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INTEGERS ARE NOT THE SUM OF PRIME AND POWER OF SOME PRIMES

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Abstract

A. de Polignac questioned if any positive odd integer could be written in the form $2^n + p$ in 1849, where n is a nonnegative integer and p is 1 or a prime.In 1950 P. Erdös demonstrated those numbers which are not of the form $(2^k + p)$ by using the covering congruences.This types of integers was focused by N. P.Romanoff, F. Cohen and J.L. Selfridge , etc. Applying analogous techniques, we examine the sequences $3^k + M, 5^k + M, 7^k + M, 11^k + M$, where M is a composite number. We will find an arithmetic progression of even numbers x such that $x - 3^k$, $x - 5^k$, $x - 7^k$, $x - 11^k$ will always be composite, for all non-negative integers of k. By applying P. Erdös style techniques, we demonstrate the existence of an arithmetic progression that contains only even integers, not of these forms $3^k + p, 5^k + p, 7^k + p, 11^k + p$, for all non-negative integers of k. Additionally, the positive density of the set containing these integers has been demonstrated.

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1. Introduction

In 1849 A. de Polignac [6] asked whether any positive odd integer can be expressed in the form $2^n + p$, where n is a non-negative integer and p is 1 or a (positive) prime. Using the Brun sieve N.P. Romanoff [2] proved that a positive proportion of the odd integers may be written in this way. On the other hand, van der Corput [7] showed that the set of positive odd integers not representable in the form has a positive density, and by means of cover of the ring Z of the integers. In 1950 P. Erdös [1] demonstrated those numbers which are not of the form $(2^k + p)$ by using the covering congruences. Also P. Erdös [3] , [4] constructed a residue class of odd numbers which contains no integers of the desired form. Inspired by the work of Erdos, in 1975 F. Cohen and J.L. Selfridge [5] observed that the 26-digit number $M = 47867742232066880047611079$ plus or minus a power of 2 can never be a prime. They then deduced that there exist odd numbers not of the form $\pm 2^a \pm q^b$ where q is a prime; $a, b \in N$. Sun [8] gave an explicit arithmetic progression with this property namely $\{m : m = M(\text{mod } \Pi_{p\in P} p)\}$, where $M : 47867742232066880047611079$ and is the set of primes $P := \{2, 3, 5, 7, 11, 13, 17, 19, 31, 31, 37, 41, 61, 73, 97, 109, 151, 241, 257, 331\}$, noted that integers in this progression are in fact not of the form $\pm p^a \pm q^b$.

Definition 1 : A finite system of congruences $k \equiv a_i (\text{mod}\, m_i), 1 \leq i \leq t$ is called covering congruences(or simply a covering) if each integer satisfies at least one congruence in the system.
For example, one can check that the system

$$\begin{aligned} k &\equiv 0 \text{ (mod 2)} \\ k &\equiv 1 \text{ (mod 3)} \\ k &\equiv 3 \text{ (mod 4)} \\ k &\equiv 5 \text{ (mod 6)} \\ k &\equiv 9 \text{ (mod 12)} \end{aligned}$$

is a covering of the integers . Although this example is a covering where the moduli are distinct , for our purpose , we don't want require that the moduli need be distinct.
Definition 2 : The sequence A of positive integers $a_1 \leq a_2 \leq ....$ has lower density δ_1 and upper density δ_2 defined by

$$\delta_1(A) = \liminf_{x \to \infty} \frac{A(n)}{n}$$

$$\delta_2(A) = \limsup_{x \to \infty} \frac{A(n)}{n}$$

where A(n) denotes the number of integers of A which are not greater than n.The value of $\delta_1(n)$ has been reffered variously as the asymptotic density, limit density, or density of A. We shall say that A has a density $\delta(A)$ only if $\delta_1(A) = \delta_2(A)$, in which case we can write

$$\delta(A) = \lim_{x \to \infty} \frac{A(n)}{n}$$

This is sometimes called natural density of A.

2. Main results

THEOREM 2.1. *Let $p_1, p_2, \ldots p_t$ be distinct primes. If there is a covering system of integers*

$$k \equiv b_i \ (mod\ c_i), \quad 1 \leq i \leq t$$

where $c_i = ord_{p_i}(b)$, then the numbers of the form $x - b^k,\ k \in \mathbb{Z}^+$ are divisible by one of the primes $p_i,\ 1 \leq i \leq t$, where x is the solution of the system of equations

$$x \equiv b^{b_i} (mod\ p_i), \quad 1 \leq i \leq t$$

.

PROOF. Let for the given $p_1, p_2, \ldots p_t$

$$k \equiv b_i (\text{mod } c_i), \quad 1 \le i \le t \tag{2.1}$$

be a covering system for an integer $k \in \mathbb{Z}^+$, then $k = b_i + nc_i$ for some $n \in \mathbb{Z}_+$. So, $b^k = b^{b_i + nc_i} \equiv b^{b_i} (\text{mod } c_i)$, since $b^{c_i} \equiv 1 (\text{mod } c_i)$.
Now consider the simultaneous system of linear congruences

$$x \equiv b^{b_i} (\text{mod } c_i), \quad 1 \le i \le t \tag{2.2}$$

By Chinese remainder theorem(CRT) this system of congruences has a unique solution in mod $p_1 p_2 \ldots p_t$. From equation(2.1) and CRT it is clear that for any $k \in \mathbb{Z}^+$ the number $x - b^k$ is divisible by any one of the p_i, $1 \le i \le t$.

□

THEOREM 2.2. *There exist an arithmetic progression consisting only even numbers, no term of this arithmetic progression is of the form* $3^k + p$

PROOF. The proof of this theorem is a direct consequence of theorem (2.1), if we take $b = 3$ and we can find a covering system of integers as given in equation (2.1) and the corresponding system of congruence as given in equation (2.2). So, the task remaining is to find a covering system of integers and their corresponding congruence system of equation as given in theorem (2.1).
We find a covering system of integers and their corresponding congruence system of equation as given in table (1).
Here we elaborate the proof of this theorem. Congruence system of first column is a covering of integers. If we take a system of congruence systems of second column, then we can show that $x - 3^k$ will always composite.
Suppose $k = 3m$, then we take congruence system $x \equiv 3^0 (\text{mod } 13)$, we can see $x - 3^k = x - 3^{3m}$ will be divisible by corresponding prime 13. $k = 6m + 1$,
Then we take congruence system $x \equiv 3^1 (\text{mod } 7)$, we can see $x - 3^k = x - 3^{(6m+1)}$ by corresponding prime 7.
In similar manner we can show for each k on the left column $x - 3^k$ will be divisible by corresponding prime.
By Chinese Remainder Theorem the congruence systems of second column has a unique solution $x \equiv A (\text{mod } M)$, where $M = \Pi_{p \in P} p$,

$$\begin{aligned} P \;=\; & \{13, 7, 757, 19, 37, 5, 73, 530713, 41, 282429005041, 1418632417, \\ & 56227703611393, 2161, 15121, 10512289, 150094634909578633, 16569793 \\ & , 3958044610033\}. \end{aligned}$$

Congruence system of x contains both odd and even numbers . Easily we can remove the odd numbers , hence we will get an arithmetic progression of even integers so that $x - 3^k$ will always composite, k is any non negative integers .

□

Table 1. covering system and their corresponding congruence system of equation

Covering congruences	**Corresponding Congrunce system**
$k \equiv 0 (\text{mod } 3)$	$x \equiv 3^0 (\text{mod } 13)$
$k \equiv 1 (\text{mod } 6)$	$x \equiv 3^1 (\text{mod } 7)$
$k \equiv 2 (\text{mod } 9)$	$x \equiv 3^2 (\text{mod } 757)$
$k \equiv 8 (\text{mod } 18)$	$x \equiv 3^8 (\text{mod } 19)$
$k \equiv 14 (\text{mod } 18)$	$x \equiv 3^{14} (\text{mod } 37)$
$k \equiv 0 (\text{mod } 4)$	$x \equiv 3^0 (\text{mod } 5)$
$k \equiv 10 (\text{mod } 12)$	$x \equiv 3^{10} (\text{mod } 73)$
$k \equiv 5 (\text{mod } 36)$	$x \equiv 3^{15} (\text{mod } 530713)$
$k \equiv 3 (\text{mod } 8)$	$x \equiv 3^3 (\text{mod } 41)$
$k \equiv 17 (\text{mod } 72)$	$x \equiv 3^{17} (\text{mod } 282429005041)$
$k \equiv 53 (\text{mod } 144)$	$x \equiv 3^{53} (\text{mod } 1418632417)$
$k \equiv 125 (\text{mod } 144)$	$x \equiv 3^{125} (\text{mod } 56227703611393)$
$k \equiv 23 (\text{mod } 216)$	$x \equiv 3^{23} (\text{mod } 2161)$
$k \equiv 95 (\text{mod } 216)$	$x \equiv 3^{95} (\text{mod } 15121)$
$k \equiv 167 (\text{mod } 216)$	$x \equiv 3^{167} (\text{mod } 10512289)$
$k \equiv 71 (\text{mod } 108)$	$x \equiv 3^{71} (\text{mod } 150094634909578633)$
$k \equiv 143 (\text{mod } 216)$	$x \equiv 3^{143} (\text{mod } 16569793)$
$k \equiv 215 (\text{mod } 216)$	$x \equiv 3^{215} (\text{mod } 3958044610033)$

Theorem 2.3. *There exist an arithmetic progression consisting only even numbers, no terms this arithmetic progression of the form* $5^k + p$

Proof. Using similar method we have a system (2).for which we can show that an arithmetic progression of even integers so that $x - 5^k$ will always composite, k is any non negative integers.

Table 2. Experimental values of CC for original and encrypted images

Covering congruences	**Corresponding Congruence system**
$k \equiv 0 (\text{mod } 3)$	$x \equiv 5^0 (\text{mod } 31)$
$k \equiv 0 (\text{mod } 2)$	$x \equiv 5^0 (\text{mod } 3)$
$k \equiv 1 (\text{mod } 6)$	$x \equiv 5^1 (\text{mod } 7)$
$k \equiv 5 (\text{mod } 12)$	$x \equiv 5^5 (\text{mod } 601)$
$k \equiv 11 (\text{mod } 24)$	$x \equiv 5^{11} (\text{mod } 390001)$
$k \equiv 23 (\text{mod } 48)$	$x \equiv 5^{23} (\text{mod } 152587500001)$
$k \equiv 47 (\text{mod } 96)$	$x \equiv 5^{47} (\text{mod } 97)$
$k \equiv 95 (\text{mod } 36)$	$x \equiv 5^{95} (\text{mod } 24003159139416881443)$

By Chinese Remainder Theorem the congruence systems of second column has a unique solution $x \equiv A(\text{mod } M)$, where $M = \Pi_{p \in P} p$,

$$P = \{31, 3, 7, 601, 3900001, 152587500001, 97, 2400315913941688144 33\}.$$

Congruence system of x contains both odd and even numbers . Easily we can remove the odd numbers , hence we will get an arithmetic progression of even integers so that $x - 5^k$ will always composite, k is any non negative integers. □

THEOREM 2.4. *There exist an arithmetic progression consisting only even numbers, no terms this arithmetic progression of the form* $7^k + p$

PROOF. Using similar method we have a system (3).for which we can show that an arithmetic progression of even integers so that $x - 7^k$ will always composite, k is any non negative integers.

TABLE 3. covering system and their corresponding congruence system of equation

Covering congruences	**Corresponding Congruence system**
$k \equiv 0(\text{mod } 3)$	$x \equiv 7^0(\text{mod } 19)$
$k \equiv 0(\text{mod } 4)$	$x \equiv 7^0(\text{mod } 5)$
$k \equiv 1(\text{mod } 6)$	$x \equiv 7^1(\text{mod } 43)$
$k \equiv 2(\text{mod } 12)$	$x \equiv 7^2(\text{mod } 13)$
$k \equiv 10(\text{mod } 12)$	$x \equiv 7^{10}(\text{mod } 181)$
$k \equiv 5(\text{mod } 24)$	$x \equiv 7^5(\text{mod } 73)$
$k \equiv 17(\text{mod } 24)$	$x \equiv 7^{17}(\text{mod } 193)$
$k \equiv 11(\text{mod } 24)$	$x \equiv 7^{11}(\text{mod } 409)$
$k \equiv 23(\text{mod } 48)$	$x \equiv 7^{23}(\text{mod } 33232924804801)$
$k \equiv 47(\text{mod } 96)$	$x \equiv 7^{47}(\text{mod } 97)$
$k \equiv 95(\text{mod } 96)$	$x \equiv 7^{95}(\text{mod } 104837857)$

By Chinese Remainder Theorem the congruence systems of second column has a unique solution $x \equiv A(\text{mod } M)$, where $M = \Pi_{p \in P} p$,

$$P = \{19, 5, 43, 13, 181, 73, 193, 409, 33232924804801, 97, 104837857\}.$$

Congruence system of x contains both odd and even numbers . Easily we can remove the odd numbers , hence we will get an arithmetic progression of even integers so that $x - 7^k$ will always composite, k is any non negative integers. □

THEOREM 2.5. *There exist an arithmetic progression consisting only even numbers, no terms this arithmetic progression of the form* $11^k + p$

PROOF. Using similar method we have a system(4). for which we can show that an arithmetic progression of even integers so that $x - 11^k$ will always composite, k is any non negative integers.

By Chinese Remainder Theorem the congruence systems of second column has a unique solution $x \equiv A(\text{mod } M)$, where $M = \Pi_{p\in P} p$,

P= { 3, 43, 45319, 1623931, 29, 1933, 55527473, 113, 449, 2521, 77001139434480073, 337, 394129, 236352238647181441, 3090443962383595123379137, 3421169496361, 19069,
520799717831587692709, 3304981, 468843103, 71596275661 }.

Congruence system of x contains both odd and even numbers . Easily we can remove the odd numbers , hence we will get an arithmetic progression of even integers so that $x - 11^k$ will always composite, k is any non negative integers . □

TABLE 4. covering system and their corresponding congruence system of equation

Covering congruences	**Corresponding Congruence system**
$k \equiv 0(\text{mod } 2)$	$x \equiv 11^0(\text{mod } 3)$
$k \equiv 0(\text{mod } 7)$	$x \equiv 11^0(\text{mod } 43)$
$k \equiv 1(\text{mod } 7)$	$x \equiv 11^1(\text{mod } 45319)$
$k \equiv 3(\text{mod } 14)$	$x \equiv 11^3(\text{mod } 1623931)$
$k \equiv 5(\text{mod } 28)$	$x \equiv 11^5(\text{mod } 29)$
$k \equiv 19(\text{mod } 28)$	$x \equiv 11^{19}(\text{mod } 1933)$
$k \equiv 9(\text{mod } 28)$	$x \equiv 11^9(\text{mod } 55527473)$
$k \equiv 23(\text{mod } 56)$	$x \equiv 11^{23}(\text{mod } 113)$
$k \equiv 51(\text{mod } 56)$	$x \equiv 11^{51}(\text{mod } 449)$
$k \equiv 11(\text{mod } 56)$	$x \equiv 11^{11}(\text{mod } 2521)$
$k \equiv 25(\text{mod } 56)$	$x \equiv 11^{25}(\text{mod } 77001139434480073)$
$k \equiv 39(\text{mod } 112)$	$x \equiv 11^{39}(\text{mod } 337)$
$k \equiv 95(\text{mod } 112)$	$x \equiv 11^{95}(\text{mod } 394129)$
$k \equiv 53(\text{mod } 112)$	$x \equiv 11^{53}(\text{mod } 236352238647181441)$
$k \equiv 109(\text{mod } 112)$	$x \equiv 11^{109}(\text{mod } 3090443962383595123379137)$
$k \equiv 13(\text{mod } 42)$	$x \equiv 11^{13}(\text{mod } 3421169496361)$
$k \equiv 27(\text{mod } 84)$	$x \equiv 11^{27}(\text{mod } 19069)$
$k \equiv 69(\text{mod } 84)$	$x \equiv 11^{69}(\text{mod } 520799717831587692709)$
$k \equiv 41(\text{mod } 126)$	$x \equiv 11^{41}(\text{mod } 3304981)$
$k \equiv 83(\text{mod } 126)$	$x \equiv 11^{83}(\text{mod } 468843103)$
$k \equiv 125(\text{mod } 126)$	$x \equiv 11^{125}(\text{mod } 71596275661)$

Finding the covering system and assigning a prime number to a class of covering system is a difficult problem.

Theorem 2.6. *The set of integers of the form* $3^k + M, 5^k + M, 7^k + M, 11^k + M$, *where* M *is a composite number has positive density.*

Proof. N.P. Romanoff [2] has proved that a positive proportion of the odd integers may be written in $(a^k + p)$, where a is any positive integer. Using Romanoff method we can say that the integers $3^k + M, 5^k + M, 7^k + M, 11^k + M$, where M is a composite number have positive density. □

3. Conclusion

In this paper different arithmetic progression have been constructed to prove that these arithmetic progression do not contain integers of the form $3^k + p, 5^k + p, 7^k + p, 11^k + p$. These results have been proved using the covering system of integers and Chinese remainder theorem. Finding the covering system and assigning a prime number to each class of covering system is a difficult problem. Using the result of first theorem and factors of a cyclotomic polynomial we have find such covering system and their corresponding system of congruence equations. Using N.P. Romanoff theorem we have also proved that the set of these type of integers have positive density.

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