(x, Tx)}{d(x, y)}| + b\,|d(x, Tx)| + c\,|d(x, y)| \text{ for all } x, y \in X.$$

In addition, we can also conclude that the above contractive condition does not make any sense in the complete partial metric space $p(x, y) = 0$ and as well as in metric space for $x = y$. So, Corollaries 5 and 6 of [5]are incorrect.

In the context of dualistic partial metric space, the contractive conditions utilized in Bakhru et. al. [4]; and $\phi - \psi$-contraction condition in Nazam and Arshad [9]:

$$\phi(|d(Tx, Ty)|) \leq \phi(\mathcal{M}(x, y)) - \psi(\mathcal{M}(x, y)), \text{ for all } x, y \in X;$$

$$\text{where, } \mathcal{M}(x, y) = \max\left\{|d(x, y)|, |\frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)}|\right\},$$

are also invalid for $d(x, y) = -1$.

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