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GENERALIZED FRACTIONAL DERIVATIVE OF MODIFIED BESSEL FUNCTION

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Abstract

In the present paper,the authors have applied the classical left and right hand sided Riemann Liouville fractional derivative operators of order α on modified Bessel function of order p. The results are expressed in terms of generalized wright functions and generalized hypergeometric functions. The compositions of fractional differential operators with cosine and sine functions are also derived in terms of generalized wright functions and generalized hypergeometric functions . The special cases of all our mains results are studied.

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1. Introduction

The fractional calculus, now a days, is one of the most rapidly growing subjects of mathematical analysis. The fractional integral operators, involving various Special functions, have found significant importance and applications in various sub field of applicable mathematical analysis. In the last three decades, a number of workers like Love [12], Mc Bride [13], Kalla [14, 15], Kalla and Saxena [16], Saigo [17, 18], and Kilbas [19], have studied the properties, applcations and different extensions of various operators of fractional calculus on a number classical & non classical Special functions & polynomials. A sufficient account of fractional calculus operators along with their properties and applications can be found in the research monographs by Miller and Ross [21] and Kiryakova [20]. In fractional calculus,the fractional derivatives are defined via fractional integrals. The applications of fractional calculus are also seen in various fields, including turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermal nuclear fusion, non-linear control theory, image processing, nonlinear biological system,astrophysics etc.. The Bessel functions are playing important role in wide range of problems of mathematical physics like problems of stochastic, radio physics, hydro dynamics, atomic & nuclear physics, which led to various research workers,who are working in the field of Special functions to explore various extensions and applications of these Bessel functions. In the recent past, various generalizations, extensions of Bessel functions have been given by many researchers [7–9], who are working in the field of Special functions and their applications. Bessel functions of the first kind of order v, defined by Watson [10] occurs frequently in the problems like electrical engineering, finite elasticity, wave mechanics, mathematical physics and chemistry, whereas the product of Bessel and modified Bessel functions of the first kind appear frequently in the problems of statistical mechanics and plasma physics [11, 23]. Very recently various researchers [10, 11] are playing fractional calculus to the Bessel functions of the first kind and second kind and also on modified Bessel functions [22]. The generalized Bessel function of the first kind $\omega_p(x)$ is defined by Saigo [2], by the following series:

$$W_p(Z) = \sum_{k=0}^{\infty}(-1)^k \frac{c^k}{\Gamma\left(P + \frac{(b+1)}{z} + k\right) K!}(\frac{z}{2})^{2k+1} \quad ; z \in C, \tag{1.1}$$

$Re(p) > -1, \quad (b, c, p) \in C$, where c denotes the set of complex numbers and Γa is a the familiar Gamma function.
Putting $b = 1$ and $c = 1$ in (1.1), we obtain the modified Bessel function of order p denoted by $I_p(z)$ as [4]:

$$I_p(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(P + 1 + k) K!}(\frac{z}{2})^{2k+1} \quad ; z \in C, \tag{1.2}$$

Motivated by the above recent applications of fractional calculus on various classical Special functions and polynomials and also by the works of various researchers [7–9], the authors in the present note have applied fractional differential operators on modified Bessel function $I_p(z)$, defined by (1.2) above.

We have also expressed our main results in terms of generalized wright function and generalized hypergeometric function [7], which are defined in the next preliminaries section. The generalized fractional calculus operators defined by Kilbas [5], are given by the following equations:

$$(I_{0^+}^{\alpha,\beta,\eta} f) = \frac{x^{-\alpha-\beta}}{\Gamma\alpha} \int_0^x (x-t)^{\alpha-1} {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}) f(t)dt, \quad (Re(\alpha) > 0), \tag{1.3}$$

$$(I_{0^-}^{\alpha,\beta,\eta} f) = \frac{x^{-\alpha-\beta}}{\Gamma\alpha} \int_0^x (t-x)^{\alpha-1} {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}) f(t)dt, \quad (Re(\alpha) > 0), \tag{1.4}$$

$$\begin{aligned}(D_{0^+}^{\alpha,\beta,\eta} f) &= (I_{0^+}^{-\alpha-\beta,\alpha+\eta} f)^{-}(x) \\ &= \left(\frac{d}{dx}\right)^n (I_{0^+}^{-\alpha-\eta,-\beta-\eta,-\alpha+\eta} f)^{-}(x),\end{aligned} \tag{1.5}$$

$$\begin{aligned}(D_{0^-}^{\alpha,\beta,\eta} f) &= (I_{0^-}^{-\alpha-\beta,\alpha+\eta} f)^{-}(x) \\ &= \left(\frac{-d}{dx}\right)^n (I_{0^-}^{-\alpha-\eta,-\beta-\eta,-\alpha+\eta} f)^{-}(x),\end{aligned} \tag{1.6}$$

where $Re(\alpha)$ denote real part of α, $\Gamma\alpha$ is the Eular gamma function [1]. It is note that:

$$(D_{0^+}^{\alpha,\beta,\eta} f) = (I_{0^+}^{\alpha,\beta,\eta})^{-1} \tag{1.7}$$

$$(D_{0^-}^{\alpha,\beta,\eta} f) = (I_{0^-}^{\alpha,\beta,\eta})^{-1} \tag{1.8}$$

when β=-α, operators (1.5) and (1.6) coincide with the classical left and right hand sided Riemann Liouville differentiation operator of order $\alpha \in$ C, $Re(\alpha) \geq 0$ and we obtain:

$$(D_{0^+}^{\alpha,-\alpha,\eta} f)(x) = (D_{0^+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{\alpha-\eta+1}} \tag{1.9}$$

$$(D_{0^+}^{\alpha,0,\eta} f)(x) = (D_{0^+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^n ((I_{0^-}^{-\alpha+\eta,-\alpha,\alpha+\eta-n} f)(x), x > 0 \tag{1.10}$$

$$(D_{0^-}^{\alpha,-\alpha,\eta} f)(x) = (D_{0^-}^{\alpha} f)(x) = -\left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{\alpha-\eta+1}} \tag{1.11}$$

$$\begin{aligned}(D_{0^+}^{\alpha,0,\eta} f)(x) &= (D_{0^+}^{\alpha} f)(x) \\ &= -\left(\frac{d}{dx}\right)^n ((I_{0^-}^{-\alpha+\eta,-\alpha,\alpha+\eta+n} f)(x), x > 0\end{aligned} \tag{1.12}$$

These operators are called as Erdelye -Kober fractional integration operators [5], given as:

$$(D_{\alpha,\eta}^{0^+} f)(x) = x^{\alpha-\eta}(\frac{d}{dx})^n \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f(t)dt}{(t-x)^{\alpha-\eta+1}} \tag{1.13}$$

The present paper is devoted to investigation of composition of generalized fractional differentiation operators (1.5) and (1.6) on modified Bessel function $I_p(z)$, defined in (1.2).

2. Preliminaries

2.1. Lemma (I) [5]

$$(I_{0^+}^{\alpha,\beta,\eta} t^{\sigma-1} f)(x) = \frac{\Gamma\sigma\Gamma\sigma-\beta+\eta}{\Gamma\sigma-\beta\Gamma\sigma+\alpha+\eta} x^{\sigma-\beta-1}, \tag{2.1}$$

where $\alpha, \beta, \eta \in C$ be such that $Re(\alpha) > 0$, $Re(\alpha) > Max[0, Re(\beta-\eta)]$

2.2. Lemma (II) [5]

$$(I_{0^-}^{\alpha,\beta,\eta} t^{\sigma-1} f)(x) = \frac{\Gamma 1-\sigma+\beta\Gamma 1-\sigma+\eta}{\Gamma 1-\sigma\Gamma 1-\sigma+\alpha+\eta+\beta} x^{\sigma-\beta-1}, \tag{2.2}$$

where $\alpha, \beta, \eta \in C$ be such that $Re(\alpha) > 0$, $Re(\alpha) > Max[0, Re(\beta-\eta)]$

2.3. Lemma (III) [5]

$$(D_{0^+}^{\alpha,\beta,\eta} t^{\sigma-1} f)(x) = \frac{\Gamma\sigma\Gamma\sigma+\alpha+\beta+\eta}{\Gamma\sigma+\beta\Gamma\sigma+\eta} x^{\sigma+\beta+1}, \tag{2.3}$$

where $\alpha, \beta, \eta \in C$ be such that $Re(\alpha) > 0$, $Re(\alpha) > -Min[0, Re(\alpha+\beta+\eta)]$
In particular for $x > 0$

$$(D_{0^+}^{\alpha} t^{\sigma-1} f)(x) = \frac{\Gamma\sigma}{\Gamma\sigma-\alpha} x^{\sigma-\alpha+1} \tag{2.4}$$

$$(D_{\eta,\alpha}^{+} t^{\sigma-1} f)(x) = \frac{\Gamma\sigma+\alpha+\eta}{\Gamma\sigma+\eta} x^{\sigma-1} \tag{2.5}$$

2.4. Lemma (IV) [5]

$$(D_{0^-}^{\alpha,\beta,\eta} t^{\sigma-1} f)(x) = \frac{\Gamma 1-\sigma-\beta\Gamma 1-\sigma-\beta}{\Gamma 1-\sigma\Gamma 1-\sigma+\eta-\beta} x^{\sigma+\beta-1} \tag{2.6}$$

In particular, for $x > 0$

$$(D_{0^-}^{\alpha} t^{\sigma-1} f)(x) = \frac{\Gamma 1-\sigma+\alpha}{\Gamma 1-\sigma} x^{\sigma-\alpha-1} \tag{2.7}$$

$$(D_{\eta,\alpha}^{-} t^{\sigma-1} f)(x) = \frac{\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma+\eta)} x^{\sigma-1} \tag{2.8}$$

2.5. Generalized Wright function${}_p\psi_q(z)$ is defined as [3] :

$$\begin{aligned} {}_p\psi_q(z) &=_p \psi_q \left[\begin{pmatrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{pmatrix} | z \right] \\ &= \sum_{k=0}^{\infty} \frac{\phi_{i=1}^{p} \Gamma(a_i+\alpha_i) z^k}{\phi_{i=1}^{q} \Gamma(b_j+\beta_j) k!} \end{aligned} \tag{2.9}$$

2.6. Generalized hypergeometric function ${}_pF_q$ is defined as [3] :

$${}_pF_q(a_1, a_2, ...a_p, b_1, b_2, ...bq) = \sum_{k=0}^{\infty} \frac{(a_1)k....(a_p)kz^k}{(b_1)k....(b_q)kk!} \tag{2.10}$$

2.7.

$$(z)_{2k} = z^{2k} (\frac{z}{2})_k (\frac{z+1}{2})_k \tag{2.11}$$

2.8.

$$\Gamma z + k = \Gamma z (z)_k \tag{2.12}$$

2.9. The series expansions of $CosZ$:

$$CosZ = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} ...... = \sum_{k=0}^{\infty} \frac{(-1)^n}{2n!} Z^{2n} \tag{2.13}$$

2.10. The series expansions of $SinZ$:

$$SinZ = z - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!}...... = \sum_{k=0}^{\infty} \frac{(-1)^n}{2n!} Z^{2n} \tag{2.14}$$

3. Repersentation in terms of generalized Wright functions

THEOREM 3.1. *The following fractional derivative holds for $I_p(z)$:*

$$(D_{0^+}^{\alpha,\beta,\eta} t^{\sigma-1} I_p(\lambda t))(x) = (\frac{\lambda}{2})^p x^{p+\sigma-\beta-1} {}_2\psi_3 \left[\begin{pmatrix} (\sigma+p,2),(\sigma+p+\alpha+\eta+\beta,2) \\ (\sigma+p+\beta,2),(\sigma+p+\eta,2),(p+1,1) \end{pmatrix} |(\frac{\lambda x}{2})^2 \right], \tag{3.1}$$

where α, β, η, $p, \lambda \in C$ be such that $Re(\alpha) \geq 0$, $Re(p) > -1$, $Re(\sigma + p) > -Min[0, Re(\alpha + \beta + \eta)]$

Proof: Applying (1.2) & (1.9) on L.H.S of (3.1), we obtain:

L.H.S. of (3.1):

$$= \sum_{k=0}^{\infty} \frac{1}{\Gamma(P+1+k)\, K!} (\frac{z}{2})^{2k+1} \quad (D_{0^+}^{\alpha,\beta,\eta} t^{p+\alpha+2k-1})(x) \tag{3.2}$$

Replacing σ by $(\sigma + p + 2k)$ in (2.3) and using (3.2), we obtain

$$= X^{\sigma+p+\beta-1} \sum_{k=0}^{\infty} \frac{\Gamma(p+\sigma+2k)\Gamma p+\sigma+\alpha+\beta+\eta+2k}{\Gamma(\sigma+p+\beta+2k)\Gamma(\sigma+\eta+p+2k)\Gamma p+k+1} \frac{((\lambda x)^2)^k}{4^k k!} \tag{3.3}$$

Using (2.9) on L.H.S. of above equation, we obtain

$$= (\frac{\lambda}{2})^p x^{p+\sigma-\beta-1} {}_2\psi_3 \left[\begin{pmatrix} (\sigma+p,2),(\sigma+p+\alpha+\eta+\beta,2) \\ (\sigma+p+\beta,2),(\sigma+p+\eta,2),(p+1,1) \end{pmatrix} |(\frac{\lambda x}{2})^2 \right] \tag{3.4}$$

=R.H.S. of (3.1), i.e desired result (3.1).

COROLLARY 3.2. *substituting β=-α in equation (3.1), we obtain the following result in terms of ${}_1\psi_2$:*

$$(D_{0^+}^{\alpha,\beta,\eta} t^{\sigma-1} I_p(\lambda t))(x) = (\frac{\lambda}{2})^p x^{p+\sigma+\alpha-1} {}_1\psi_2 \left[\begin{pmatrix} (\sigma+p,2), \\ (\sigma+p-\alpha,2),,(p+1,1) \end{pmatrix} |(\frac{\lambda x}{2})^2 \right] \tag{3.5}$$

COROLLARY 3.3. *substituting β=0 in (3.1),we obtain the following result in terms of ${}_1\psi_2$:*

$$(D_{0^+}^{\alpha,\beta,\eta}t^{\sigma-1}I_p(\lambda t))(x) = (\frac{\lambda}{2})^p x^{p+\sigma-1}{}_1\psi_2\left[\begin{pmatrix}(\sigma+p+\alpha+\eta,2),\\(\sigma+p+\eta,2),,(p+1,1)\end{pmatrix}|(\frac{\lambda x}{2})^2\right] \quad (3.6)$$

THEOREM 3.4. *The following fractional derivative holds for $I_p(\frac{\lambda}{t})$:*

$$(D_{0^-}^{\alpha,\beta,\eta}t^{\sigma-1}I_p(\frac{\lambda}{t}))(x) = (\frac{\lambda}{2})^p x^{p+\sigma-\beta-1}{}_2\psi_3\left[\begin{pmatrix}(1-\sigma+p-\beta,2),(1-\sigma+p+\alpha+\eta,2)\\(1-\sigma+p,2),(1-\sigma+p+\eta-\beta,2),(p+1,1)\end{pmatrix}|(\frac{\lambda}{2x})^2\right], \quad (3.7)$$

where α, β, η, $p,\lambda \in C$ be such that $Re(\alpha) \geq 0$, $Re(p) > -1$, $Re(\sigma-p) < 1 + Min[0, Re(-\beta-\eta), Re(\alpha+\eta)]$

Proof: Applying (1.2) & (1.11) on L.H.S of (3.7), we obtain:

L.H.S. of (3.7):

$$= \sum_{k=0}^{\infty}\frac{1}{\Gamma(P+1+k)K!}(\frac{z}{2})^{2k+1} \quad (D_{0^-}^{\alpha,\beta,\eta}t^{\sigma-p-2k-1})(x) \quad (3.8)$$

Replacing σ by $(\sigma-P-2k)$ in (2.6) and using (3.8), we obtain

$$= X^{\sigma-p+\beta-1}\sum_{k=0}^{\infty}\frac{\Gamma(1-\sigma+p-\beta+2k)\Gamma p+1-\sigma+\alpha+\eta+2k}{\Gamma(1-\sigma+p+\beta+2k)\Gamma(1-\sigma+\eta-\beta+p+2k)\Gamma p+k+1}\frac{((\lambda)^2)^k}{4x^k k!} \quad (3.9)$$

Using (2.9) on L.H.S. of above equation, we obtain

$$= (\frac{\lambda}{2})^p x^{p+\sigma-\beta-1}{}_2\psi_3\left[\begin{pmatrix}(1-\sigma+p-\beta,2),(1-\sigma+p+\alpha+\eta,2)\\(1-\sigma+p,2),(1-\sigma+p+\eta-\beta,2),(p+1,1)\end{pmatrix}|(\frac{\lambda}{2x})^2\right] \quad (3.10)$$

=R.H.S. of (3.7), i.e desired result (3.7).

COROLLARY 3.5. *Substituting $\beta = -\alpha$ in (3.7), we obtain the following result in terms of ${}_1\psi_2$:*

$$(D_{-}^{\alpha}t^{\sigma-1}I_p(\frac{\lambda}{t}))(x) = (\frac{\lambda}{t})^p x^{\sigma-p-\alpha-1}{}_1\psi_2\left[\begin{pmatrix}(1-\sigma+p+\alpha,2),\\(1-\sigma+p,2),(p+1,1)\end{pmatrix};(\frac{\lambda}{2x})^2\right] \quad (3.11)$$

COROLLARY 3.6. *Substituting $\beta = 0$ in (3.7), we obtain the following result in terms of ${}_1\psi_2$:*

$$(D_{-}^{\alpha}t^{\sigma-1}I_p(\frac{\lambda}{t}))(x) = (\frac{\lambda}{t})^p x^{\sigma-p-\alpha-1}{}_1\psi_2\left[\begin{pmatrix}(1+\alpha-\sigma+p+\eta,2),\\(1-\sigma+p+\eta,2),(p+1,1)\end{pmatrix};(\frac{\lambda}{2x})^2\right] \quad (3.12)$$

4. Repersentation in terms of generalized hypergeometric functions

Theorem 4.1. *The following fractional derivative holds for $I_p(\lambda t)(x)$:*

$$\begin{aligned}
&(D_{0^+}^{\alpha,\beta,\eta} t^{\sigma-1} I_p(\lambda t))(x) \\
&= \left(\frac{\lambda}{2}\right)^p x^{\sigma+p+\beta-1} \frac{\Gamma(\sigma+p)\Gamma(\sigma+p+\alpha+\eta+\beta)}{\Gamma(\sigma+p+\beta)\Gamma(\sigma+p+\eta)\Gamma(p+1)} {}_4F_5 \\
&\left[\begin{pmatrix} \frac{\sigma+p}{2}, \frac{1+\sigma+p}{2}, \frac{\sigma+p+\beta+\alpha+\eta}{2}, \frac{1+\sigma+p+\alpha+\eta+\beta}{2}, \\ p+1, \frac{\beta+\sigma+p}{2}, \frac{1+\beta+\sigma+p}{2} \frac{\sigma+p+\eta}{2}, \frac{1+\sigma+p+\eta}{2} \end{pmatrix} \left(\frac{\lambda x}{2}\right)^2\right],
\end{aligned} \tag{4.1}$$

where, $Re(\sigma+p) > 0, Re(\alpha+\beta+\eta+\sigma) > 0, Re(p+\beta+\sigma) > 0, Re(p+\eta+\sigma) > 0.$

Proof: Applying (2.3) & (2.12) on L.H.S of (4.1), we obtain:

L.H.S. of (4.1):

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{1}{\Gamma(P+1+k)\,K!} (\frac{\lambda}{2})^{2k+p} X^{\sigma+2k+\beta+1} \frac{\Gamma(p+\sigma)\Gamma p+\sigma+\alpha+\beta+\eta}{\Gamma(\sigma+p+\beta)\Gamma(\sigma+\eta+p)} \\
&\frac{(p+\sigma)_{2k}(p+\sigma+\alpha+\beta+\eta)_{2k}}{(\sigma+p+\beta)_{2k}(\sigma+\eta+p)_{2k}}
\end{aligned} \tag{4.2}$$

Using (2.11) on L.H.S. of above equation, we obtain

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{1}{\Gamma(P+1+k)\,K!} (\frac{\lambda}{2})^{2k+p} X^{\sigma+2k+\beta+1} \frac{\Gamma(p+\sigma)\Gamma p+\sigma+\alpha+\beta+\eta}{\Gamma(\sigma+p+\beta)\Gamma(\sigma+\eta+p)\Gamma p+1} \\
&\frac{(\frac{p+\sigma}{2})_k(\frac{p+\sigma+1}{2})_k(\frac{p+\sigma+\alpha+\beta+\eta}{2})_k(\frac{p+\sigma+\alpha+\beta+\eta+1}{2})_k}{(\frac{\sigma+p+\beta}{2})_k(\frac{\sigma+p+\beta+1}{2})_k(\frac{\sigma+\eta+p}{2})_k(\frac{\sigma+\eta+p+1}{2})_k(p+1)_k k!}
\end{aligned} \tag{4.3}$$

Using (2.10) on L.H.S. of above equation, we obtain

$$\begin{aligned}
= \quad &\left(\frac{\lambda}{2}\right)^p x^{p+\sigma+\beta-1} \frac{\Gamma(\sigma+p)\Gamma(\sigma+p+\alpha+\beta+\eta)}{\Gamma(\sigma+p+\beta)\Gamma(\sigma+p+\eta)\Gamma(p+1)} {}_4F_5 \\
&\left[\begin{pmatrix} \frac{\sigma+p}{2}, \frac{\sigma+p+1}{2}, \frac{\sigma+p+\alpha+\eta+\beta}{2}, \frac{\sigma+\beta+p+\eta+\alpha+1}{2}, \\ p+1, \frac{\sigma+p+\beta}{2}, \frac{\sigma+p+\beta+1}{2}, \frac{\sigma+p+\eta}{2}, \frac{\sigma+p+\eta+1}{2} \end{pmatrix} \left(\frac{\lambda x}{2}\right)^2\right]
\end{aligned} \tag{4.4}$$

=R.H.S. of (4.1), i.e desired result (4.1)

Corollary 4.2. *Substituting $\beta = -\alpha$ in (4.1), we obtain the following result in terms of ${}_2F_3$:*

$$(D_{0^+}^{\alpha,\beta,\eta} t^{\sigma-1} I_p(\lambda t))(x) = \left(\frac{\lambda}{2}\right)^p x^{p+\sigma-\alpha-1} \frac{\Gamma(\sigma+p)}{\Gamma(\sigma+p-\alpha)\Gamma(p+1)} {}_2F_3 \left[\begin{pmatrix} \frac{\sigma+p}{2}, \frac{\sigma+p+1}{2}, \\ p+1, \frac{\sigma+p-\alpha}{2}, \frac{\sigma+p-\alpha+1}{2} \end{pmatrix} \left(\frac{\lambda x}{2}\right)^2\right] \tag{4.5}$$

Corollary 4.3. *Substituting $\beta = 0$ in (4.1), we obtain the following result in terms of ${}_2F_3$:*

$$(D_{0^+}^{\alpha,\beta,\eta} t^{\sigma-1} I_p(\lambda t))(x) = \left(\frac{\lambda}{2}\right)^p x^{p+\sigma-1} \frac{\Gamma(\sigma+p+\alpha+\eta)}{\Gamma(\sigma+p+\eta)\Gamma(p+1)} {}_2F_3 \left[\begin{pmatrix} \frac{\sigma+p+\alpha+\eta}{2}, \frac{\sigma+p+\alpha+\eta+1}{2}, \\ p+1, \frac{\sigma+p+\eta}{2}, \frac{\sigma+p+\eta+1}{2} \end{pmatrix} \left(\frac{\lambda x}{2}\right)^2\right] \tag{4.6}$$

Theorem 4.4. *The following fractional derivative holds for $I_p(\frac{\lambda}{t})(x)$:*

$$(D_{0^-}^{\alpha,\beta,\eta} t^{\sigma-1} I_p(\frac{\lambda}{t})(x) = \left(\frac{\lambda}{2}\right)^p x^{\sigma-p+\beta-1} \frac{\Gamma(1-\sigma+p-\beta)\Gamma(1-\sigma+p+\alpha+\eta)}{\Gamma(1-\sigma+p)\Gamma(1-\sigma+p+\eta-\beta)\Gamma(p+1)} {}_4F_5 \left[\begin{pmatrix} \frac{1-\sigma+p-\beta}{2}, \frac{2-\sigma+p-\beta}{2}, \frac{1-\sigma+p+\alpha+\eta}{2}, \frac{2-\sigma+\beta+p+\eta}{2}, \\ p+1, \frac{1-\sigma+p}{2}, \frac{2-\sigma+p}{2} \frac{1-\sigma+p+\eta-\beta}{2}, \frac{2-\sigma+p+\eta-\beta}{2} \end{pmatrix} \left(\frac{\lambda}{2x}\right)^2\right], \tag{4.7}$$

where, $Re(\sigma+p) > 0, Re(\alpha+\beta+\eta+\sigma) > 0, Re(p+\beta+\sigma) > 0, Re(p+\eta+\sigma) > 0$.

Proof: Applying (2.6) & (2.12) on L.H.S of (4.7), we obtain :

L.H.S. of (4.7):

$$= (\frac{\lambda}{2})^p X^{\sigma-p+\beta-1} \frac{\Gamma(p+1-\sigma-\beta)\Gamma p+1-\sigma+\alpha+\eta}{\Gamma(1-\sigma+p)\Gamma(1-\sigma+\eta+p-\beta)\Gamma p+1} \frac{(1+p-\sigma-\beta)_{2k}(p+1-\sigma+\alpha+\eta)_{2k}}{(1-\sigma+p)_{2k}(1-\sigma+\eta+p-\beta)_{2k}(p+1)_{2k}} \tag{4.8}$$

Using (2.11) on L.H.S. of above equation, we obtain

$$= (\frac{\lambda}{2})^p \frac{\Gamma(p+1-\sigma-\beta)\Gamma p+1-\sigma+\alpha+\eta}{\Gamma(1-\sigma+p)\Gamma(1-\sigma+\eta+p-\beta)\Gamma p+1} \left[\frac{(\frac{p+1-\sigma-\beta}{2})_{2k}(\frac{p+2-\sigma-\beta}{2})_{2k}(\frac{p+1-\sigma+\alpha+\eta}{2})_{2k}(\frac{p+2-\sigma+\alpha+\eta}{2})_{2k}}{(\frac{\sigma+p+\beta}{2})_{2k}(\frac{\sigma+p+\beta+1}{2})_{2k}(\frac{\sigma+\eta+p}{2})_{2k}(\frac{\sigma+\eta+p+1}{2})_{2k}(p+1)_k k!}\left(\frac{\lambda}{2x}\right)^2\right] \tag{4.9}$$

Using (2.10) on L.H.S. of above equation, we obtain

$$=\left(\frac{\lambda}{2}\right)^{p} x^{\sigma-p+\beta-1}\frac{\Gamma(1-\sigma+p-\beta)\Gamma(1-\sigma+p+\alpha+\eta)}{\Gamma(1-\sigma+p)\Gamma(1-\sigma+p+\eta-\beta)\Gamma(p+1)}{}_4F_5 \left[\begin{pmatrix} \frac{1-\sigma+p-\beta}{2}, \frac{2-\sigma+p-\beta}{2}, \frac{1-\sigma+p+\alpha+\eta}{2}, \frac{2-\sigma+\beta+p+\eta}{2}, \\ p+1, \frac{1-\sigma+p}{2}, \frac{2-\sigma+p}{2} \frac{1-\sigma+p+\eta-\beta}{2}, \frac{2-\sigma+p+\eta-\beta}{2} \end{pmatrix}\left(\frac{\lambda}{2x}\right)^{2}\right] \tag{4.10}$$

=R.H.S. of (4.7), i.e desired result (4.7)

Corollary 4.5. *Substituting $\beta = -\alpha$ in (4.7), we obtain the following result in terms of ${}_2F_3$:*

$$(D^{\alpha}_{0^-} t^{\sigma-1} I_p(\frac{\lambda}{t})(x) = \left(\frac{\lambda}{2}\right)^{p} x^{-p+\sigma-\alpha-1}\frac{\Gamma(-\sigma+p+1+\alpha)}{\Gamma(-\sigma+p+1)\Gamma(p+1)}{}_2F_3 \left[\begin{pmatrix} \frac{1-\sigma+p+\alpha}{2}, \frac{2-\sigma+p+\alpha}{2}, \\ p+1, \frac{1-\sigma+p}{2}, \frac{2-\sigma+p}{2} \end{pmatrix}\left(\frac{\lambda}{2x}\right)^{2}\right] \tag{4.11}$$

Corollary 4.6. *Substituting $\beta = 0$ in (4.7), we obtain the following result in terms of ${}_2F_3$:*

$$(D^{0}_{0^-} t^{\sigma-1} I_p(\frac{\lambda}{t})(x) = \left(\frac{\lambda}{2}\right)^{p} x^{-p+\sigma-1}\frac{\Gamma(-\sigma+p+1+\alpha)}{\Gamma(-\sigma+p+1\eta)\Gamma(p+1)}{}_2F_3 \left[\begin{pmatrix} \frac{1-\sigma+p+\alpha+\eta}{2}, \frac{2-\sigma+p+\alpha+\eta}{2}, \\ p+1, \frac{1-\sigma+p+\eta}{2}, \frac{2-\sigma+p+\eta}{2} \end{pmatrix}\left(\frac{\lambda}{2x}\right)^{2}\right] \tag{4.12}$$

5. Fractional differentiation of the cosine function

Substituting $p = \frac{-1}{2}$, the modified Bessel function $I_P(z)$ in (1.2) coincides with Cosine function, apart from the multiplier $(\frac{2}{\pi z})^{\frac{1}{z}}$:

$$I_{\frac{-1}{2}}(z) = (\frac{2}{\pi z})^{\frac{1}{2}} Cos(Z) \tag{5.1}$$

Theorem 5.1. *The following fractional derivative holds for $Cos(\lambda t)$:*

$$(D^{\alpha,\beta,\eta}_{0^+} t^{\sigma-1} Cos(\lambda t))(x) = (\pi)^{(}\frac{1}{2}) x^{\sigma+\beta-1} {}_2\psi_3 \left[\begin{pmatrix} (\sigma,2),(\sigma+\alpha+\eta+\beta,2) \\ (\sigma+\beta,2),(\sigma+\eta,2),(\frac{1}{2},1) \end{pmatrix} |(\frac{\lambda x}{2})^{2}\right], \tag{5.2}$$

where α, β, η, $p, \lambda \in C$ be such that $Re(\alpha) \geq 0$, $Re(p) > -1$, $Re(\sigma + p) > -Min[0, Re(\alpha + \beta + \eta)]$

Proof: Let us denote left hand side of (5.2) by P then substituting p=- ($\frac{1}{2}$) in (3.1) and using (5.1), we obtain :

L.H.S. of (5.2):

$$P = (\pi)^{(\frac{1}{2})} x^{(\frac{-3}{2})+\sigma-\beta} {}_2\psi_3 \left[\begin{pmatrix} (\sigma + (\frac{-1}{2}), 2), (\sigma + (\frac{-1}{2}) + \alpha + \eta, 2) \\ (\sigma + (\frac{-1}{2}) + \beta, 2), (\sigma + (\frac{-1}{2}) + \eta, 2), ((\frac{1}{2}), 1) \end{pmatrix} |(\frac{\lambda x}{2})^2 \right] \quad (5.3)$$

replace σ by (σ + ($\frac{1}{2}$) in (5.3)

$$= (\pi)^{(\frac{1}{2})} x^{\sigma+\beta-1} {}_2\psi_3 \left[\begin{pmatrix} (\sigma, 2), (\sigma + \alpha + \eta + \beta, 2) \\ (\sigma + \beta, 2), (\sigma + \eta, 2), (\frac{1}{2}, 1) \end{pmatrix} |(\frac{\lambda x}{2})^2 \right] \quad (5.4)$$

=R.H.S. of (5.2), i.e desired result (5.2).

Corollary 5.2. *Substituting β=-α in (5.2), we obtain the following result in terms of ${}_1\psi_2$:*

$$(D^{\alpha}_{0^+} t^{\sigma-1} Cos(\lambda t))(x) = (\pi)^{(\frac{1}{2})} x^{\sigma+\beta-1} {}_2\psi_3 \left[\begin{pmatrix} (\sigma, 2), \\ (\sigma - \alpha, 2), (\sigma + \eta, 2), (\frac{1}{2}, 1) \end{pmatrix} |(\frac{\lambda x}{2})^2 \right] \quad (5.5)$$

Theorem 5.3. *The following fractional derivative holds for $Cos(\lambda t)$:*

$$(D^{\alpha,\beta,\eta}_{0^+} t^{\sigma-1} Cos(\lambda t))(x) = x^{\sigma+\beta-1} \frac{\Gamma(\sigma)\Gamma(\sigma + \alpha + \eta + \beta)}{\Gamma(\sigma + \beta)\Gamma(\sigma + \eta)} {}_4F_5 \left[\begin{pmatrix} \frac{\sigma}{2}, \frac{1+\sigma}{2}, \frac{\sigma+\beta+\alpha+\eta}{2}, \frac{1+\sigma+\alpha+\eta+\beta}{2}, \\ (\frac{1}{2}), \frac{\beta+\sigma}{2}, \frac{1+\beta+\sigma}{2} \frac{\sigma+\eta}{2}, \frac{1+\sigma+\eta}{2} \end{pmatrix} \left(\frac{\lambda x}{2}\right)^2 \right], \quad (5.6)$$

where, $Re(\sigma) > 0, Re(\alpha + \beta + \eta + \sigma) > 0, Re(\beta + \sigma) > 0, Re(\eta + \sigma) > 0$.

Proof: Applying (4.1) & (5.1) on L.H.S of (5.6), we obtain:

L.H.S. of (5.6):

$$= \left(\frac{\lambda}{2}\right)^p x^{p+\sigma+\beta-1} \frac{\Gamma(\sigma + p)\Gamma(\sigma + p + \alpha + \beta + \eta)}{\Gamma(\sigma + p + \beta)\Gamma(\sigma + p + \eta)\Gamma(p + 1)} {}_4F_5 \left[\begin{pmatrix} \frac{\sigma+p}{2}, \frac{\sigma+p+1}{2}, \frac{\sigma+p+\alpha+\eta+\beta}{2}, \frac{\sigma+\beta+p+\eta+\alpha+1}{2}, \\ p + 1, \frac{\sigma+p+\beta}{2}, \frac{\sigma+p+\beta+1}{2}, \frac{\sigma+p+\eta}{2}, \frac{\sigma+p+\eta+1}{2} \end{pmatrix} \left(\frac{\lambda x}{2}\right)^2 \right] \quad (5.7)$$

Substituting p=0 in (5.7), we obtain the results in terms of ${}_4F_5$

$$=x^{\sigma+\beta-1}\frac{\Gamma(\sigma)\Gamma(\sigma+\alpha+\eta+\beta)}{\Gamma(\sigma+\beta)\Gamma(\sigma+\eta)}{}_4F_5 \left[\begin{pmatrix} \frac{\sigma}{2}, \frac{1+\sigma}{2}, \frac{\sigma+\beta+\alpha+\eta}{2}, \frac{1+\sigma+\alpha+\eta+\beta}{2}, \\ (\frac{1}{2}), \frac{\beta+\sigma}{2}, \frac{1+\beta+\sigma}{2} \frac{\sigma+\eta}{2}, \frac{1+\sigma+\eta}{2} \end{pmatrix}\left(\frac{\lambda x}{2}\right)^2\right], \tag{5.8}$$

=R.H.S. of (5.6), i.e desired result (5.6)

Corollary 5.4. *Substituting $\beta = -\alpha$ in (5.6), we obtain the following result in terms of ${}_2F_3$:*

$$\begin{aligned}&(D^{\alpha}_{0^+}t^{\sigma-1}Cos(\lambda t))(x)\\ &= x^{\sigma-\alpha-1}\frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)}{}_2F_3\\ &\left[\begin{pmatrix} \frac{\sigma}{2}, \frac{\sigma+1}{2} \\ \frac{1}{2}, \frac{\sigma-\alpha}{2}, \frac{\sigma-\alpha+1}{2}\end{pmatrix}\left(\frac{\lambda x}{2}\right)^2\right]\end{aligned} \tag{5.9}$$

6. Fractional differentiation of the sine function

Substituting $p = \frac{-1}{2}$, the modified Bessel Function $I_P(z)$ in (1.2) coincides with Sine Function, apart from the multiplier $(\frac{2}{\pi z})^{\frac{1}{z}}$.

$$I_{\frac{1}{2}}(z) = (\frac{2}{\pi z})^{\frac{1}{2}} S in(Z) \tag{6.1}$$

Theorem 6.1. *The following fractional derivative holds for $S in(\lambda t)$:*

$$(D^{\alpha,\beta,\eta}_{0^+}t^{\sigma-2}S in(\lambda t))(x) = (\frac{\lambda}{2})(\pi)^{(\frac{1}{2})}x^{\sigma+\beta-1}{}_2\psi_3\left[\begin{pmatrix}(\sigma,2),(\sigma+\alpha+\eta+\beta,2)\\(\sigma+\beta,2),(\sigma+\eta,2),(\frac{3}{2},1)\end{pmatrix}|(\frac{\lambda x}{2})^2\right], \tag{6.2}$$

where α, β, η, $p, \lambda \in C$ be such that $Re(\alpha) \geq 0$, $Re(p) > -1$, $Re(\sigma + p) > -Min[0, Re(\alpha+\beta+\eta)]$

Proof: Let us denote left hand side of (6.2) by Q then substituting p= $(\frac{1}{2})$ in (3.1) and using (6.1), we obtain:

L.H.S. of (6.2):

$$Q = (\frac{\lambda}{2})(\pi)^{(\frac{1}{2})} x^{(\frac{-3}{2})+\sigma-\beta} {}_2\psi_3 \left[\begin{pmatrix} (\sigma + (\frac{1}{2}), 2), (\sigma + (\frac{1}{2}) + \alpha + \eta, 2) \\ (\sigma + (\frac{1}{2}) + \beta, 2), (\sigma + (\frac{3}{2}) + \eta, 2), ((\frac{1}{2}), 1) \end{pmatrix} | (\frac{\lambda x}{2})^2 \right] \quad (6.3)$$

replace σ by $(\sigma - (\frac{1}{2})$ in (6.3)

$$= (\frac{\lambda}{2})(\pi)^{(\frac{1}{2})} x^{\sigma+\beta-1} {}_2\psi_3 \left[\begin{pmatrix} (\sigma, 2), (\sigma + \alpha + \eta + \beta, 2) \\ (\sigma + \beta, 2), (\sigma + \eta, 2), (\frac{3}{2}, 1) \end{pmatrix} | (\frac{\lambda x}{2})^2 \right] \quad (6.4)$$

=R.H.S. of (6.2), i.e desired result (6.2).

Corollary 6.2. *substituting β=-α in (6.2), we obtain the following result in terms of ${}_1\psi_2$:*

$$(D^{\alpha}_{0^+} t^{\sigma-1} Sin(\lambda t))(x) = (\frac{\lambda}{2})(\pi)^{(\frac{1}{2})} x^{\sigma-\alpha-1} {}_2\psi_3 \left[\begin{pmatrix} (\sigma, 2), (\sigma + \alpha + \eta - \alpha, 2) \\ (\sigma - \alpha, 2), (\sigma + \eta, 2), (\frac{3}{2}, 1) \end{pmatrix} | (\frac{\lambda x}{2})^2 \right]$$

(6.5)

Theorem 6.3. *The following fractional derivative holds for $Sin(\lambda t)$:*

$$(D^{\alpha,\beta,\eta}_{0^+} t^{\sigma-1} Sin(\lambda t))(x) = x^{\sigma+\beta-1} \frac{\Gamma(\sigma)\Gamma(\sigma + \alpha + \eta + \beta)}{\Gamma(\sigma + \beta)\Gamma(\sigma + \eta)} {}_4F_5 \left[\begin{pmatrix} \frac{\sigma}{2}, \frac{1+\sigma}{2}, \frac{\sigma+\beta+\alpha+\eta}{2}, \frac{1+\sigma+\alpha+\eta+\beta}{2}, \\ \frac{3}{2}, \frac{\beta+\sigma}{2}, \frac{1+\beta+\sigma}{2}, \frac{\sigma+\eta}{2}, \frac{1+\sigma+\eta}{2} \end{pmatrix} \left(\frac{\lambda x}{2}\right)^2 \right], \quad (6.6)$$

where, $Re(\sigma) > 0, Re(\alpha + \beta + \eta + \sigma) > 0, Re(\beta + \sigma) > 0, Re(\eta + \sigma) > 0$.

Proof: Applying (4.1) & (6.1) on L.H.S of (6.6), we obtain :

L.H.S. of (6.6):

$$= \left(\frac{\lambda}{2}\right)^p x^{p+\sigma+\beta-1} \frac{\Gamma(\sigma + p)\Gamma(\sigma + p + \alpha + \beta + \eta)}{\Gamma(\sigma + p + \beta)\Gamma(\sigma + p + \eta)\Gamma(p + 1)} {}_4F_5 \left[\begin{pmatrix} \frac{\sigma+p}{2}, \frac{\sigma+p+1}{2}, \frac{\sigma+p+\alpha+\eta+\beta}{2}, \frac{\sigma+\beta+p+\eta+\alpha+1}{2}, \\ p + 1, \frac{\sigma+p+\beta}{2}, \frac{\sigma+p+\beta+1}{2}, \frac{\sigma+p+\eta}{2}, \frac{\sigma+p+\eta+1}{2} \end{pmatrix} \left(\frac{\lambda x}{2}\right)^2 \right] \quad (6.7)$$

Substituting p=0 in (6.7), we obtain the results in terms of ${}_4F_5$

$$=x^{\sigma+\beta-1}\frac{\Gamma(\sigma)\Gamma(\sigma+\alpha+\eta+\beta)}{\Gamma(\sigma+\beta)\Gamma(\sigma+\eta)}{}_4F_5 \left[\left(\begin{matrix}\frac{\sigma}{2}, \frac{1+\sigma}{2}, \frac{\sigma+\beta+\alpha+\eta}{2}, \frac{1+\sigma+\alpha+\eta+\beta}{2}, \\ \frac{3}{2}, \frac{\beta+\sigma}{2}, \frac{1+\beta+\sigma}{2} \frac{\sigma+\eta}{2}, \frac{1+\sigma+\eta}{2}\end{matrix}\right)\left(\frac{\lambda x}{2}\right)^2\right], \tag{6.8}$$

= R.H.S. of (6.6), i.e desired result (6.6)

Corollary 6.4. *Substituting $\beta = -\alpha$ in (6.6), we obtain the following result in terms of ${}_2F_3$:*

$$(D^{\alpha}_{0^+}t^{\sigma-1}Sin(\lambda t))(x) = x^{\sigma-\alpha-1}\frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)}{}_2F_3\left[\left(\begin{matrix}\frac{\sigma}{2}, \frac{\sigma+1}{2} \\ \frac{3}{2}, \frac{\sigma-\alpha}{2}, \frac{\sigma-\alpha+1}{2}\end{matrix}\right)\left(\frac{\lambda x}{2}\right)^2\right] \tag{6.9}$$

7. Concluding Remarks

It is evident that the study of fractional calculus open the minds to infinity new branches of thought. It fills in the gapes of traditional calculus. The fractional derivatives and integrals are being used to formulate various phenomena in physics, biology, electrical system and thermodynamics etc., during the last three decades. In the recent past, various fractional and integral derivatives are being generalized and applied on various Special functions. In the present paper, the generalized fractional derivatives operators are applied on modified Bessel function of order p. Fractional differentiation of Cosine and Sine functions which are related to modified Bessel function, are also derived. Some special cases are also discussed. All main results are derived in terms of generalized wright function and generalized hypergeometric function.

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