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ON RETRO BANACH FRAMES IN SEPARABLE BANACH SPACES

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Abstract

We study retro Banach frames on the unit sphere of a separable Banach space. A characterization of retro Banach frames in terms of sets associated with retro Banach frames and unit balls in Banach spaces is given. A dual property of retro Banach frames is also discussed.

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1. Introduction

Duffin and Schaeffer [6], while addressing some difficult problems from the theory of nonharmonic Fourier series introduced frames for Hilbert spaces. The theory of frames for Hilbert spaces was revived by Daubechies, Grossmann and Meyer in [4]. Let $\mathcal{H}$ be a separable real (or complex) Hilbert space with the inner product $\langle ., . \rangle$. A sequence $\{f_k\}_{k=1}^{\infty} \subset \mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$, if there exist positive constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2 \text{ for all } f \in \mathcal{H}.$$

The scalars A and B are called *lower* and *upper bounds* of the frame $\{f_k\}_{k=1}^{\infty}$, respectively. They are not unique. A frame $\{f_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ is said to be *tight* (or *Parseval frame*), if it is possible to choose $A = B$, *normalized tight* if, $A = B = 1$ and, *standard normalized tight*, if, it is normalized tight and $\|f_k\| = 1$ for all $k \in \mathbb{N}$. A frame $\{f_k\}$ for $\mathcal{H}$ is said to be *exact*, if it ceases to be a frame for $\mathcal{H}$ after removal of any element f_j from the collection $\{f_k\}_{k=1}^{\infty}$. The *frame operator* $S =: \mathcal{H} \to \mathcal{H}$ is given by

$$S(f) = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \ f \in \mathcal{H}.$$

The frame operator S is a positive, continuous and invertible linear operator on $\mathcal{H}$. This gives the *reconstruction formula* for all $f \in \mathcal{H}$,

$$f = SS^{-1}f = \sum_{k=1}^{\infty} \langle S^{-1}f, f_k \rangle f_k. \tag{1.1}$$

The scalars $\{\langle S^{-1}f, f_k\rangle\}_{k=1}^{\infty}$ are called *frame coefficients* of the vector $f \in \mathcal{H}$. The sequence $\{S^{-1}f_k\}_{k=1}^{\infty}$ also form a frame for $\mathcal{H}$, the *canonical dual frame* of the frame $\{f_k\}_{k=1}^{\infty}$. A frame $\{g_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ is called an *alternate dual frame* (or simply *dual frame*) for $\{f_k\}_{k=1}^{\infty}$ if, for all $f \in \mathcal{H}$, we have

$$f = \sum_{k=1}^{\infty} \langle f, g_k\rangle f_k. \tag{1.2}$$

The dual of a frame need not be unique. More precisely, the dual of a frame is unique if and only it is exact. Furthermore, dual frames can be written in an explicit algebraic and parametric formula. Depending upon applications, we choose certain type a dual frame [2]. Duality principles in Gabor theory such as the Ron-Shen duality principle and the Wexler-Raz biorthogonality relations play a fundamental role for analyzing Gabor systems, see [5].

Frames were extended to Banach spaces by Gröchenig in [8]. Casazza, Han and Larson introduced and studied various types of frames and framing models in Banach spaces in [3]. For fundamental properties of Banach frames, we refer to [1, 3, 10, 12, 15**?** –17]. The Banach frame for a Banach space ensure reconstruction of a vector (signal) via a bounded linear operator (or the synthesis operator). The basic theory of frames can be found in the text [9].

Throughout this paper $\mathcal{X}$ will denote an infinite dimensional separable Banach space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $\mathcal{X}^*$ the conjugate space of $\mathcal{X}$. For a sequence $\{\Phi_n\} \subset \mathcal{X}$, $[\Phi_n]$ denotes the closure of the linear hull of $\{\Phi_n\}$ in the norm topology of $\mathcal{X}$; and $S_{\mathcal{X}} = \{\Phi \in \mathcal{X} : \|\Phi\| = 1\}$, $B_{\mathcal{X}} = \{\Phi \in \mathcal{X} : \|\Phi\| \leq 1\}$ denote the unit sphere and unit ball in $\mathcal{X}$, respectively. As usual, $\delta_{j,k}$ denotes the Kronecker delta. Unless stated otherwise, all sequences are indexed by the set of positive integers $\mathbb{N}$. Jain, Kaushik and Vashisht [10] introduced the notion of retro Banach frames for conjugate of separable Banach spaces.

Definition 1.1. [10] A pair $\mathcal{F} \equiv (\{f_k\}, \Theta)$ (where, $\{f_k\} \subset \mathcal{X}, \Theta : \mathcal{Z}_d \to \mathcal{X}^*$) is called a *retro Banach frame* for $\mathcal{X}^*$ with respect to an associated sequence space $\mathcal{Z}_d$, if

(*i*) $\{f^*(f_k)\} \in \mathcal{Z}_d$, for each $f^* \in \mathcal{X}^*$.

(*ii*) There exist positive constants $0 < \mathrm{A}_0 \leq \mathrm{B}_0 < \infty$ such that

$$\mathrm{A}_0\|f^*\| \leq \|\{f^*(f_k)\}\|_{\mathcal{Z}_d} \leq \mathrm{B}_0\|f^*\|, \text{ for each } f^* \in \mathcal{X}^*.$$

(*iii*) Θ is a bounded linear operator such that $\Theta(\{f^*(f_k)\}) = f^*$, for each $f^* \in \mathcal{X}^*$.

Definition 1.2. A retro Banach frame $(\{f_k\}, \Theta)$ for $\mathcal{X}^*$ is said to be exact if for any $j \in \mathbb{N}$, there is no reconstruction operator Ξ such that $(\{f_k\}_{k\neq j}, \Xi)$ is not a retro Banach frame for $\mathcal{X}^*$.

Remark 1.3. *A retro Banach frame* $(\{f_k\}, \Theta)$ *for* $\mathcal{X}^*$ *is exact if and only if* $f_j \neq [f_k]_{k\neq j}$ *for all* $j \in \mathbb{N}$. *Its proof can be found in [10].*

The paper is motivated by the recent work of Freeman, Hotovy and Martin for the construction of moving finite unit norm tight frames in the sphere [7]. They showed in [7] that the sphere S^n has a finite unit norm tight frames (FUNTF) if and only if n is odd. They also obtained upper bound on the minimal size of such a frame. In this paper, we study retro Banach on spheres in Banach space under the name retro Banach frames of type S, see Definition 2.1. A characterization of retro Banach frames which lies on the unit sphere in Banach spaces in terms of sets associated with retro Banach frames is given, see Theorem 2.5. In Proposition 2.6, a dual property of retro Banach frames of type S in Banach spaces is given.

2. The Main Results

Definition 2.1. A retro Banach frame $\mathcal{F} = (\{f_k\}, \Theta)$ for $\mathcal{X}^*$ is said to be of type S, if there exists a sequence $\{f_n^*\} \subset \mathcal{X}^*$ such that

(*i*) $(\{f_k\}, \{f_k^*\}) \subset S_{\mathcal{X}} \times S_{\mathcal{X}^*}$,
(*ii*) $f_j^*(f_k) = \delta_{j,k}$ for all $j, k \in \mathbb{N}$.

Remark 2.2. *The concept of retro Banach frame of type S is similar to the normalized Schauder bases for separable Banach spaces, see [11, 13] for details. We recall that a sequence $(\{f_k\} \subset \mathcal{X}$ is called a Schauder basis for $\mathcal{X}$, if for each $f \in \mathcal{X}$ there is a unique sequence of scalars $\{a_k\}_{k=1}^{\infty}$ such that*

$$f = \sum_{k=1}^{\infty} a_k f_k, \tag{2.1}$$

where the series in (2.1) converges in the norm topology of $\mathcal{X}$. That is, $\|f - \sum_{k=1}^{n} a_k f_k\| \to 0$ as $n \to \infty$. In this case, a_k are bounded linear functional on $\mathcal{X}$ such that $a_j(f_k) = \delta_{j,k}$ for all $j, k \in \mathbb{N}$. If $(\{f_k\}, \{a_k\}) \subset S_{\mathcal{X}} \times S_{\mathcal{X}^*}$, then we say that $\{f_k\}_{k=1}^{\infty}$ is a normalized Schauder basis for $\mathcal{X}$. One of the differences between retro Banach frames and normalized bases is related to series representation of vectors in the underlying space. To be precise, if $(\{f_k\}, \Theta)$ is a retro Banach frame for $\mathcal{X}^*$, then each $f \in \mathcal{X}$ may not have series expansion. Indeed, consider the separable Banach space $\mathcal{X} = (C(\boldsymbol{T}), \|.\|_{\infty})$ consisting of scalar-valued continuous 1-periodic functions on $\mathbb{R}$. Define $\chi_k(t) = e^{2\pi ikt}$, $k \in \mathbb{Z}$. Note that $\{\chi_k\}_{k=-\infty}^{\infty}$ define bounded linear functionals on $\mathcal{X}$ by the formula:

$$\chi_k^*(f) = \langle \chi_k, f \rangle = \int_0^1 f(t) e^{-2\pi ikt} dt, \ f \in \mathcal{X}.$$

It is easy to see that $\chi_k^*(\chi_m) = \delta_{k,m}$ for all $k, m \in \mathbb{Z}$. Furthermore, it is well known that there exists continuous functions in $\mathcal{X} = (C(\boldsymbol{T}), \|.\|_{\infty})$ which can not be expressed as infinite series expansion over $\{\chi_k\}_{k=-\infty}^{\infty}$, see [9, p. 153*] for technical details. That is, $\{\chi_k\}_{k=-\infty}^{\infty}$ is not a normalized Schauder basis for $\mathcal{X}$. Now $\mathcal{Z}_d = \left\{ \{f^*(\chi_k)\}_{k=-\infty}^{\infty} : f^* \in \mathcal{X}^* \right\}$ is a Banach space with the norm given by*

$$\left\| \{f^*(\chi_k)\}_{k=-\infty}^{\infty} \right\|_{\mathcal{Z}_d} = \|f^*\|_{\mathcal{X}^*}, f^* \in \mathcal{X}^*.$$

Define $\Theta_o : \mathcal{Z}_d \to \mathcal{X}^*$ *by* $\Theta : \{f^*(\chi_k)\}_{k=-\infty}^{\infty} \mapsto f^*, f^* \in \mathcal{X}^*$ *Then,* Θ_o *is a bounded linear operator such that* $\mathcal{F} \equiv (\{\chi_k\}_{k=-\infty}^{\infty}, \Theta_o)$ *is a retro Banach frame for* $\mathcal{X}^*$ *which is of type* S.

Remark 2.3. *As in the case of Hilbert frames, retro Banach frames are complete in* $\mathcal{X}$. *This gives an approximation (not necessarily unique),* $f = \lim_{n\to\infty} \sum_{j=1}^{m_n} \alpha_j^{(n)} f_j$ *of each vector* f *in* $\mathcal{X}$.

Remark 2.4. *Every finite dimensional Banach space has a retro Banach frame of type* S. *Furthermore, in finite dimensional Banach spaces, the concept of retro Banach frame of type* S *is equivalent to an Auerbach basis.*

Next, we give a geometrical characterization of retro Banach frames of type S in separable Banach spaces in terms of sets associated with unit balls in Banach spaces. This is inspired by the work of Karlin in [11] for normalized Schauder bases in separable Banach spaces.

Theorem 2.5. *Let* $\mathcal{F} = (\{f_k\}, \Theta)$ *be a retro Banach frame for* $\mathcal{X}^*$. *Then, the following statements are equivalent.*

(i) $\mathcal{F}$ *is of type* S.

(ii) $\mathcal{F}$ *is exact and the unit ball* $B_{\mathcal{X}}$ *of* $\mathcal{X}$ *lies between the sets*

$$\mathcal{Z}_1 = \left\{ f_0 = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i \in \mathcal{X}, \{\alpha_i^{(n)}\} \subset \mathbb{K} : \lim_{n\to\infty} \sum_{i=1}^{m_n} |\alpha_i^{(n)}| \leq 1 \right\},$$

and

$$\mathcal{Z}_2 = \left\{ f_0 = \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i \in \mathcal{X}, \{\alpha_i^{(n)}\} \subset \mathbb{K} : \max_{\substack{1\leq i\leq m_n \\ 1\leq n<\infty}} |\alpha_i^{(n)}| \leq 1 \right\}.$$

Proof. (i) $\Longrightarrow$ (ii) : By (i), there exists a sequence $\{f_k^*\} \subset \mathcal{X}^*$ such that $f_j^*(f_k) = \delta_{j,k}$ for all $j, k \in \mathbb{N}$ (this gives $f_j \notin [f_k]_{k\neq j}$ for all $j \in \mathbb{N}$) and $(\{f_n\}, \{f_n^*\}) \subset S_{\mathcal{X}} \times S_{\mathcal{X}^*}$. Since $f_j \notin [f_k]_{k\neq j}$ for all $j \in \mathbb{N}$, $\mathcal{F}$ is exact. To show $\mathcal{Z}_1 \subset B_{\mathcal{X}} \subset \mathcal{Z}_2$. Let

$f = \lim_{n\to\infty} \sum_{j=1}^{m_n} \alpha_j^{(n)} f_j \in \mathcal{X}$ be arbitrary. For this f, we compute

$$\max_{\substack{1\le i\le m_n\\ 1\le n<\infty}} |\alpha_i^{(n)}| = \max_{\substack{1\le i\le m_n\\ 1\le n<\infty}} \left| f_i^*\Big(\lim_{n\to\infty} \sum_{j=1}^{m_n} \alpha_j^{(n)} f_j \Big) \right| \tag{2.2}$$

$$\le \|f_i^*\| \left\| f = \lim_{n\to\infty} \sum_{j=1}^{m_n} \alpha_j^{(n)} f_j \right\|$$

$$= \|f\| \tag{2.3}$$

$$\le \lim_{n\to\infty} \sum_{j=1}^{m_n} \left|\alpha_j^{(n)}\right| \|f_j\|$$

$$= \lim_{n\to\infty} \sum_{j=1}^{m_n} \left|\alpha_j^{(n)}\right|. \tag{2.4}$$

If $f = \lim\limits_{n\to\infty} \sum\limits_{j=1}^{m_n} \alpha_j^{(n)} f_j \in \mathcal{Z}_1$, then $\lim\limits_{n\to\infty} \sum\limits_{j=1}^{m_n} \left|\alpha_j^{(n)}\right| \le 1$. Thus, by using (2.3) and (2.4), we obtain $\|f\| \le 1$, so $f \in B_{\mathcal{X}}$. On the other hand, if $f_0 = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{m_n} \alpha_i^{(n)} f_i$ is a vector in $B_{\mathcal{X}}$, then by using (2.2) and (2.3), we have $\max\limits_{\substack{1\le i\le m_n\\ 1\le n<\infty}} |\alpha_i^{(n)}| \le 1$. Therefore, $f_0 \in \mathcal{Z}_2$. Hence $\mathcal{Z}_1 \subset B_{\mathcal{X}} \subset \mathcal{Z}_2$.

$(ii) \Longrightarrow (i)$: Since $\mathcal{F} \equiv (\{f_k\}, \Theta)$ is exact, there exists a sequence $\{f_k^*\} \subset \mathcal{X}^*$ such that $f_j^*(f_k) = \delta_{j,k}$ for all $j, k \in \mathbb{N}$. To show $(\{f_k\}, \{f_k^*\}) \subset S_{\mathcal{X}} \times S_{\mathcal{X}^*}$. Let us write $f_k = \sum_{i=1}^{\infty} \delta_{k,i} f_i = \lim\limits_{n\to\infty} \sum_{i=1}^{n(=m_n)} \delta_{k,i} f_i$ $(k \in \mathbb{N})$. Therefore, $f_k \in \mathcal{Z}_1 \subset B_{\mathcal{X}}$ for all $k \in \mathbb{N}$. Thus, $\|f_k\| \le 1$ for all $k \in \mathbb{N}$.

Fix $j \in \mathbb{N}$. Then, $\frac{f_j}{\|f_j\|} \in B_{\mathcal{X}} \subset \mathcal{Z}_2$. Therefore, $\frac{1}{\|f_j\|} = \max\limits_{\substack{1\le i\le m_n\\ 1\le n<\infty}} \frac{|\beta_i^{(n)}|}{\|f_j\|}$, where $\beta_{j,n} = \delta_{j,n}$ for all $n \in \mathbb{N}$. This gives $\|f_j\| \ge 1$. Hence $\{f_k\} \subset S_{\mathcal{X}}$. Next, we show that $\{f_k^*\} \subset S_{\mathcal{X}^*}$. Let $\lim\limits_{n\to\infty} \sum\limits_{l=1}^{m_n} \alpha_i^{(n)} f_i = f_0 \in \mathcal{X}$ $(f_0 \neq 0)$ be arbitrary. Then, by hypothesis, we have

$$\lim_{n\to\infty} \sum_{j=1}^{m_n} \frac{\alpha_j^{(n)}}{\|f_0\|} f_j \in B_{\mathcal{X}} \subset \mathcal{Z}_2.$$

This gives

$$\max_{\substack{1\le j\le m_n\\ 1\le n<\infty}} \frac{\alpha_j^{(n)}}{\|f_0\|} f_j \le 1. \tag{2.5}$$

For $1 \leq j \leq m_n,\ n \in \mathbb{N}$, we have

$$\begin{aligned}\|\alpha_j^{(n)} f_j\| &\leq |\alpha_j^{(n)}|\|f_j\| \\ &\leq |\alpha_j^{(n)}| \\ &\leq \|f_0\| \quad \big(\text{by using}(2.5)\big) \\ &= \Big\| \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i \Big\|.\end{aligned}$$

Therefore

$$\|\alpha_j^{(n)} f_j\| \leq \Big\| \lim_{n\to\infty} \sum_{i=1}^{m_n} \alpha_i^{(n)} f_i \Big\|, \quad 1 \leq j \leq m_n,\ n \in \mathbb{N}. \tag{2.6}$$

By using (2.6), we obtain

$$\begin{aligned}1 &= \|f_j\| \\ &\leq \inf_{\{\alpha_i^{(n)}\}} \Big\| f_j - \lim_{n\to\infty} \sum_{\substack{i=1 \\ i\neq j}}^{m_n} \alpha_i^{(n)} f_i \Big\| \\ &= \text{dist}(f_j, [f_1, f_2, \ldots, f_{j-1}, f_{j+1}, \ldots]) \\ &\leq \|f_j\| = 1 \text{ for all } j \in \mathbb{N}.\end{aligned} \tag{2.7}$$

Let $g^* \in \mathcal{X}^*$, $\mathcal{Z} = \{f \in \mathcal{X} : g^*(f) = 0\}$ and let $f_0 \in \mathcal{X}$. Then, by using the Hahn-Banach Theorem, we can show that $\text{dist}(f_0, \mathcal{Z}) = \frac{|g^*(f_0)|}{\|g^*\|}$. Choose $g^* = f_j^*$ ($j \in \mathbb{N}$ is arbitrary but fixed). Then, $\mathcal{Z} = [f_1, f_2, \ldots, f_{j-1}, f_{j+1}, \ldots]$. Therefore, for $f_0 = f_j$, by using (2.7), we obtain

$$1 = \text{dist}(f_j, \mathcal{Z}) = \frac{|f_j^*(f_j)|}{\|f_j^*\|} \quad \left(= \frac{|g^*(f_0)|}{\|g^*\|}\right) = \frac{1}{\|f_j^*\|}.$$

That is, $\{f_k^*\} \subset S_{\mathcal{X}^*}$. Hence, $(\{f_k\}, \{f_k^*\}) \subset S_{\mathcal{X}} \times S_{\mathcal{X}^*}$. □

The Dual Version of Theorem 2.5: Recall that the dual frame of a frame for a Hilbert space preserve the property of reconstruction via an infinite series (need not be uniquely). Not only this, the dual of a Hilbert frame has a relation with the original frame (see (1.2)). In the definition of a retro Banach frame there are three Banach spaces, the duality of retro Banach frames is not so natural as in case of frames for Hilbert spaces. The following result give a type of duality for retro Banach frames of type S.

Theorem 2.6. *Let $\mathcal{F} = (\{f_k\}, \Theta)$ be a retro Banach frame of type S for $\mathcal{X}^*$. Then, there exists a reconstruction operator Ξ such that $\mathcal{G} = (\{f_k^*\}, \Xi)$ is a retro Banach frame of type S for $\mathcal{W}^{**}$, where $\mathcal{W}^* = [f_k^*] \subset \mathcal{X}^*$.*

Proof. Since $\mathcal{F}$ is a retro Banach frame of type S for $\mathcal{X}^*$ with admissible sequence $\{f_k^*\} \subset \mathcal{X}^*$, we have $(\{f_k\}, \{f_k^*\}) \subset S_{\mathcal{X}} \times S_{\mathcal{X}^*}$ and $f_j^*(f_k) = \delta_{j,k}$ for all $j, k \in \mathbb{N}$. Let $\mathcal{E}_0 = \{\{\psi(f_k^*)\} : \psi \in \mathcal{W}^{**}\}$. Then, $\mathcal{E}_0$ is a Banach space with the norm given by

$$\|\{\psi(f_k^*)\}\|_{\mathcal{E}_0} = \|\psi\|_{\mathcal{W}^{**}}, \psi \in \mathcal{W}^{**}.$$

Define $\Xi : \mathcal{E}_0 \to \mathcal{W}^*$ by $\Xi(\{\psi(f_k^*)\}) = \psi, \psi \in \mathcal{W}^{**}$. Then, Ξ is a bounded linear operator such that $\mathcal{G} \equiv (\{f_k^*\}, \Xi)$ is a retro Banach frame for $\mathcal{W}^{**}$ with bounds $A = B = 1$.

Next we show that $\mathcal{G}$ is of type S. Choose $f_k^{**} = u(f_k)$ for all $k \in \mathbb{N}$, where u is the canonical mapping of $\mathcal{X}$ into $\mathcal{W}^{**}$. Then, $\{f_k^{**}\} \subset \mathcal{W}^{**}$ and

$$f_j^{**}(f_k^*) = u(f_j)(f_k^*) = f_k^*(f_j) = \delta_{j,k} \text{ for all } j, k \in \mathbb{N}.$$

Now

$$\|f_k^{**}\| \geq \frac{|f_k^{**}(f_k^*)|}{\|f_k^*\|} = |f_k^{**}(f_k^*)| = |u(f_k)(f_k^*)| = |f_k^*(f_k)| = 1, \ k \in \mathbb{N}. \tag{2.8}$$

We compute

$$\begin{aligned} \|f_k^{**}\| &= \sup_{\|f^*\|\leq 1} |f_k^{**}(f^*)| \\ &= \sup_{\|f^*\|\leq 1} |u(f_k)(f_k^*)| \\ &= \sup_{\|f^*\|\leq 1} |f_k^*(f_k)| \\ &\leq \|f_k\| = 1, \ k \in \mathbb{N}. \end{aligned} \tag{2.9}$$

By using (2.8) and (2.9), we have $\{f_k^{**}\} \subset S_{\mathcal{W}^{**}}$. Therefore, $(\{f_k^*\}, \{f_k^{**}\}) \subset S_{\mathcal{W}^*} \times S_{\mathcal{W}^{**}}$. Hence $\mathcal{G}$ is retro Banach frame of type S for $\mathcal{W}^{**}$. □

Remark 2.7. *The retro Banach frame $\mathcal{G}$ in Theorem 2.6 is the dual retro Banach frame for $\mathcal{F}$ (it is unique). Recall that if, $\{g_k\}$ is a dual frame of an exact Hilbert frame $\{f_k\}$ for $\mathcal{H}$, then $\{g_k\}$ is also exact and vice-versa. This dual property is not true, in general, for retro Banach frames of type S. This is justified in the following example.*

Example 2.8. *Let $\mathcal{X} = (c_0, \|.\|_\infty)$ and let $\chi_k = \{\delta_{j,k}\}_{j=1}^{\infty}$, $k \in \mathbb{N}$. Define $\{f_k\} \subset \mathcal{X}$ by $f_k = 2\sum_{j=1}^{k} \chi_j$, $k \in \mathbb{N}$. Then, there exists a reconstruction operator Θ such that $\mathcal{F} \equiv (\{f_k\}, \Theta)$ be the retro Banach frame for $\mathcal{X}^*$ with respect to $\mathcal{E}_d = \{\{f^*(f_k)\} : f^* \in \mathcal{X}^*\}$ and with frame bounds $A_o = B_o = 1$. One may observe that $\mathcal{F}$ is not of type S. Furthermore, $\mathcal{F}$ is exact with the admissible sequence $\{f_k^*\} \subset \mathcal{X}^*$ given by*

$$f_k^*(f) = \begin{cases} 0, & 1 \leq j \leq k-1 \\ \frac{1}{2}\xi_j & j = k \\ -\frac{1}{2}\xi_j & j = k+1 \\ 0 & j > k+1 \end{cases}, \quad f = \{\xi_i\} \in \mathcal{X}, \ k \in \mathbb{N}.$$

Define $\Xi : \mathcal{E}_d = \{\{\psi(f_k^*)\} : \psi \in \mathcal{W}^{**}\} \to \mathcal{W}^{**}$ *by* $\Xi(\{\psi(f_k^*)\}) = \psi$, $\psi \in \mathcal{W}^{**}$. *Here* $\mathcal{W}^* = [f_k^*]$. *Then,* Ξ *is a bounded linear operator such that* $\mathcal{G} \equiv (\{f_k^*\}, \Xi)$ *is a retro Banach frame for* $\mathcal{W}^{**}$ *with respect to* $\mathcal{E}_d$ *with retro frame bounds* $A = B = 1$.

Next we show the dual frame $\mathcal{G}$ *of* $\mathcal{F}$ *is of type* S. *Choose* $f_k^{**} = u(f_k)$, $k \in \mathbb{N}$, *where* u *is the canonical mapping of* $\mathcal{X}$ *into* $\mathcal{W}^{**}$. *Then,* $f_j^{**}(f_k^*) = u(f_j)(f_k^*) = f_k^*(f_j) = \delta_{j,k}$ *for all* $j, k \in \mathbb{N}$.

For any $f^* = \{\xi_i\} \in \mathcal{W}^*$, *we have*

$$|f^*(f_k)| = \left|\sum_{i=1}^{k} 2\xi_i\right| \leq \sum_{i=1}^{\infty} |\xi_i| = \|f^*\|. \tag{2.10}$$

Therefore, by using (2.10), *we have*

$$\begin{aligned} \|f_k^{**}\| &= \sup_{\substack{f^* \in \mathcal{W}^* \\ \|f^*\| \leq 1}} |f^{**}(f^*)| \\ &= \sup_{\substack{f^* \in \mathcal{W}^* \\ \|f^*\| \leq 1}} |u(f_k)(f^*)| \\ &= \sup_{\substack{f^* \in \mathcal{W}^* \\ \|f^*\| \leq 1}} |f^*(f_k)| \\ &\leq 1 \text{ for all } k \in \mathbb{N}. \end{aligned} \tag{2.11}$$

On the other hand, by definition of $f_k^* \in \mathcal{W}^*$, $\{f_k^*\} \subset S_{\mathcal{W}^*}$. *Therefore, for all* $k \in \mathbb{N}$, *we have*

$$\|f_k^{**}\| = \|u(f_k)\| \geq |u(f_k)(f_k^*)| = |f_k^*(f_k)| = 1. \tag{2.12}$$

By using (2.11) *and* (2.12), *we have* $\{f_k^{**}\} \subset S_{\mathcal{W}^{**}}$. *Therefore,* $(\{f_k^*\}, \{f_k^{**}\}) \subset S_{\mathcal{W}^*} \times S_{\mathcal{W}^{**}}$. *Hence* $\mathcal{G}$ *is a retro Banach frame of type* S *for* $\mathcal{W}^{**}$.

REMARK 2.9. *If* $\mathcal{X}$ *is reflexive, then* $\mathcal{F}$ *is a retro Banach frame of type* S *for* $\mathcal{X}^*$ *if and only if its dual frame* $\mathcal{G}$ *is a retro Banach frame of type* S *for* $\mathcal{W}^{**}$.

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