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QUASI-ISOMETRY BETWEEN TWO ALMOST CONTACT METRIC MANIFOLDS

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Abstract

In this paper the notion of quasi-isometry between two Riemannian manifolds has been introduced. This idea is also imposed to study quasi-isometry between two almost contact metric manifolds. Moving further, some curvature properties of two quasi-isometrically embedded almost contact metric manifolds, $N(k)-$contact metric manifolds and Sasakian manifolds are investigated. Next, an illustrative example of a quasi-isometry between two Sasakian structures is constructed. Finally, a relation between the scalar curvature and the quasi-isometric constants for two quasi-isometric Riemannian manifolds has been established.

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1. Introduction

The notion quasi-isometry was first introduced by the American mathematician G.D. Mostow [3] in 1973 and later it was Gromov [8] who studied quasi-isometry to a much further extent in the context of geometric group theory. But Mostow used the term pseudo-isometry and this notion was little bit different from the one that we will be discussing here (See [1]). Let (X, d_x) and (Y, d_y) be two metric spaces and $f : (X, d_x) \longrightarrow (Y, d_y)$ be a map. Then the map f is said to be an (L, C) quasi-isometric embedding, if there exist constants $L \geq 1$ and $C \geq 0$ such that for all $p, q \in (X, d_x)$

$$\frac{1}{L} d_x(p, q) - C \leq d_y(f(p), f(q)) \leq L d_x(p, q) + C. \tag{1.1}$$

Moreover, if the quasi-isometric embedding f has a quasi dense image, i.e if there is a constant $D \geq 0$ such that $\forall y \in Y$, $\exists x \in X$ for which $d_y(f(x), y) \leq D$, then the map f is called a quasi isometry and we call that the two metric spaces (X, d_x) and (Y, d_y) are quasi-isometric. For example, it can shown that the grid $\mathbb{Z}^2$ with the taxicab metric is quasi-isometric to the plane $\mathbb{R}^2$ with the usual Euclidean metric via the natural

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inclusion map as a Quasi-isometry [4]. Also it is easy to see that any metric space of finite diameter is quasi-isometric to a point. In that manner we can say that, all metric spaces of finite diameter are same in the sense of quasi-isometry.

We say that a map $f : X \longrightarrow Y$ has finite distance from a map $g : X \longrightarrow Y$ if there is a constant $M \geq 0$ such that, for all $x \in X$ we have $d_x(g(x), f(x)) \leq M$ and $f \sim g$ if f is at finite distance from g. Then it is easy to check that $' \sim'$ is an equivalence relation. We denote $QI(X)$ be the set of all quasi-isometries from $X \longrightarrow X$, and let $QI(X)/\sim$ be the set of all quasi isometries of X modulo finite distance. Moreover the composition $([f], [g]) \mapsto [f \circ g]$ on the set of all equivalence classes $QI(X)$ forms a group, called the quasi-isometry group of X. It is a major problem in geometric group theory, to find the quasi-isometry groups of spaces.

In geometric group theory the main idea is to see how groups can be viewed as geometric objects. To be more precise on which geometric object can act as a group in a 'nice way' so that the interplay between the group and the space reveals the algebraic properties of the group. In this direction one fundamental result is the Švarc-Milnor lemma, which says that if a group G acts properly and co-compactly by isometries on a non-empty proper geodesic metric space (X, d), then G is finitely generated and for all $x \in X$ the map

$$G \longrightarrow X$$

$$g \mapsto g.x$$

is a quasi-isometry (a metric space is proper if all balls of finite radius are compact in the metric topology and an action of a group G on a topological space X is co-compact if the quotient space X/G is compact with respect to the quotient topology).

One of the central theorems in geometric group theory is Gromov's polynomial growth theorem (See [8]), which says that finitely generated groups have polynomial growth if and only if they are virtually nilpotent (i.e if the group has a subgroup of finite index that is nilpotent). Then using this theorem an interesting result can be proved that, if a group G is quasi-isometric to $\mathbb{Z}^n$ then G has a subgroup of finite index which is isomorphic to $\mathbb{Z}^n$.

In Riemannian Geometry, two Riemannian manifolds $(M_1^{m_1}, g_1)$ and $(M_2^{m_2}, g_2)$ are said to be isometric if there exists a diffeomorphism $f : M_1 \rightarrow M_2$ such that $g_2(f_*X, f_*Y) = g_1(X, Y)$ for all $X, Y \in \chi(M_1)$, where $f_* : \chi(M_1) \rightarrow \chi(M_2)$ is the differential of f, where $\chi(M_1)$ and $\chi(M_2)$ are set of all vector fields of M_1 and M_2 respectively. Such a map f is called isometry. This motivates us to define quasi-isometry between two Riemannian manifolds.

Definition 1.1. Let $(M_1^{m_1}, g_1)$ and $(M_2^{m_2}, g_2)$ be two Riemannioan manifolds of respective dimensions m_1 and m_2. Let $\chi(M_1)$ and $\chi(M_2)$ be the set of all vector fields associated to M_1 and M_2 respectively. A diffeomorphism $f : M_1^{m_1} \rightarrow M_2^{m_2}$ is said to be a quasi-isometric embedding between M_1 and M_2 if there exist constants $A \geq 1, B \geq 0$ such that for all $X, Y \in \chi(M_1)$,

$$\frac{1}{A} g_1(X, Y) - B \leq g_2(f_*(X), f_*(Y)) \leq A g_1(X, Y) + B. \tag{1.2}$$

Moreover, if for all $Z \in \chi(M_2)$ there exists $X \in \chi(M_1)$ and a constant $D \geq 0$ such that

$$g_2(Z, f_*(X)) \leq D, \tag{1.3}$$

then f is called quasi-isometry between the manifolds M_1 and M_2.
The two manifolds M_1 and M_2 are called quasi-isometric if there exists such a quasi-isometry f between M_1 and M_2.

The definition given in (1.1) is based on usual metric of the metric space whereas in Definition 1.1 we have considered the Riemannian metric for the inequalities, which is more generalized form than the usual metric d.

In this paper we have introduced the concept of quasi-isometry for almost contact metric manifolds, for $N(k)-$contact metric manifolds and for Sasakian manifolds of same dimensions and established some inequalities between two quasi-isometric metric manifolds for various cases like when the ambient manifold is conformally flat, concircularly flat, etc. We have given an example of a quasi-isometry between two Sasakian manifolds. And finally we find a relationship between the scalar curvature and the quasi-isometric constants for two Riemannian manifolds to be quasi-isometric.

2. Preliminaries

A *contact manifold* M^{2n+1} is a C^∞ manifold together with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. More specifically, $\eta \wedge (d\eta)^n$ is a volume element on M, which is non-zero everywhere on M^{2n+1} so that the manifold M is orientable.

Let M^{2n+1} be a $(2n+1)$ dimensional manifold and let there exist a $(1,1)$ tensor field ϕ, a vector field ξ and a global 1-form η on M such that

$$\phi^2 = -I + \eta \otimes \xi, \tag{2.1}$$
$$\eta(\xi) = 1, \tag{2.2}$$

then we say that M has an *almost contact structure* (ϕ, ξ, η). And the manifold M equipped with this almost contact structure (ϕ, ξ, η) is called an *almost contact manifold* (See [2]).

Here the vector field ξ is called the *characteristic vector field* or *Reeb vector field.*

Proposition 2.1. *[2] For an almost contact structure (ϕ, ξ, η) the following relations hold:*

$$\phi \circ \xi = 0, \tag{2.3}$$
$$\eta \circ \phi = 0, \tag{2.4}$$
$$Rank\phi = 2n. \tag{2.5}$$

Theorem 2.2. *[2] Every almost contact structure (ϕ, ξ, η) on a manifold M^{2n+1} admits a Riemannian metric g satisfying:*

$$\eta(X) = g(X, \xi), \tag{2.6}$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.7}$$

And the metric g is called *compatible* with the almost contact structure (ϕ, ξ, η) and the manifold M^{2n+1} with the almost contact metric structure (ϕ, ξ, η, g) is called an *almost contact metric manifold*.

In 1988, S. Tanno [5] introduced the notion of $k-$nullity distribution on a contact metric manifold which is defined as follows: The $k-$nullity distribution of a Riemannian manifold (M, g) for a real number k is a distribution,

$$N(k) : p \longrightarrow N_p(k) = [Z \in T_pM : R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}], \quad (2.8)$$

for any $X, Y \in T_pM$, where R is the *Riemannian curvature tensor* and T_pM denotes the tangent vector space of M^{2n+1} at point $p \in M$.

If the characteristic vector field of a contact metric manifold belongs to the $k-$nullity distribution, then the relation,

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] \quad (2.9)$$

holds. A contact metric manifold with $\xi \in N(k)$ is called a $N(k)-$*contact metric manifold*.

Proposition 2.3. *[2] Let $M^{2n+1}(\phi, \xi, \eta, g)(n \geq 2)$ be a $N(k)-$contact metric manifold. Then the following relations hold:*

$$Q\xi = (2nk)\xi, \quad (2.10)$$
$$S(X,\xi) = 2nk\eta(X), \quad (2.11)$$
$$\eta(R(X,Y)Z) = k[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)], \quad (2.12)$$

where, R is the Riemannian curvature tensor, S is the Ricci tensor of type $(0,2)$ and Q is the Ricci operator or the symmetric endomorphism of the tangent space T_pM at the point $p \in M$ and is given by $S(X,Y) = g(QX,Y)$.

Next we recall a very important manifold named Sasakian manifold which was introduced by the Japanese mathematician S. Sasaki [6] in the year 1960. Later, the works of Boyer, Galicki [7] and other mathematicians have made a substantial progress in the study of Sasakian manifolds. In mathematical physics Sasakian manifolds and more specifically Sasakian space forms are widely used. Sasakian manifolds or normal contact metric manifolds are an odd-dimensional counterpart of the Kähler manifolds in complex geometry.

An almost contact manifold M^{2n+1} together with the almost contact structure (ϕ, ξ, η) is said to be a *Sasakian manifold* or a *normal contact metric manifold* if

$$[\phi, \phi](X,Y) + 2d\eta(X,Y)\xi = 0,$$

where, $[\phi, \phi]$ is the *Nijenhuis torsion tensor field* of ϕ and is given by,

$$[\phi, \phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi([\phi X, Y]) - \phi([X, \phi Y]).$$

Theorem 2.4. *An almost contact metric manifold M^{2n+1} with the structure (ϕ, ξ, η, g) is Sasakian if and only if*

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

where, ∇ is the Levi-Civita connection on M^{2n+1} (See [2]).

Proposition 2.5. *[2] Let M^{2n+1} be a Sasakian manifold with the structure (ϕ, ξ, η, g), then the following relations are true:*

$$\nabla_X \xi = -\phi X, \tag{2.13}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.14}$$

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.15}$$

$$\eta(R(X, Y)Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z), \tag{2.16}$$

where, R is the Riemannian curvature tensor of M^{2n+1} and is given by,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for all vector fields X,Y,Z on M.

The theorems and results that are stated above will be used frequently in proofs of the next chapters. For a detailed discussion and proofs of these we refer to the text [2].

3. Quasi-isometry between two almost contact metric manifolds

Consider two odd dimensional almost contact metric manifolds M_1 and M_2 with the structure $(\phi_1, \xi_1, \eta_1, g_1)$ and $(\phi_1, \xi_1, \eta_1, g_1)$ respectively. In this section, we study quasi-isometry between two almost contact metric manifolds M_1 and M_2.

As in (1.2), for all $X, Y \in \chi(M_1)$, we have,

$$\frac{1}{A} g_1(X, Y) - B \leq g_2(f_*(X), f_*(Y)) \leq A g_1(X, Y) + B.$$

For $Y = \xi_1$, we get using (2.6),

$$\frac{1}{A} \eta_1(X) - B \leq \quad g_2(f_*(X), f_*(\xi_1)) \quad \leq A\eta_1(X) + B.$$

If the function f_* preserves the structure vector field between the two manifolds M_1 and M_2, that is, if $f_*(\xi_1) = \xi_2$, then

$$g_2(f_*(X), f_*(\xi_1)) = g_2(f_*(X), \xi_2) = \eta_2(f_*(X)),$$

so that,

$$\frac{1}{A} \eta_1(X) - B \leq \eta_2(f_*(X)) \leq A\eta_1(X) + B, \quad \forall X \in \chi(M_1). \tag{3.1}$$

Since the tensor field ϕ is anti-symmetric with respect to the Riemannian metric g, that is,

$$g(\phi X, Y) = -g(X, \phi Y),$$

we have,

$$g_1(\phi_1 X, X) = 0.$$

So, replacing X by $\phi_1 X$, we get from (1.2),

$$\frac{1}{A} g_1(\phi_1 X, Y) - B \leq g_2(f_*(\phi_1 X), f_*(Y)) \leq A g_1(\phi_1 X, Y) + B.$$

Again for $Y = X$,

$$-B \leq g_2(f_*(\phi_1 X), f_*(X)) \leq B, \quad \forall X \in \chi(M_1). \tag{3.2}$$

Now replacing X by $\phi_1 X$ and Y by $\phi_1 Y$ we get from (1.2),

$$\frac{1}{A} g_1(\phi_1 X, \phi_1 Y) - B \leq g_2(f_*(\phi_1 X), f_*(\phi_1 Y)) \leq A g_1(\phi_1 X, \phi_1 Y) + B.$$

The left inequality of the above implies

$$\frac{1}{A} g_1(X, Y) - B \leq g_2(f_*(\phi_1 X), f_*(\phi_1 Y)) + \frac{1}{A} \eta_1(X)\eta_1(Y). \tag{3.3}$$

Similarly, the right inequality gives

$$g_2(f_*(\phi_1 X), f_*(\phi_1 Y)) + A\eta_1(X)\eta_1(Y) \leq A g_1(X, Y) + B. \tag{3.4}$$

Since $A \geq 1$, we have

$$A \geq \frac{1}{A}. \tag{3.5}$$

Using (3.5), from (3.3) and (3.4), we get for all $X, Y \in \chi(M_1)$,

$$\frac{1}{A} g_1(X, Y) - B \leq g_2(f_*(\phi_1 X), f_*(\phi_1 Y)) + \frac{1}{A} \eta_1(X)\eta_1(Y) \leq A g_1(X, Y) + B. \tag{3.6}$$

Replacing X by $\phi_1 X$ and Y by $\phi_1 Y$, the above implies

$$\frac{1}{A} g_1(X, Y) - B \leq g_2(f_*({\phi_1}^2 X), f_*({\phi_1}^2 Y)) + \frac{1}{A} \eta_1(X)\eta_1(Y) \leq A g_1(X, Y) + B.$$

Now using the linearity of the differential f_* and (2.1) and also if f_* preserves the structure vector field between the two manifolds, a simple calculation leads to

$$\frac{1}{A} g_1(X, Y) - B \leq g_2(f_*(X), f_*(Y)) - \eta_1(X) g_2(\xi_2, f_*(Y))$$
$$-\eta_1(Y) g_2(f_*(X), \xi_2) + \eta_1(X)\eta_1(Y)(g_2(\xi_2, \xi_2) + \frac{1}{A}) \leq A g_1(X, Y) + B,$$

which implies

$$\frac{1}{A} g_1(X, Y) - B \leq g_2(f_*(X), f_*(Y)) - \eta_1(X)\eta_2(f_*(Y))$$
$$-\eta_1(Y)\eta_2(f_*(X)) + \eta_1(X)\eta_1(Y)(1 + \frac{1}{A}) \leq A g_1(X, Y) + B. \tag{3.7}$$

So collecting all these results, we can state that:

Theorem 3.1. *Let $M_1(\phi_1, \xi_1, \eta_1, g_1)$ and $M_2(\phi_2, \xi_2, \eta_2, g_2)$ be two odd dimensional almost contact metric manifolds and let $f : M_1 \to M_2$ be a quasi-isometric embedding. Also consider that f_* preserves the structure vector field between the two manifolds. Then for all $X, Y \in \chi(M_1)$, the following relations hold:*

1. $\frac{1}{A}\eta_1(X) - B \le \eta_2(f_*(X)) \le A\eta_1(X) + B$,
2. $-B \le g_2(f_*(\phi_1 X), f_*(X)) \le B$,
3. $\frac{1}{A}g_1(X, Y) - B \le g_2(f_*(\phi_1 X), f_*(\phi_1 Y)) + \frac{1}{A}\eta_1(X)\eta_1(Y) \le Ag_1(X, Y) + B$,
4. $\frac{1}{A}g_1(X, Y) - B \le g_2(f_*(X), f_*(Y)) - \eta_1(X)\eta_2(f_*(Y)) - \eta_1(Y)\eta_2(f_*(X)) + \eta_1(X)\eta_1(Y)(1 + \frac{1}{A}) \le Ag_1(X, Y) + B$.

4. Quasi-isometry between two $N(k)-$contact metric manifolds

In this section we deal with the quasi-isometry between two $N(k)-$ contact metric manifolds in a similar way and establish some interesting results.

Recall that if a transformation does not change the angle between the tangent vectors of a manifold, it is called a conformal transformation. The Weyl conformal curvature tensor C of a Riemannian manifold (M, g) of dimension $2n + 1$ $(n \ge 1)$ is an invariant under any conformal transformation of the metric g and it is defined by

$$\begin{aligned} C(X, Y)Z = R(X, Y)Z - \tfrac{1}{(2n-1)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ -g(X, Z)QY] + \tfrac{r}{2n(2n-1)}[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{4.1}$$

where R is *Riemannian curvature tensor*, S is the *Ricci tensor of type* $(0, 2)$, Q is the *Ricci operator* given by $S(X, Y) = g(QX, Y)$, and r is the *scalar curvature* of the manifold M.

Next, let the manifold M_1 be *conformally flat* i.e; $C_1(X, Y)Z = 0$ for all $X, Y, Z \in \chi(M_1)$. Then from the equation (4.1) we get,

$$\begin{aligned} R_1(X, Y)Z \quad = \tfrac{1}{(2n-1)}[S_1(Y, Z)X - S_1(X, Z)Y + g_1(Y, Z)Q_1X \\ -g_1(X, Z)Q_1Y] - \tfrac{r}{2n(2n-1)}[g_1(Y, Z)X - g_1(X, Z)Y]. \end{aligned} \tag{4.2}$$

Putting $Z = \xi_1$ and using (2.8),(2.9) and the relation $S_1(X, \xi_1) = 2nk\eta_1(X)$, we get after some calculations

$$R_1(X, Y)\xi_1 = \frac{2nk}{r - 2nk}[\eta_1(Y)Q_1X - \eta_1(X)Q_1Y]. \tag{4.3}$$

And for $Y = \xi_1$,

$$Q_1X = (\frac{r - 2nk}{2n})X + [(2n + 1)k - \frac{r}{2n}]\eta_1(X)\xi_1. \tag{4.4}$$

Now since $R_1(X, Y)Z \in \chi(M_1)$, for all $X, Y, Z, W \in \chi(M_1)$, the left side inequality of (1.2) implies

$$\frac{1}{A}g_1(R_1(X, Y)Z, W) - B \le g_2(f_*(R_1(X, Y)Z), f_*(W)). \tag{4.5}$$

Using (4.2), the above inequality becomes

$$\frac{1}{A}[\frac{1}{(2n-1)}\{S_1(Y,Z)g_1(X,W) - S_1(X,Z)g_1(Y,W) + g_1(Y,Z)g_1(Q_1X,W) - g_1(X,Z)g_1(Q_1Y,W)\} - \frac{r}{2n(2n-1)}\{g_1(Y,Z) g_1(X,W) - g_1(X,Z)g_1(Y,W)\}] - B \leq g_2(f_*(R_1(X,Y)Z), f_*(W)). \tag{4.6}$$

For $Z = \xi_1$, the above gives

$$\frac{1}{A}[\frac{1}{(2n-1)}\{2nk\eta_1(Y)g_1(X,W) - 2nk\eta_1(X)g_1(Y,W) +\eta_1(Y)S_1(X,W) - \eta_1(X), S_1(Y,W)\} - \frac{r}{2n(2n-1)}\{\eta_1(Y) g_1(X,W) - \eta_1(X)g_1(Y,W)\}] - B \leq g_2(f_*(R_1(X,Y)\xi_1), f_*(W)). \tag{4.7}$$

Simplifying after some steps and assuming $[\frac{1}{(2n-1)}(2nk - \frac{r}{2n})] = l_1$ and $\frac{1}{(2n-1)} = l_2$ we get

$$\frac{1}{A}[l_1\{\eta_1(Y)g_1(X,W) - \eta_1(X)g_1(Y,W)\} + l_2\{\eta_1(Y)S_1(X,W) -\eta_1(X)S_1(Y,W)\}] - B \leq g_2(f_*(R_1(X,Y)\xi_1), f_*(W)). \tag{4.8}$$

Setting $Y = Z = \xi_1$ in (4.2), it can be shown that conformally flat $N(k)-$ contact metric manifold M_1 becomes η-Einstein manifold, that is,

$$S_1(X,Y) = ag_1(X,Y) + b\eta_1(X)\eta_1(Y),$$

where, $a = [\frac{r}{2n} - k]$ and $b = [(2n+1)k - \frac{r}{2n}]$. Then putting this value of S_1 in (4.8) and after simplification we have

$$\frac{1}{A}[l_1\{\eta_1(Y)g_1(X,W) - \eta_1(X)g_1(Y,W)\} + l_2\{\eta_1(Y)(ag_1(X,W) +b\eta_1(X)\eta_1(W)) - \eta_1(X)(ag_1(Y,W) + b\eta_1(Y)\eta_1(W))\}] -B \leq g_2(f_*(R_1(X,Y)\xi_1), f_*(W)). \tag{4.9}$$

Now, using (2.12) and observing that $(l_1 + al_2) = k$, the above inequality becomes

$$\frac{1}{A}\eta_1(R_1(Y,X)W) - B \leq g_2(f_*(R_1(X,Y)\xi_1), f_*(W)). \tag{4.10}$$

Reminding the linearity of f_* and using the relation (2.9), the last inequality leads to

$$\frac{1}{A}\eta_1(R_1(Y,X)W) - B \leq k[\eta_1(Y)g_2(f_*(X), f_*(W)) - \eta_1(X)g_2(f_*(Y), f_*(W))]. \tag{4.11}$$

Similarly, taking the right side inequality of the (1.2) and proceeding as above we get

$$k[\eta_1(Y)g_2(f_*(X), f_*(W)) - \eta_1(X)g_2(f_*(Y), f_*(W))] \leq A\eta_1(R_1(Y,X)W)] + B. \tag{4.12}$$

So, combining the inequalities (4.11) and (4.12), we can write

THEOREM 4.1. *Let $M_1(\phi_1,\xi_1,\eta_1,g_1)$ and $M_2(\phi_2,\xi_2,\eta_2,g_2)$ be two odd dimensional $N(k)-$contact metric manifolds with* $\dim M_1 = 2n+1$ $(n > 1)$. *Suppose $f : M_1 \to M_2$ be a quasi-isometric embedding with the constants $A \geq 1, B \geq 0$. Furthermore, if* *the manifold M_1 is conformally flat, then for all $X, Y, W \in \chi(M_1)$, the metric g_2 of the manifold M_2 satisfies*

$$\frac{1}{A}\eta_1(R_1(Y,X)W) - B \leq k[\eta_1(Y)g_2(f_*(X), f_*(W)) \\ -\eta_1(X)g_2(f_*(Y), f_*(W))] \leq A\eta_1(R_1(Y,X)W)] + B, \tag{4.13}$$

where R_1 is the Riemannian curvature tensor of the manifold M_1.

Remark 4.2. *Now consider f is a quasi-isometry between M_1 and M_2. Also consider $f_*(X) = Z_1$ and $f_*(Y) = Z_2$. Then there exists some $W \in \chi(M_1)$ such that $g_2(Z_1, f_*(W)) \leq D$ and $g_2(Z_2, f_*(W)) \leq D$, where $D \geq 0$. So from (4.11), we get*

$$\frac{1}{A}\eta_1(R_1(Y,X)W) - B \leq kD\eta_1(Y - X).$$

After a small calculation we can remark that,

$$R_1(Y,X)W \leq A(B\xi_1 + kD(Y - X)).$$

The following corollary can also be demonstrated:

Corollary 4.3. *Let f be a quasi-isometric embedding between two $N(k)-$ contact metric manifolds $M_1(\phi_1, \xi_1, \eta_1, g_1)$ and $M_2(\phi_2, \xi_2, \eta_2, g_2)$ with M_1 conformally flat. If f_* preserves the structure vector field, then for some $A \geq 1$ and $B_1 \geq 0$ we have*

$$-B_1 \leq \eta_1(Y)g_2(f_*(X), \xi_2) - \eta_1(X)g_2(f_*(Y), \xi_2) \leq B_1. \tag{4.14}$$

Proof. The proof of this corollary follows from the equation (4.13) after putting $W = \xi_1$ and using (2.9). □

Next, we consider the manifold M_1 to be *conformally flat Einstein manifold*, then its Ricci tensor S_1 satisfies $S_1(X,Y) = \frac{r}{2n+1}g_1(X,Y)$. Now using this in (4.8) we get

$$\frac{1}{A}[l_1\{\eta_1(Y)g_1(X,W) - \eta_1(X)g_1(Y,W)\} + l_2\frac{r}{2n+1}\{\eta_1(Y)g_1(X,W) \\ -\eta_1(X)g_1(Y,W)\}] - B \leq g_2(f_*(R_1(X,Y)\xi_1), f_*(W)). \tag{4.15}$$

Then after simplification this yields

$$\frac{1}{Ak}(l_1 + \frac{r}{2n+1}l_2)\eta_1(R_1(Y,X)W) - B \leq g_2(f_*(R_1(X,Y)\xi_1), f_*(W)).$$

Now considering $a_1 = \frac{1}{k}(l_1 + \frac{r}{2n+1}l_2) = \frac{1}{k(2n-1)}(2nk - \frac{r}{2n} + \frac{r}{2n+1})$, the above inequality transforms into

$$\frac{a_1}{A}\eta_1((R_1(Y,X)W)) - B \leq g_2(f_*(R_1(X,Y)\xi_1), f_*(W)).$$

Applying the linearity of f_* the last inequality becomes

$$\tfrac{a_1}{A}\eta_1((R_1(Y,X)W)) - B \leq k[\eta_1(Y)g_2(f_*(X), f_*(W)) \\ -\eta_1(X)g_2(f_*(Y), f_*(W))]. \tag{4.16}$$

Again proceeding similarly with the right side inequality, we have

$$k[\eta_1(Y)g_2(f_*(X), f_*(W)) - \eta_1(X)g_2(f_*(Y), f_*(W))] \\ \leq a_1 A\eta_1((R_1(Y, X)W)) + B. \tag{4.17}$$

Hence, from (4.16) and (4.17), we can state the following corollary

Corollary 4.4. *Let f be a quasi-isometric embedding between two $N(k)-$ contact metric manifolds $M_1(\phi_1, \xi_1, \eta_1, g_1)$ and $M_2(\phi_2, \xi_2, \eta_2, g_2)$. Moreover, if the manifold M_1 be conformally flat Einstein manifold, then for some $A \geq 1, B \geq 0$ and for all $X, Y, W \in \chi(M_1)$ the metric g_2 of M_2 satisfies*

$$\tfrac{a_1}{A}\eta_1((R_1(Y, X)W)) - B \leq k[\eta_1(Y)g_2(f_*(X), f_*(W)) \\ -\eta_1(X)g_2(f_*(Y), f_*(W))] \leq a_1 A\eta_1((R_1(Y, X)W)) + B, \tag{4.18}$$

where R_1 is the Riemannian curvature tensor of the manifold M_1 and $a_1 = \frac{1}{k(2n-1)}(2nk - \frac{r}{2n} + \frac{r}{2n+1})$.

The concircular curvature tensor of a manifold (M^{2n+1}, g) is given by

$$\bar{C}(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n+1)}[g(Y, Z)X - g(X, Z)Y].$$

Now, if our ambient manifold M_1 be *concircularly flat* i.e $\bar{C}(X, Y)Z = 0$, then from above we have

$$R_1(X, Y)Z = \frac{r}{2n(2n+1)}[g(Y, Z)X - g(X, Z)Y].$$

Putting this value in the left inequality of (1.2), we get

$$\tfrac{r}{2An(2n+1)}[g_1(Y, Z)g_1(X, W) - g_1(X, Z)g_1(Y, W)] \\ -B \leq g_2(f_*(R_1(X, Y)Z), f_*(W)).$$

Then for $Z = \xi_1$ and using (2.6), (2.12) with the linearity of f_* and simplifying we have

$$\frac{b_1}{A}\eta_1(R_1(Y, X)W) - B \leq k[\eta_1(Y)g_2(f_*(X), f_*(W)) - \eta_1(X)g_2(f_*(Y), f_*(W))], \tag{4.19}$$

with $b_1 = \frac{r}{2nk(2n+1)}$.

Again proceeding similarly as above, from the right side inequality of (1.2), we have

$$k[\eta_1(Y)g_2(f_*(X), f_*(W)) - \eta_1(X)g_2(f_*(Y), f_*(W))] \leq b_1 A\eta_1(R_1(Y, X)W) + B. \tag{4.20}$$

So, combining (4.19) and (4.20) we get

Theorem 4.5. *Let $M_1(\phi_1, \xi_1, \eta_1, g_1)$ and $M_2(\phi_2, \xi_2, \eta_2, g_2)$ be two $N(k)-$ contact metric manifolds with* $\dim M_1 = 2n + 1$ $(n \geq 1)$. *Suppose $f : M_1 \to M_2$ be a quasi-isometric embedding with the constants $A \geq 1, B \geq 0$. Moreover, if M_1 is concircularly flat, then for all $X, Y, W \in \chi(M_1)$ we have*

$$\tfrac{b_1}{A}\eta_1(R_1(Y, X)W) - B \leq k[\eta_1(Y)g_2(f_*(X), f_*(W)) \\ -\eta_1(X)g_2(f_*(Y), f_*(W))] \leq b_1 A\eta_1(R_1(Y, X)W) + B, \tag{4.21}$$

where $b_1 = \frac{r}{2nk(2n+1)}$.

We have the conharmonic curvature tensor for a manifold (M^{2n+1}, g) given by,

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \tfrac{1}{(2n-1)}[S(Y,Z)X - S(X,Z)Y \\ +g(Y,Z)QX - g(X,Z)QY],$$

where Q is the *Ricci operator* and is given by $g(QX,Y) = S(X,Y)$. Now consider the manifold M_1 *conharmonically flat* i.e $\tilde{C}(X,Y)Z = 0$. Then using the value of $R_1(X,Y)Z$ from above, we get from the left side inequality of (1.2),

$$\tfrac{l_2}{A}[S_1(Y,Z)g_1(X,W) - S_1(X,Z)g_1(Y,W) + g_1(Y,Z)g_1(Q_1X,W) \\ -g_1(X,Z)g_1(Q_1Y,W)] - B \leq g_2(f_*(R_1(X,Y)Z), f_*(W)).$$

Then putting $Z = \xi_1$ and using (2.6), $g_1(X,\xi) = 2nk\eta_1(X)$ and $S_1(Q_1X,Y) = S_1(X,Y)$, we get

$$\tfrac{l_2}{A}[2nk\{\eta_1(Y)g_1(X,W) - \eta_1(X)g_1(Y,W)\} + [\eta_1(Y)S_1(X,W) \\ -\eta_1(X)S_1(Y,W)]] - B \leq g_2(f_*(R_1(X,Y)\xi_1), f_*(W)).$$

Moreover if M_1 is an Einstein manifold, then the above inequality becomes

$$\tfrac{l_2}{A}(2nk + \tfrac{r}{2n+1})[\eta_1(Y)g_1(X,W) - \eta_1(X)g_1(Y,W)] \\ -B \leq g_2(f_*(R_1(X,Y)\xi_1), f_*(W)).$$

Finally, using (2.12) and the linearity of f_*, the above yields

$$\frac{c_1}{A}\eta_1(R_1(Y,X)W) - B \leq k[\eta_1(Y)g_2(f_*(X), f_*(W)) - \eta_1(X)g_2(f_*(Y), f_*(W))]. \quad (4.22)$$

where, $c_1 = \frac{l_2}{k}(2nk + \frac{r}{2n+1})$.
Similarly the right inequality of (1.2) gives

$$k[\eta_1(Y)g_2(f_*(X), f_*(W)) - \eta_1(X)g_2(f_*(Y), f_*(W))] \leq c_1A\eta_1(R_1(Y,X)W) + B. \quad (4.23)$$

Therefore from the inequalities (4.22) and (4.23) we can state

Theorem 4.6. *Let $M_1(\phi_1, \xi_1, \eta_1, g_1)$ and $M_2(\phi_2, \xi_2, \eta_2, g_2)$ be two $N(k)-$ contact metric manifolds with* $\dim M_1 = 2n+1$ $(n \geq 1)$*. Suppose $f : M_1 \to M_2$ be the quasi-isometric embedding. Furthermore, if the manifold M_1 is conharmonically flat Einstein manifold, then for all $X, Y, W \in \chi(M_1)$, the metric g_2 of the manifold M_2 satisfies:*

$$\frac{c_1}{A}\eta_1(R_1(Y,X)W) - B \leq k[\eta_1(Y)g_2(f_*(X), f_*(W)) \\ -\eta_1(X)g_2(f_*(Y), f_*(W))] \leq c_1A\eta_1(R_1(Y,X)W) + B, \quad (4.24)$$

where, $c_1 = \frac{l_2}{k}(2nk + \frac{r}{2n+1}) = \frac{4n}{2n-1}$, since we have $k = \frac{r}{2n(2n-1)}$ for $N(k)-$ contact Einstein manifolds.

Recall that the Weyl projective curvature tensor P of type $(1,3)$ on a Riemannian manifold (M^{2n+1}, g) can be defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y].$$

In a similar calculation, if M_1 is *projectively flat* $N(k)-$contact Einstein manifold, i.e. if $P_1 = 0$ and $S_1(X,Y) = \frac{r}{2n+1} g_1(X,Y)$, then we write

THEOREM 4.7. *Let $M_1(\phi_1, \xi_1, \eta_1, g_1)$ and $M_2(\phi_2, \xi_2, \eta_2, g_2)$ be two quasi- isometrically embedded $N(k)-$contact metric manifolds with* $\dim M_1 = 2n+1)$ $(n \geq 1)$. *Suppose $f M_1 \to M_2$ be such embedding between M_1 and M_2 with the constants $A \geq 1, B \geq 0$. Furthermore, if the manifold M_1 is projectively flat Einstein manifold, then we have*

$$\begin{aligned} &\frac{1}{A}\eta_1(R_1(Y,X)W) - B \leq k[\eta_1(Y)g_2(f_*(X), f_*(W)) \\ &-\eta_1(X)g_2(f_*(Y), f_*(W))] \leq A\eta_1(R_1(Y,X)W) + B. \end{aligned} \tag{4.25}$$

5. Quasi-isometry between two Sasakian manifolds

Some basic introductory details about the Sasakian manifold is given in the preliminary section. Now we recall an important theorem to establish the rest of the results.

THEOREM 5.1. *[2] A $N(k)-$contact metric manifold is Sasakian if and only if $k = 1$.*

Using this theorem we can imply the following result from the previous results for $N(k)-$contact metric manifold.

THEOREM 5.2. *Let $M_1(\phi_1, \xi_1, \eta_1, g_1)$ and $M_2(\phi_2, \xi_2, \eta_2, g_2)$ be two Sasakian manifolds with dimension of $M_1 = 2n+1$ $(n \geq 1)$. Let $f : M_1 \to M_2$ be a quasi-isometry embedding between M_1 and M_2 with constants $A \geq 1, B \geq 0$. Then the following inequalities hold in the respective following cases:*

1. *If M_1 is conformally flat or conformally flat Einstein or concircularly flat or projectively flat manifold, then*

$$\begin{aligned} &\frac{1}{A}\eta_1(R_1(Y,X)W) - B \leq \eta_1(Y)g_2(f_*(X), f_*(W)) \\ &-\eta_1(X)g_2(f_*(Y), f_*(W)) \leq A\eta_1(R_1(Y,X)W) + B. \end{aligned} \tag{5.1}$$

2. *If M_1 is conharmonically flat Einstein manifold, then*

$$\begin{aligned} &\frac{c_1}{A}\eta_1(R_1(Y,X)W) - B \leq \eta_1(Y)g_2(f_*(X), f_*(W)) \\ &-\eta_1(X)g_2(f_*(Y), f_*(W)) \leq c_1 A\eta_1(R_1(Y,X)W) + B, \end{aligned} \tag{5.2}$$

   where $c_1 = \frac{4n}{2n-1}$.

Example: Consider $M_1 = \mathbb{R}^3$ with the Euclidean metric g_1. Let $\alpha = \frac{1}{2}(dz - ydx)$, $\xi = \frac{\partial}{\partial z}$ and $g_1 = \alpha \otimes \alpha + \frac{1}{4}(dx^2 + dy^2)$. Then we take $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$ and $e_3 = \frac{\partial}{\partial z}$ as a set of linearly independent basis vectors for the set of vector fields $\chi(M_1)$ of the manifold M_1. Also consider the $(1,1)$ tensor field ϕ be given as, $\phi_1(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$, $\phi_1(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial x}$ and $\phi_1(\frac{\partial}{\partial z}) = 0$. Then it can be easily checked that the manifold (M_1, g_1) with the above defined structure is a Sasakian manifold.

Take another manifold $M_2 = \{(x, y, z) \in \mathbb{R}^3 : 1 < y < 2, z \neq 0\}$, where (x, y, z) are the standard co-ordinates of $\mathbb{R}^3$. Then the linearly independent vector fields are given by $f_1 = \frac{\partial}{\partial y}$, $f_2 = z^2(\frac{\partial}{\partial z} + 2y\frac{\partial}{\partial x})$ and $f_3 = \frac{\partial}{\partial x}$. Let g_2 be the Riemannian metric defined by: $g_{ij} = 1$ for $i = j$ and $g_{ij} = 0$ for $i \neq j$. Let ϕ be the $(1,1)$ tensor field defined by; $\phi_2(f_1) = f_3$, $\phi_2(f_2) = 0$ and $\phi_2(f_3) = -f_1$. Thus for taking $\xi = f_2$, we can show that the manifold (M_2, g_2) with this structure is a Sasakian manifold.

Now we define a map $f_* : \chi(M_1) \longrightarrow \chi(M_2)$ on the basis vector fields by,

$$f_*(e_1) = \frac{1}{2}(yf_3 + \frac{1}{\sqrt{y}}f_1), \;\; f_*(e_2) = \frac{1}{2}f_2, \;\; f_*(e_3) = -\frac{1}{2}f_3.$$

The f is a quasi-isometry between the two Sasakian manifolds M_1 and M_2 with the constants $A = 2$ and $B = 1$.

6. Quasi-isometric inequality between two Riemannian manifolds

We will conclude this article with the following result. This theorem concerns between any two Riemannian manifold which have a quasi-isometric structure among them.

Theorem 6.1. *Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of same dimension n and let f be the quasi-isometric embedding between them with some constants $A \geq 1$ and $B \geq 0$. Then*

$$\frac{r_1}{A} - n^2B \leq g_2(f_*(R_1(e_i, e_j)e_j), f_*(e_i)) \leq Ar_1 + n^2B,$$

r_1 being the scalar curvature of the manifold M_1.

Proof. For all X, Y, Z and W in $\chi(M_1)$, $R_1(X, Y)Z$ is also in $\chi(M_1)$ and for f being the quasi-isometry between M_1 and M_2, we get

$$\begin{aligned} \frac{1}{A}g_1(R_1(X, Y)Z, W) - B \leq & \quad g_2(f_*(R_1(X, Y)Z), f_*(W)) \\ \leq & \quad Ag_1(R_1(X, Y)Z, W) + B. \end{aligned} \tag{6.1}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space $T_p(M_1)$ at $p \in M_1$. Then for $X = W = e_i$, we get from the left inequality of (6.1),

$$\frac{1}{A}S_1(Y, Z) - nB \leq g_2(f_*(R_1(e_i, Y)Z), f_*(e_i)).$$

Again putting $Y = Z = e_j$ we get

$$\frac{1}{A}r - n^2B \leq g_2(f_*(R_1(e_i, e_j)e_j), f_*(e_i)). \tag{6.2}$$

Similarly, right inequality gives

$$g_2(f_*(R_1(e_i, e_j)e_j), f_*(e_i)) \leq Ar_1 + n^2B. \tag{6.3}$$

Finally (6.2) and (6.3) together complete the proof. □

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