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MODELLING THE USE OF TECHNOLOGY TO IMPROVE DEGRADED LAND AND BIOMASS DENSITY

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Abstract

Land degradation occurs when land loses its natural productivity due to various human activities and natural hazards. It eventually leads to a decrease in biomass density. This paper proposes nonlinear mathematical models to study the increase in biomass density by using technology to improve degraded land. The models have four variables, namely, the biomass density, the degraded land area, the fertile land area and the technology to improve degraded land. Two types of technological efforts are considered: (I) linearly varying with degraded land, (II) logistically varying with degraded land. The stability theory of differential equations is used to analyze these models. The local & global stability of the non-trivial equilibrium points have been studied by using suitable Lyapunov functions. In both the cases, analysis shows that as the technology increases the degraded land becomes fertile, leading to an increase in the biomass density. The results are confirmed using numerical simulation. The comparative analysis of the effect of doubling vital parameters on the state variables has been examined using a basic sensitivity analysis corresponding to the differential equations in the models.

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1. Introduction

Land degradation is the process in which both biotic and abiotic population is tremendously affected by the direct and indirect impact of human activities on the land. Various factors are responsible for land degradation like wind, storm, drought, acid rain, flood, water, pollution, mechanical erosion, excessive use of chemicals in agricultural activities, mining, road construction etc. Important forms of land degradation are aridity and salinization of soil. The most affected countries by degraded land are India, China, Argentina, Russia, Brazil, the United States, and Australia. Better management is needed for the world's arable systems to combat multidimensional land degradation [1]. India suffers the loss of 0.8 mt of nitrogen (N), 26.3 mt of potassium (K) and 1.8 mt of phosphorous (P) every year. As per data available from ICAR (Indian Council of Agricultural Research) and NAAS (National Academy of Agricultural Sciences), about 120.72 million hectares (MHA) of land is degraded every year. In India, states

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like Telangana, M.P., Jammu & Kashmir, Odisha, Maharashtra, Jharkhand, Gujarat, Karnataka and Rajasthan which are affected by it [2]. There is a decrease in Earth's resources due to deforestation, agriculture depletion, overgrazing, and quarrying of sand, stone, ore and minerals. Continuous deforestation also leads to soil erosion and the formation of barren lands. Doon Valley in India is an example of a situation occurring due to limestone quarriers and industrialization. To overcome, the consequences of deforestation and industrialization, reforestation is recommended which is capable of improving the resource biomass [3–6]. The emission of primary and secondary toxicants also affect the density of forest resources and soil fertility [7].

Various technological efforts like genetically engineered plants and sustainable harvesting techniques are adopted to conserve resources [8–10, 23, 24]. Soil fertility is maintained by adding N and P individually or their combination in soil and maintaining the moisture in soil. There is a need to conserve soil organic matter as well as water [11]. Land degradation can be reversed or degraded land can be improved by adding required nutrients to soil, buffering soil acidity, rebuilding topsoil, desalinization, adopting various technologies like green manuring, use of local products as sources of rock phosphate and lime, employing techniques for improving irrigation and water harvesting [12, 23]. This can be done through increasing awareness among society and employing more technological efforts. In Central Asia, endorsement of sustainable land management technologies like improving irrigation technology, laser land leveling, zero tillage, mulching, crop diversification with legumes, use of phosphogypsum for sodic soils, use of fertilizers, planting halophytic plants, strip cropping, drip irrigation and terracing are seen to have positive impact [13, 14]. In India, various policies are adopted to improve ecological and agricultural growth like Green India Mission 2015, Pradhan Mantri Krishi Sinchayee Yojana (PMKSY) 2015, Soil Health Card Scheme 2015, Neem Coated Urea Scheme, National Project on Organic Farming Scheme 2004, etc [2].

The drop-off of forest assets caused by growing population, industrial society, pollution leading to soil erosion, top-soil depletion and soil acidification have been studied by many researchers [3–6, 8, 10, 15]. The consequences of wind and rainwater on fertile topsoil are studied by B. Dubey [16, 17]. Some recent contributions in this area are the ecological mathematical models proposed Ivanyo et al. and Apazhev et al. First one is to optimize agricultural and livestock production, taking into account soil degradation in rain-fed and irrigated areas by reducing losses in extreme condition and minimizing the environmental damage during the cultivation [18]. Another is to use reclaimed land by maintaining the groundwater level, the salinity of the soil, the quality of irrigation water, etc [19]. In this study, our objective is to propose and analyse models to increase the fertile land that is re-obtained from degraded land by applying technology. As the fertile land increases, there is an increase in biomass production.

1.1. Comparison with Previous Models Mathematical modeling has been widely used to understand land degradation, biomass growth, and environmental resource management. Several ecological models have attempted to study the impact of soil degradation on agricultural productivity and biomass density. However, our proposed models introduce technological interventions as a dynamic variable, which is a novel approach. Below, we compare our work with key previous studies:

1. Ivanyo et al. (2022) developed a model to optimize agricultural and livestock production by minimizing environmental damage. However, their study focused on soil degradation in extreme conditions without explicitly incorporating the role of technology in land restoration [18].

2. Apazhev et al. (2019) proposed models for reclaimed land, accounting for groundwater levels, soil salinity, and irrigation quality. While these factors are crucial, their model did not explore the direct effects of technological advancements in land rehabilitation [19].

3. Dubey investigated the impact of wind and rainwater on topsoil degradation but did not propose an active restoration mechanism [16, 17].

4. Yasin Rustamov et al. have proposed a mathematical model states various factors like water, nutrients, humus in appropriate proportion responsible for soil's fertility [25].

5. A mathematical model by Sapna Devi et al. investigated the impact of fertilizer, earthworms on increasing the fertility of soil to increase in agricultural production [22].

6. Our work extends these studies by introducing technology as a control mechanism to reverse land degradation and improve biomass density. Unlike previous models, we examine two different technological growth patterns (linear and logistic) and their stability using Lyapunov functions.

Our approach fills an important research gap by providing a dynamic framework to analyze the role of technological interventions in land restoration and biomass improvement. The numerical simulations and sensitivity analyses further validate the effectiveness of these technological strategies.

2. Notation and Assumption

To develop a mathematical model for analyzing the impact of technology on improving degraded land and biomass density, we define the following variables and parameters:

2.1. Notation

Symbol	Description
$B(t)$	Biomass density at time t
$L_1(t)$	Area of degraded land at time t
$L_2(t)$	Area of fertile land at time t
$T(t)$	Technological efforts applied at time t
s	Intrinsic growth rate of biomass
L	Carrying capacity of biomass
s_1	Growth rate of biomass due to fertile land
s_2	Contribution to carrying capacity of biomass due to fertile land
s_0	Natural depletion coefficient of biomass
Q	Constant rate of increase in degraded land
δ_0	Natural depletion coefficient of degraded land
δ_1	Depletion rate coefficient of degraded land due to technology
θ	Conversion rate coefficient of degraded land to fertile land
θ_0	Natural depletion rate coefficient of fertile land
μ	Growth rate coefficient of technology concerning degraded land
μ_0	Natural depletion rate of technology

2.2. Assumptions

1. Biomass density (B) follows a logistic growth pattern, where its growth rate and carrying capacity increase with the area of fertile land (L_2).
2. The area of degraded land (L_1) increases at a constant rate due to natural and anthropogenic factors but can be reduced by technological efforts.
3. Fertile land (L_2) is obtained by applying technology (T) to degraded land (L_1), making it directly proportional to both.
4. Technological efforts (T) are modeled in two ways:
   (a) Scenario I: Technology varies linearly with degraded land.
   (b) Scenario II: Technology varies logistically with degraded land.
5. Natural depletion affects all state variables (B, L_1, L_2, T), and these effects are captured by respective depletion coefficients.

3. Scenario I. Technology (T) linearly varying with degraded land (L_1)

3.1. Mathematical Model Biomass density can be increased by using technology to improve degraded land. To model this scenario, let us consider four variables which are non-linearly interacting, namely; biomass density of resource $B(t)$, the area of degraded land $L_1(t)$, the area of obtained fertile land $L_2(t)$ and the magnitude of technology (machinery, fertilizer etc.) $T(t)$. The model which governs the dynamics of the system consists of nonlinear ordinary differential equations as follows:

$$\frac{dB}{dt} = s(B - \frac{B^2}{L}) - s_0 B + s_1 B L_2 + s_2 B^2 L_2 \quad (3.1)$$
$$\frac{dL_1}{dt} = Q - \delta_0 L_1 - \delta_1 T L_1 \quad (3.2)$$
$$\frac{dL_2}{dt} = \theta \delta_1 T L_1 - \theta_0 L_2 \quad (3.3)$$
$$\frac{dT}{dt} = \mu L_1 - \mu_0 T \quad (3.4)$$

where, $B(0) \geq 0$, $L_1(0) \geq 0$, $L_2(0) \geq 0$, $T(0) \geq 0$, $s - s_0 > 0$.

The equation (3.1) of the above model governs the biomass density varying logistically where s is the intrinsic growth rate of B and L is the carrying capacity of B. The growth rate of B and carrying capacity of B increase with the increase of fertile land L_2. s_1 denotes the intrinsic growth rate of B due to L_2 and s_2 denotes the rate of contribution to carrying capacity of B due to L_2. s_0 is natural depletion coefficient in B (biomass density). In the equation (3.2), it is supposed that degraded land L_1 increases constantly with rate Q, δ_0 is the coefficient due to natural depletion and δ_1 denotes depletion rate coefficient of L_1 due to technology (T). In the equation (3.3), L_2 (fertile land) is obtained by applying technology on degraded land L_1 and hence, L_2 is directly proportional to both T and L_1 i.e. $\delta_1 T L_1$. Here, θ denotes the conversion rate coefficient for degraded land L_1 using technology, and θ_0 is the rate coefficient of natural depletion in L_2. In the equation (3.4), the growth rate of technology is directly proportional to the area of degraded land L_1, where, μ denotes the growth rate coefficient of T with respect to L_1 and μ_0 is its natural depletion. Here, technology is linearly varying with L_1. The model (3.1) - (3.4) is compatible with the real-world situation and the state variables of the system represented by the flow chart depicted in fig. 1

3.2. Bounds of variables Bounds of variables for the model system (3.1) - (3.4) are given by the lemma, which is as follows:

Lemma 3.1. *The set Δ denotes the region of attraction. All solutions commencing inside Δ, remain inside it.*
$\Delta = \{(B, L_1, L_2, T) \in R_+^4 : 0 \leq B \leq B_{max}, 0 \leq L_1 \leq L_{1_{max}}, 0 \leq L_2 \leq L_{2_{max}}, 0 \leq T \leq T_{max}\}$,
where, $B_{max} = \dfrac{s + s_1 L_{2_{max}}}{\frac{s}{L} - s_2 L_{2_{max}}}$, *provided* $\dfrac{s}{L} - s_2 L_{2_{max}} > 0$,
$L_{1_{max}} = \dfrac{Q}{\delta_0}$, $L_{2_{max}} = \dfrac{\theta \delta_1 \mu Q^2}{\mu_0 \delta_0^2 \theta_0}$, $T_{max} = \dfrac{\mu Q}{\mu_0 \delta_0}$.

Proof. From equation (3.1) we get,
$\dfrac{dB}{dt} \leq sB(1 - \dfrac{B}{L}) + s_1 B L_2 + s_2 B^2 L_2$

[1] Note: Fig.1 shows the effect of applying technology on degraded land

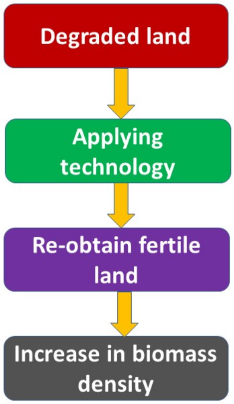


FIGURE 1: Increase in Biomass density improving degraded land

$$\implies \lim_{t\to\infty} supB(t) \leq \frac{s + s_1 L_{2_{max}}}{\frac{s}{L} - s_2 L_{2_{max}}} = B_{max} \text{ (say)},$$

Again, from equation (3.2) we obtain,

$$\frac{dL_1}{dt} \leq Q - \delta_0 L_1$$

$$\implies \lim_{t\to\infty} supL_1(t) \leq \frac{Q}{\delta_0} = L_{1_{max}} \text{ (say)},$$

Further, from the equation (3.3) we get,

$$\frac{dL_2}{dt} \leq \theta\delta_1 T L_1 - \theta_0 L_2$$

$$\implies \lim_{t\to\infty} supL_2(t) \leq \frac{\theta\delta_1\mu Q^2}{\mu_0\delta_0^2\theta_0} = L_{2_{max}} \text{ (say)},$$

Now, from equation (3.4) we get,

$$\frac{dT}{dt} \leq \mu L_1 - \mu_0 T$$

$$\implies \lim_{t\to\infty} supT(t) \leq \frac{\mu Q}{\mu_0\delta_0} = T_{max} \text{ (say)}.$$

Hence, the proof. □

3.3. Equilibrium analysis In a dynamical system, the state which remains consistent across time represents the equilibrium point. There are two non-negative equilibrium points in $B - L_1 - L_2 - T$ space denoted by $E_0(0, \hat{L_1}, \hat{L_2}, \hat{T})$ and $E^*(B^*, L_1^*, L_2^*, T^*)$. The equilibrium points of the system (3.1) - (3.4) are obtained by equating to zero the right hand sides of equations (3.1) - (3.4).

3.3.1. Existence of E_0 The existence of E_0 is obvious. we get E_0 after solving algebraic equations, which are as follows:

$$B = 0, \tag{3.5}$$

$$Q - \delta_0 L_1 - \delta_1 T L_1 = 0, \tag{3.6}$$

$$\theta\delta_1 T L_1 - \theta_0 L_2 = 0, \tag{3.7}$$

$$\mu L_1 - \mu_0 T = 0, \tag{3.8}$$

on solving (3.6) - (3.8) we get
$L_1 = \dfrac{Q}{\delta_0 + \delta_1 T}$, $L_2 = \dfrac{\theta\delta_1 T Q}{\theta_0(\delta_0 + \delta T)}$ and a quadratic equation in T i.e.

$$\mu_0\delta_1 T^2 + \mu_0\delta_0 T - Q\mu = 0$$

which gives a unique positive root. Now, the expressions of the variables at equilibrium point E_0 are as follows:

$$\hat{B} = 0,$$

$$\hat{T} = \frac{-\delta_0\mu_0 + \sqrt{\delta_0^2\mu_0^2 + 4\delta_1\mu_0\mu Q}}{2\delta_1\mu_0}.$$

$$\hat{L_1} = \frac{Q}{\delta_0 + \delta_1\hat{T}},$$

$$\hat{L_2} = \frac{\theta\delta_1\hat{T}Q}{\theta_0(\delta_0 + \delta_1\hat{T})},$$

3.3.2. Existence of E^* The following algebraic equations can be solved to get E^*:

$$s(1 - \frac{B}{L}) - s_0 + s_1 L_2 + s_2 B L_2 = 0, \tag{3.9}$$

$$Q - \delta_0 L_1 - \delta_1 T L_1 = 0, \tag{3.10}$$

$$\theta\delta_1 T L_1 - \theta_0 L_2 = 0, \tag{3.11}$$

$$\mu L_1 - \mu_0 T = 0, \tag{3.12}$$

on solving (3.9) - (3.12) we get
$B = \dfrac{s - s_0 + s_1 L_2}{(\frac{s}{L} - s_2 L_2)}$, $L_1 = \dfrac{Q}{\delta_0 + \delta_1 T}$, $L_2 = \dfrac{\theta\delta_1 T Q}{\theta_0(\delta_0 + \delta T)}$ and a quadratic equation in T i.e.

$$\mu_0\delta_1 T^2 + \mu_0\delta_0 T - Q\mu = 0$$

which gives a unique positive root. Now, the value of the variables at equilibrium point E^* is as follows:

$$B^* = \frac{s - s_0 + s_1 L_2^*}{(\frac{s}{L} - s_2 L_2^*)},$$

$$\text{provided } \frac{s}{L} - s_2 L_2^* > 0 \text{ which holds since } \frac{s}{L} > s_2 L_{2max}$$

$$L_1^* = \frac{Q}{\delta_0 + \delta_1 T^*},$$

$$L_2^* = \frac{\theta \delta_1 T^* Q}{\theta_0(\delta_0 + \delta_1 T^*)},$$

$$T^* = \frac{-\delta_0\mu_0 + \sqrt{\delta_0^2\mu_0^2 + 4Q\mu\mu_0\delta_1}}{2\delta_1\mu_0}.$$

Remark 3.2. *We see that at equilibrium point* $\frac{dL_1}{d\mu} < 0$*, which shows that as the rate coefficient of technology increases at the equilibrium level there is a decrease in infertile land area. From the model system* (3.1) - (3.4)*, we have*

$$Q - \delta_0 L_1 - \delta_1 T L_1 = 0,$$
$$\mu L_1 - \mu_0 T = 0,$$

differentiating above equations with respect to μ *we get,*

$$-\delta_0 \frac{dL_1}{d\mu} - \delta_1 \frac{dT}{d\mu} L_1 - \delta_1 T \frac{dL_1}{d\mu} = 0, \tag{3.13}$$

$$\mu \frac{dL_1}{d\mu} + L_1 - \mu_0 \frac{dT}{d\mu} = 0, \tag{3.14}$$

on solving (3.13) *and* (3.14) *we get,*

$$\frac{dL_1}{d\mu} = -\frac{\delta_1 \frac{dT}{d\mu} L_1}{(\delta_0 + \delta_1 T)}, \tag{3.15}$$

since, $\frac{dT^*}{d\mu} = \frac{Q}{(2\mu_0\delta_1 T^* + \mu_0\delta_0)} > 0$*. Clearly,* $\frac{dL_1}{d\mu} < 0$ *at* $L_1 = L_1^*$*, which shows that the area of infertile land decreases as the rate of technology increases at the equilibrium point* L_1^**.*

3.4. Stability analysis The system (3.1) - (3.4) is non-linear so, we can't find its exact solution but long term behavior of the system can be analyzed by using stability analysis.

3.4.1. Local stability The local stability of the system is determined by signs the eigenvalues of the Jacobian matrix corresponding to the equilibrium state. The following theorem states the local stability analysis of E_0 and E^*.

Theorem 3.3 (Local stability). *E_0 is always unstable equilibrium point. There is not any constraint required for the interior (non-trivial) equilibrium point E^* to be locally stable.*

Proof. For the model system (3.1) - (3.4), J_0 (Jacobian matrix) at E_0 is given as follows:

$$J_0 = \begin{pmatrix} s - s_0 + s_1\hat{L}_2 & 0 & 0 & 0 \\ 0 & -\delta_0 - \delta_1\hat{T} & 0 & -\delta_1\hat{L}_1 \\ 0 & \theta\delta_1\hat{T} & -\theta_0 & \theta\delta_1\hat{L}_1 \\ 0 & \mu & 0 & -\mu_0 \end{pmatrix}$$

now, $s - s_0 + s_1\hat{L}_2$ is positive eigen value of the matrix J_0. Therefore, E_0 is unstable in B-direction. Jacobian matrix J^* associated to E^* (nontrivial equilibrium point) is as follows:

$$J^* = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & -\delta_0 - \delta_1 T^* & 0 & -\delta_1 L_1^* \\ 0 & \theta\delta_1 T^* & -\theta_0 & \theta\delta_1 L_1^* \\ 0 & \mu & 0 & -\mu_0 \end{pmatrix}$$

where,

$a_{11} = B^*(-\dfrac{s}{L} + s_2 L_2^*)$, $a_{13} = s_1 B^* + s_2 B^{*2}$.

The eigen value for this matrix can't be easily evaluated by Routh-Hurwitz criteria. Hence, we use some suitable Lyapunov function to determine local stability of E^*.

First, we linearize the system (3.1) - (3.4) about $E^*(B^*, L_1^*, L_2^*, T^*)$ by using the transformations:

$$B = B^* + b, L_1 = L_1^* + l_1, L_2 = L_2^* + l_2, T = T^* + t_1, \tag{3.16}$$

where, b, l_1, l_2, t_1 are small perturbation around E^*.

Take a positive definite function about E^*

$$V = \frac{1}{2}\frac{m_1 b^2}{B^*} + \frac{1}{2}m_2 l_1^2 + \frac{1}{2}m_3 l_2^2 + \frac{1}{2}m_4 t_1^2, \tag{3.17}$$

where, $m_1 > 0$, $m_2 > 0$, $m_3 > 0$ and $m_4 > 0$, to be chosen properly.

Differentiating V taken in equation (3.17) with respect to t yields,

$$\frac{dV}{dt} = \frac{m_1 b}{B^*}\frac{db}{dt} + m_2 l_1\frac{dl_1}{dt} + m_3 l_2\frac{dl_2}{dt} + m_4 t_1\frac{dt_1}{dt}, \tag{3.18}$$

using the model equations in (3.18) and after simplification we get,

$$\frac{dV}{dt} = -m_1[\frac{s}{L} - s_2L_2^*]b^2 - m_2[\delta_0 + \delta_1 T^*]l_1^2 - m_3\theta_0 l_2^2 - m_4\mu_0 t_1^2 + [m_1(s_1 + s_2B^*)]bl_2 + [m_4\mu - m_2\delta_1 L_1^*]l_1t_1 + m_3[\theta\delta_1 L_1^*]l_2t_1 + m_3[\theta\delta_1 T^*]l_1l_2. \quad (3.19)$$

$\frac{dV}{dt}$ comes out to be negative definite if, the some constraints are satisfied, which are as follows:

$$[m_1(s_1 + s_2B^*)]^2 < \frac{4}{3}m_1m_3\theta_0(\frac{s}{L} - s_2L_2^*) \quad (3.20)$$

$$[m_4\mu - m_2\delta_1L_1^*]^2 < m_2m_4\mu_0(\delta_0 + \delta_1T^*) \quad (3.21)$$

$$m_3(\theta\delta_1L_1^*)^2 < \frac{2}{3}m_4\theta_0\mu_0 \quad (3.22)$$

$$m_3(\theta\delta_1T^*)^2 < \frac{2}{3}m_2\theta_0(\delta_0 + \delta_1T^*) \quad (3.23)$$

choosing, $m_2 = 1, m_3 = 1, m_4 = \frac{\delta_1L_1^*}{\mu}$, we get,

$$m_1 < \frac{8}{9}\frac{(\frac{s}{L} - s_2L_2^*)\theta_0^2}{(s_1 + s_2B^*)^2\theta^2\delta_1}Min\{\frac{\mu_0}{\mu L_1^*}, \frac{(\delta_0 + \delta_1T^*)}{\delta_1T^{*2}}\}. \quad (3.24)$$

Thus, $\frac{dV}{dt}$ is negative definite and there is not any constraint required for the interior (non-trivial) equilibrium point E^* to be locally stable.

□

3.4.2. Global stability By employing a suitable positive definite Lyapunov function, global stability of interior equilibrium can be obtained in the region of attraction Δ. The summarization of global stability behavior of E^* is provided in the following theorem.

THEOREM 3.4 (Global stability). *There is not any constraint required for the interior (non-trivial) equilibrium point E^* to be globally stable inside the region of attraction Δ.*

PROOF. Take a positive definite function

$$W = k_1(B - B^* - B^*\ln\frac{B}{B^*}) + \frac{k_2}{2}(L_1 - L_1^*)^2 + \frac{k_3}{2}(L_2 - L_2^*)^2 + \frac{k_4}{2}(T - T^*)^2, \quad (3.25)$$

where, $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, $k_4 > 0$, to be chosen properly. The function W is considered after checking that it is zero at equilibrium $E^*(B^*, L_1^*, L_2^*, T^*)$ and it is positive for all different positive values of B, L_1, L_2 and T.
Now, differentiating W taken in equation (3.25) with respect to t we get,

$$\begin{aligned}\dot{W} = &-k_1(\frac{s}{L} - s_2L_2^*)(B - B^*)^2 - k_2(\delta_0 + \delta_1 T^*)(L_1 - L_1^*)^2 - k_3\theta_0(L_2 - L_2^*)^2 \\ &- k_4\mu_0(T - T^*)^2 + [k_1(s_1 + s_2B)](B - B^*)(L_2 - L_2^*) \\ &+ [k_4\mu - k_2\delta_1 L_1](L_1 - L_1^*)(T - T^*) + k_3\theta\delta_1 L_1(T - T^*)(L_2 - L_2^*) \\ &+ k_3\theta\delta_1 T^*(L_1 - L_1^*)(L_2 - L_2^*).\end{aligned} \tag{3.26}$$

$\dot{W}$ comes out to be positive definite [21] in the region of attraction Δ if, the following constraints are satisfied:

$$[(s_1 + s_2B_{max})k_1]^2 < \frac{4}{3}k_1k_3\theta_0(\frac{s}{L} - s_2L_2^*) \tag{3.27}$$

$$[k_4\mu - k_2\delta_1\frac{Q}{\delta_0}]^2 < k_2k_4\mu_0(\delta_0 + \delta_1T^*) \tag{3.28}$$

$$k_3(\theta\delta_1\frac{Q}{\delta_0})^2 < \frac{2}{3}k_4\mu_0\theta_0 \tag{3.29}$$

$$k_3(\theta\delta_1T^*)^2 < \frac{2}{3}k_2\theta_0(\delta_0 + \delta_1T^*) \tag{3.30}$$

Choosing, $k_2 = 1$, $k_3 = 1$, $k_4 = \dfrac{\delta_1 Q}{\mu\delta_0}$, we get,

$$k_1 < \frac{8}{9}\frac{\theta_0^2(\frac{s}{L} - s_2L_2^*)}{(s_1 + s_2B_{max})^2(\theta^2\delta_1)}Min\{\frac{\mu_0\delta_0}{Q\mu}, \frac{(\delta_0 + \delta_1T^*)}{\delta_1T^{*2}}\}. \tag{3.31}$$

Thus, $\dfrac{dW}{dt}$ is negative definite and there is not any constraint required for the interior (non-trivial) equilibrium point E^* to be globally stable inside the region of attraction Δ.

□

3.5. Model's Persistence In a real-world situation, state variables remain positive for any period if initially those were positive. Mathematically, if $Z(0) > 0$, then $Z(t) > 0 \ \forall \ t$ and $\lim_{t\to\infty} inf Z(t) > 0$. If we can find some $\epsilon > 0$ independent of $Z(0)$, which is the initial condition, such that $\lim_{t\to\infty} inf Z(t) > \epsilon$ then the system satisfies the persistence condition. The persistence of the system (3.1) - (3.4) is provided in the following theorem:

Theorem 3.5 (Persistence). *The uniform persistence of the system* (3.1) - (3.4) *for increasing biomass density by improving degraded land holds if* $\frac{s}{L} - s_2 L_{2_{max}} > 0$ *where,* $L_{2_{max}}$ *is the upper bound of the fertile land.*

Proof. To prove system persist uniformly, from (3.1) of the model (3.1) - (3.4) we get,

$$\frac{dB}{dt} \geq (s - s_0 + s_1 L_{2_{min}})B - (\frac{s}{L} - s_2 L_{2_{max}})B^2$$

$\Longrightarrow$

$$\lim_{t\to\infty} inf B(t) \geq \frac{(s - s_0 + s_1 L_{2_{min}})}{(\frac{s}{L} - s_2 L_{2_{max}})} = B_{min}(say),$$

Here, $B_{min} > 0$ if $\frac{s}{L} - s_2 L_{2_{max}} > 0$. Also, $0 < B_{min} < B_{max}$.
Now, from (3.2) we get,

$$\frac{dL_1}{dt} \geq \frac{Q}{\delta_0 + \delta_1 T_{max}} = L_{1_{min}}$$

$\Longrightarrow$

$$\lim_{t\to\infty} inf L_1(t) \geq \frac{Q}{\delta_0 + \delta_1 T_{max}} = L_{1_{min}}(say),$$

Clearly, $L_{1_{min}} > 0$ and $L_{1_{min}} < L_{1_{max}} = \frac{Q}{\delta_0}$. From (3.4) we get,

$$\frac{dT}{dt} \geq \frac{\mu L_{1_{min}}}{\mu_0} = T_{min}$$

$\Longrightarrow$

$$\lim_{t\to\infty} inf T(t) \geq \frac{\mu L_{1_{min}}}{\mu_0} = T_{min}(say),$$

Here,
$0 < T_{min} = \frac{\mu Q}{\mu_0(\delta_0 + \delta_1 T_{max})} < \frac{\mu Q}{\mu_0 \delta_0} = T_{max}$.
From (3.3) we get,

$$\frac{dL_2(t)}{dt} \geq \frac{\theta \delta_1 \mu Q^2}{\mu_0(\delta_0 + \delta_1 T_{max})^2 \theta_0}$$

$\Longrightarrow$

$$\lim_{t\to\infty} inf L_2(t) \geq \frac{\theta \delta_1 \mu Q^2}{\mu_0(\delta_0 + \delta_1 T_{max})^2 \theta_0} = L_{2_{min}}(say)$$

Thus, $0 < L_{2_{min}} = \dfrac{\theta\delta_1\mu Q^2}{\mu_0(\delta_0 + \delta_1 T_{max})^2\theta_0} < \dfrac{\theta\delta_1\mu Q^2}{\mu_0\delta_0^2\theta_0} = L_{2_{max}}$.
Hence, from above using Lemma 3.1 we can easily obtain the following conditions:

$$B_{min} \leq \lim_{t\to\infty} inf B(t) \leq \lim_{t\to\infty} sup B(t) \leq B_{max},$$

$$L_{1_{min}} \leq \lim_{t\to\infty} inf L_1(t) \leq \lim_{t\to\infty} sup L_1(t) \leq L_{1_{max}},$$

$$L_{2_{min}} \leq \lim_{t\to\infty} inf L_2(t) \leq \lim_{t\to\infty} sup L_2(t) \leq L_{2_{max}},$$

$$T_{min} \leq \lim_{t\to\infty} inf T(t) \leq \lim_{t\to\infty} sup T(t) \leq T_{max},$$

which shows that the system (3.1) - (3.4) persists uniformly. This is the proof of the theorem. □

3.6. Numerical Simulation The feasibility of analytical results for the system (3.1) - (3.4) is checked using numerical simulation in this section.The description of the model's variable and parameters is given the table 1. For the model, we consider the following suitable set of parameter values,
$s = 0.05$, $L = 10000$, $s_0 = 0.001$, $s_1 = 0.00001$, $s_2 = 0.0000000002$, $Q = 10$, $\delta_0 = 0.05$, $\delta_1 = 0.00001$, $\theta = 0.2$, $\theta_0 = 0.00001$, $\mu = 0.01$, $\mu_0 = 0.005$.
Considering the aforemention set of parameters, the values of variables B, L_1, L_2, T in E^* (non-trivial equilibrium point) are obtained as $B^* = 84193.10711$, $L_1^* = 186.1406$, $L_2^* = 13859.3384$, $T^* = 372.2813$. To understand the variation in variables like B, L_1

Table 1: Description table for various parameters and variables

Description of parameter	Symbol	Units and values
Intrinsic growth rate of biomass density	s	0.05 per year
Carrying capacity of biomass density	L	10000 kg per cubic metre
Natural depletion in biomass density	s_0	0.001 per year
Intrinsic growth rate of biomass density due to fertile land	s_1	0.00001 per square km per year
Rate of contribution to carrying capacity of biomass density due to fertile land	s_2	0.0000000002 cubic meter per kg per square km per year
Constant growth rate of degraded land	Q	10 square km per year
Natural depletion in degraded land	δ_0	0.05 per year
Depletion rate coefficient of fertile land due to technology applied on degraded land	δ_1	0.00001 square metre per Technological effort per year
Growth rate coefficient of fertile land due to technology applied on degraded land	θ	0.2 unit less quantity lying between 0 & 1
Natural depletion in fertile land	θ_0	0.00001 per year
Growth rate coefficient of technology	μ	0.01 Technological efforts per square metre per square km per year
Natural depletion in technology	μ_0	0.005 per year
Time	t	year
Density of biomass resource	B	kg per cubic metre
The area of degraded land	L_1	square km
The area of fertile land	L_2	square km
The magnitude of technology	T	Technological efforts per square metre

& L_2 with time for distinct values of parameters, plots of these variables with time are shown in fig. 2 to fig. 6 . In fig. 2, there is an increase in biomass density as the parameter μ increases while the other parameters are kept fixed. In fig. 3, the fertile land is seen to increase as the parameter θ increases considering the other parameters fixed. In fig. 4, we have shown an increase in biomass density as θ increases taking other parameters fixed. In fig. 5, we see that as we increase the parameter μ and take the remaining parameters fixed, there is a decline in the degraded land. From fig. 6, we conclude that fertile land increases as μ increases taking other parameters fixed.
In figs. 7 and 8, plots of the solutions commencing inside the region of attraction Δ are shown. All the solution curves tend towards the equilibrium value showing the global stability in $L_1 - T$-space and $L_1 - L_2 - T$-space.

3.7. Sensitivity Analysis Herein, we have discussed the semi-relative basic differential sensitivity analysis of the state variable of the model (3.1) - (3.4) with respect to θ, δ_1, μ [20].
Consider, a state variable Y and a parameter u then the semi-relative sensitivity function of Y with respect to u is $Y_u(t,u) = \dfrac{\partial Y(t,u)}{\partial u}$, where

$$\frac{d}{dt}\left(\frac{\partial Y(t)}{\partial u}\right) = \frac{\partial f}{\partial Y}\frac{\partial Y(t)}{\partial u} + \frac{\partial f}{\partial u} \tag{3.32}$$

with initial condition $\dfrac{\partial Y(0)}{\partial u} = 0$. Here, $\dfrac{\partial f}{\partial Y}$ denotes the Jacobian of the model (3.1) - (3.4) and $\dfrac{\partial f}{\partial u}$ denotes derivative of R.H.S. of the model system (3.1) - (3.4). The sensitivity solutions are obtained by solving system (3.32) for $Y_u(t,u)$ after coupling it with the system (3.1) - (3.4). Now, $uY_u(t,u)$ represents the semi-relative basic differential sensitivity solution. The effect of doubling the parameter u on the state variable at different time interval is shown in the plots obtained from semi-relative sensitivity analysis with respect to u.
Now, the semi-relative sensitivity system for state variables with respect to parameter θ is as follows:

$$\begin{aligned}
\dot{B_\theta}(t,\theta) &= s[B_\theta(t,\theta) - 2\frac{B(t,\theta)}{L}B_\theta(t,\theta)] - s_0 B_\theta(t,\theta) + s_1 B_\theta(t,\theta)L_2(t,\theta)\\
&\quad + s_1 B(t,\theta)L_{2_\theta}(t,\theta) + 2s_2 B(t,\theta)B_\theta(t,\theta)L_2(t,\theta) + s_2 B^2(t,\theta)L_{2_\theta}(t,\theta)\\
\dot{L_{1_\theta}}(t,\theta) &= -\delta_0 L_{1_\theta}(t,\theta) - \delta_1 T_\theta(t,\theta)L_1(t,\theta) - \delta_1 T(t,\theta)L_{1_\theta}(t,\theta)\\
\dot{L_{2_\theta}}(t,\theta) &= \delta_1 T(t,\theta)L_1(t,\theta) + \theta\delta_1 T(t,\theta)L_{1_\theta}(t,\theta) + \theta\delta_1 T_\theta(t,\theta)L_1(t,\theta) - \theta_0 L_{2_\theta}(t,\theta)\\
\dot{T_\theta}(t,\theta) &= \mu L_{1_\theta}(t,\theta) - \mu_0 T_\theta(t,\theta)
\end{aligned} \tag{3.33}$$

Similarly, we can find the semi-relative system for state variables with respect to δ_1 and μ. The basic differential sensitivity solution of four state variables B, L_1, L_2, T are plotted relative to the parameters θ, δ_1, μ. These plots are illustrated in fig. 9. From

[2] Note: Table 1 describe the values and units of the various parameters and variables

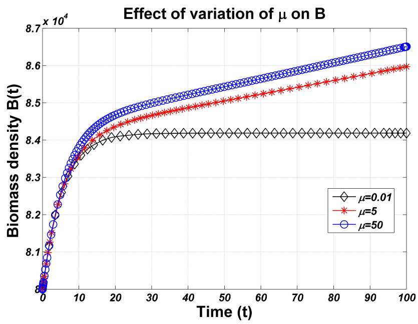


FIGURE 2: Variation in biomass density B for distinct values of μ.

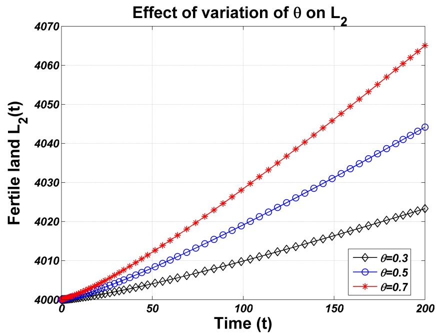


FIGURE 3: Variation in fertile land L_2 for distinct values of θ.

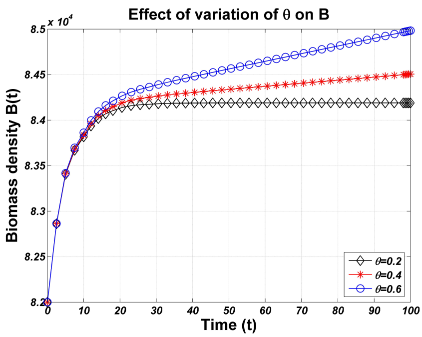


FIGURE 4: Variation in biomass density B for distinct values of θ.

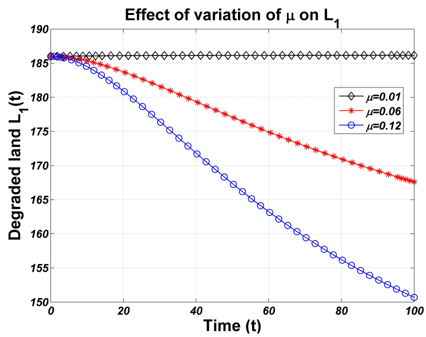


FIGURE 5: Variation in degraded land L_1 for distinct values of μ.

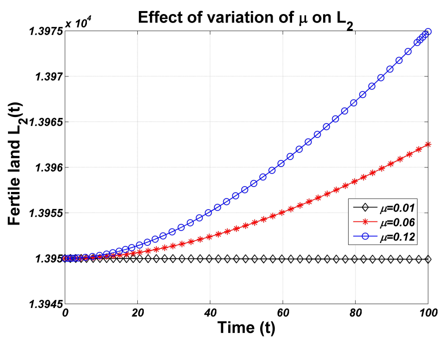


FIGURE 6: Variation in fertile land L_2 for distinct values of μ.

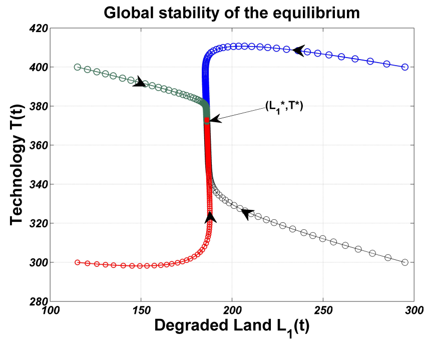


FIGURE 7: Global stability of L_1^*-T^* in $L_1 - T$-space.

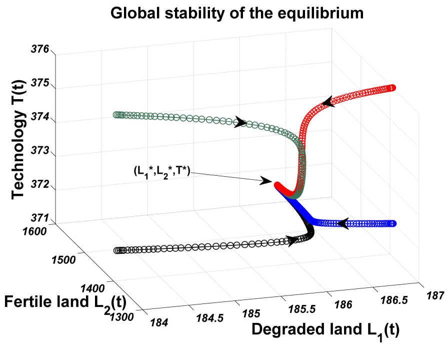
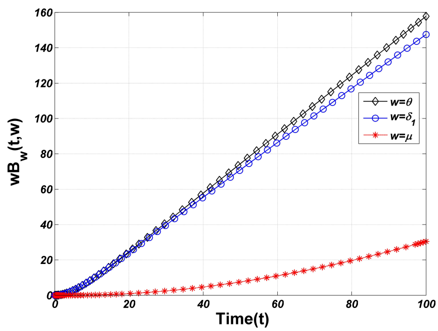
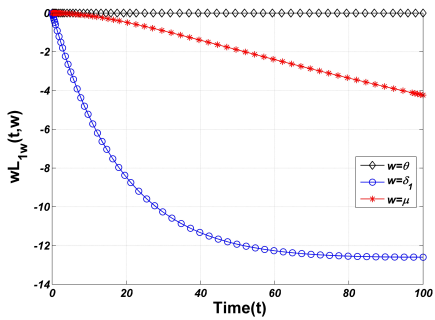


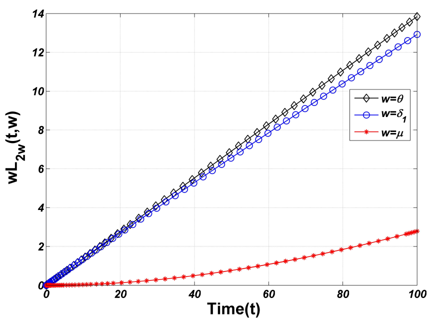
FIGURE 8: Global stability of L_1^*-L_2^*-T^* in $L_1 - L_2 - T$-space.



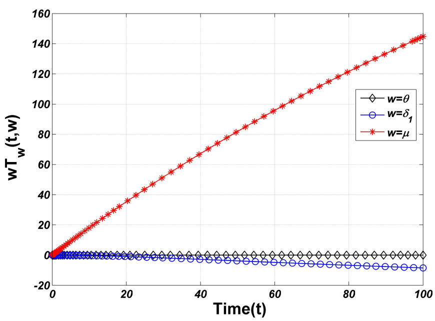
(a) Effect of doubling of parameters on $B(t)$



(b) Effect of doubling of parameters on $L_1(t)$



(c) Effect of doubling of parameters on $L_2(t)$



(d) Effect of doubling of parameters on $T(t)$

FIGURE 9: plots of semi-relative basic differential senstivity taking parameters θ, δ_1, μ for different state variables

plot 9a of fig. 9, we observe an increase in the biomass density due to the doubling of θ, δ_1, μ. Here, we notice that biomass density increases by 74.4 kg/m^3, 70.7 kg/m^3, 7.5 kg/m^3 concerning θ, δ_1, μ respectively, in 50 years. We notice from plot 9b of fig. 9 that there is a negative impact on the area of degraded land due to the doubling of θ, δ_1, μ, this is because fertile land is obtained from degraded land. From plot 9b, we note that L_1 decreases by 11.9 km^2 due to doubling of δ_1 and 1.9 km^2 due to doubling of μ in 50 years. In plot 9c, we observe that L_2 increases by 6.94 km^2, 6.6 km^2 and 0.8 km^2 due to θ, δ_1, μ respectively, in 50 years. From the last plot 9d, we detect that δ_1 and θ decrease the magnitude of technology because degraded land is converted into fertile land by using technology while μ increases the magnitude of technology by 81.5 kg/m^2 in 50 years.

Remark 3.6. *θ is the most influential coefficient for biomass density and fertile land in the system. The policymakers can choose the appropriate value of θ according to the rate of degradation in different conditions like hilly, desert and coastal areas to increase the technological efforts to improve degraded land.*

4. Scenario II. Technology (T) is varying logistically with degraded land (L_1)

4.1. Mathematical Model In this case, technology is varying logistically with degraded land. The model which governs the dynamics of the system consists of nonlinear ordinary differential equations as follows:

$$\frac{dB}{dt} = s(B - \frac{B^2}{L}) - s_0 B + s_1 B L_2 + s_2 B^2 L_2 \tag{4.1}$$
$$\frac{dL_1}{dt} = Q - \delta_0 L_1 - \delta_1 T L_1 \tag{4.2}$$
$$\frac{dL_2}{dt} = \theta\delta_1 T L_1 - \theta_0 L_2 \tag{4.3}$$
$$\frac{dT}{dt} = \mu L_1 T - \mu_0 T^2 \tag{4.4}$$

where, $B(0) \geq 0$, $L_1(0) \geq 0$, $L_2(0) \geq 0$, $T(0) \geq 0$, $s - s_0 > 0$.

Equations (4.1) - (4.4) in the model are the same as the earlier scenario (I) 1.1 but equation (4.4) governs the technology varying logistically with L_1. Here, μL_1 is the intrinsic growth rate and $\frac{\mu L_1}{\mu_0}$ is the carrying capacity. Both carrying capacity & intrinsic growth rate of T increase with an increase in the degraded land L_1.

4.2. Bounds of variables Bounds of variables for the model system (4.1) - (4.4) are given by the lemma, which is as follows:

Lemma 4.1. *The set Δ denotes the region of attraction. All solutions commencing inside Δ remain inside it.*
$\Delta=\{(B,L_1,L_2,T)\in R^4_+: 0 \leq B \leq B_{max},\ 0 \leq L_1 \leq L_{1_{max}},\ 0 \leq L_2 \leq L_{2_{max}},\ 0 \leq T \leq T_{max}\}$,

where, $B_{max} = \dfrac{s + s_1 L_{2_{max}}}{\dfrac{s}{L} - s_2 L_{2_{max}}}$, *provided* $\dfrac{s}{L} - s_2 L_{2_{max}} > 0$,
$L_{1_{max}} = \dfrac{Q}{\delta_0}$, $L_{2_{max}} = \dfrac{\theta\delta_1\mu Q^2}{\mu_0\delta_0^2\theta_0}$, $T_{max} = \dfrac{\mu Q}{\mu_0\delta_0}$.

We skip the proof of lemma 4.1 since its proof is similar to the proof of lemma 3.1 provided in 3.2.

4.3. Equilibrium analysis There are four non-negative real equilibrium points in $B - L_1 - L_2 - T$ space denoted by $E_0(0, \frac{Q}{\delta_0}, 0, 0)$, $E_1(0, \hat{L_1}, \hat{L_2}, \hat{T})$, $E_2(\bar{B}, \frac{Q}{\delta_0}, 0, 0)$ and $E^*(B^*, L_1^*, L_2^*, T^*)$ obtained by putting right hand sides of equations (4.1) - (4.4) equal to zero.

4.3.1. Existence of E_0 Taking $B = 0$ & $T = 0$ we get, $L_1 = \dfrac{Q}{\delta_0}$ and $L_2 = 0$.

4.3.2. Existence of E_1 The existence of E_1 is obvious which can be obtained by taking $B = 0$ & $T \neq 0$ and solving the following algebraic equations:

$$Q - \delta_0 L_1 - \delta_1 T L_1 = 0, \tag{4.5}$$
$$\theta\delta_1 T L_1 - \theta_0 L_2 = 0, \tag{4.6}$$
$$\mu L_1 - \mu_0 T = 0, \tag{4.7}$$

on solving (4.5) - (4.7) we get
$L_1 = \dfrac{Q}{\delta_0 + \delta_1 T}$, $L_2 = \dfrac{\theta\delta_1 T Q}{\theta_0(\delta_0 + \delta T)}$ and a quadratic equation in T i.e.

$$\mu_0\delta_1 T^2 + \mu_0\delta_0 T - Q\mu = 0$$

which gives a unique positive root. Now, the value of the variables at equilibrium point E_1 is as follows:

$$\hat{B} = 0,$$
$$\hat{T} = \frac{-\delta_0\mu_0 + \sqrt{\delta_0^2\mu_0^2 + 4\delta_1\mu_0\mu Q}}{2\delta_1\mu_0}.$$
$$\hat{L_1} = \frac{Q}{\delta_0 + \delta_1\hat{T}},$$
$$\hat{L_2} = \frac{\theta\delta_1\hat{T}Q}{\theta_0(\delta_0 + \delta_1\hat{T})},$$

4.3.3. Existence of E_2 Take $B \neq 0$ & $T = 0$ we get, $\bar{L_1} = \dfrac{Q}{\delta_0}$, $\bar{L_2} = 0$ and $\bar{B} = \dfrac{s - s_0 + s_1 L_2^*}{(\dfrac{s}{L} - s_2 L_2^*)}$.

4.3.4. Existence of E^* The following algebraic equations can be solved to get E^*:

$$s(1-\frac{B}{L}) - s_0 + s_1 L_2 + s_2 B L_2 = 0, \tag{4.8}$$
$$Q - \delta_0 L_1 - \delta_1 T L_1 = 0, \tag{4.9}$$
$$\theta\delta_1 T L_1 - \theta_0 L_2 = 0, \tag{4.10}$$
$$\mu L_1 - \mu_0 T = 0, \tag{4.11}$$

on solving (4.8) - (4.11) we get
$B = \dfrac{s - s_0 + s_1 L_2}{(\frac{s}{L} - s_2 L_2)}$, $L_1 = \dfrac{Q}{\delta_0 + \delta_1 T}$, $L_2 = \dfrac{\theta\delta_1 T Q}{\theta_0(\delta_0 + \delta T)}$ and a quadratic equation in T i.e.

$$\mu_0\delta_1 T^2 + \mu_0\delta_0 T - Q\mu = 0$$

Now, the value of the variables at equilibrium point E^* is as follows:

$$B^* = \frac{s - s_0 + s_1 L_2^*}{(\frac{s}{L} - s_2 L_2^*)}, \text{ provided } \frac{s}{L} - s_2 L_2^* > 0$$
$$L_1^* = \frac{Q}{\delta_0 + \delta_1 T^*},$$
$$L_2^* = \frac{\theta\delta_1 T^* Q}{\theta_0(\delta_0 + \delta_1 T^*)},$$
$$T^* = \frac{-\delta_0\mu_0 + \sqrt{\delta_0^2\mu_0^2 + 4Q\mu\mu_0\delta_1}}{2\delta_1\mu_0}.$$

4.4. Stability analysis

4.4.1. Local stability The following theorem states the local stability analysis of E_0,E_1,E_2 and E^*.

Theorem 4.2 (Local stability). *E_0,E_1,E_2 are always unstable equilibrium points. There is not any constraint required for the interior (non-trivial) equilibrium point E^* to be locally stable.*

Proof. For the model system (4.1) - (4.4), J_0 (Jacobian matrix) at E_0 is given as follows:

$$J_0 = \begin{pmatrix} s - s_0 & 0 & 0 & 0 \\ 0 & -\delta_0 & 0 & -\delta_1\frac{Q}{\delta_0} \\ 0 & 0 & -\theta_0 & \theta\delta_1\frac{Q}{\delta_0} \\ 0 & 0 & 0 & \mu\frac{Q}{\delta_0} \end{pmatrix}$$

Here, $s - s_0$, $-\delta_0$, $-\theta_0$, $\mu\dfrac{Q}{\delta_0}$ are eigen values of J_0. Clearly, $s - s_0$ and $\mu\dfrac{Q}{\delta_0}$ are positive eigen values. Therefore, E_0 is unstable in B & T-direction.
Now, for E_1, the Jacobian matrix J_1 is given as follows:

$$J_1 = \begin{pmatrix} s - s_0 + s_1\hat{L}_2 & 0 & 0 & 0 \\ 0 & -\delta_0 - \delta_1\hat{T} & 0 & -\delta_1\hat{L}_1 \\ 0 & \theta\delta_1\hat{T} & -\theta_0 & \theta\delta_1\hat{L}_1 \\ 0 & \mu & 0 & -\mu_0 \end{pmatrix}$$

Here, $s - s_0 + s_1\hat{L}_2$ is positive eigen value of the matrix J_1. Therefore, E_1 is unstable in B-direction.
Again, for E_2, the Jacobian matrix J_2 is given as follows:

$$J_2 = \begin{pmatrix} s - s_0 & 0 & s_1B + s_2B^2 & 0 \\ 0 & -\delta_0 & 0 & -\delta_1\dfrac{Q}{\delta_0} \\ 0 & 0 & -\theta_0 & \theta\delta_1\dfrac{Q}{\delta_0} \\ 0 & 0 & 0 & \mu\dfrac{Q}{\delta_0} \end{pmatrix}$$

Here, $s - s_0$, $-\delta_0$, $-\theta_0$, $\mu\dfrac{Q}{\delta_0}$ are eigen values of J_2. Clearly, $s - s_0$ and $\mu\dfrac{Q}{\delta_0}$ are positive eigen values. Therefore, E_2 is unstable in B & T-direction.
Now, Jacobian matrix J^* associated to E^* (nontrivial equilibrium point), is as follows:

$$J^* = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & -\delta_0 - \delta_1T^* & 0 & -\delta_1L_1^* \\ 0 & \theta\delta_1T^* & -\theta_0 & \theta\delta_1L_1^* \\ 0 & \mu & 0 & -\mu_0 \end{pmatrix}$$

where,
$a_{11} = B^*(-\dfrac{s}{L} + s_2L_2^*)$, $a_{13} = s_1B^* + s_2B^{*2}$.
The eigen value for this matrix can't be easily evaluated by Routh-Hurwitz criteria. Hence, we use some suitable Lyapunov function to determine local stability of E^*.
First, we linearize the system (4.1) - (4.4) about $E^*(B^*, L_1^*, L_2^*, T^*)$ by using the transformations

$$B = B^* + b, L_1 = L_1^* + l_1, L_2 = L_2^* + l_2, T = T^* + t_1, \tag{4.12}$$

where, b, l_1, l_2, t_1 are small perturbation around E^*.
Take a positive definite function about E^*

$$V = \frac{1}{2}\frac{m_1b^2}{B^*} + \frac{1}{2}m_2l_1^2 + \frac{1}{2}m_3l_2^2 + \frac{1}{2}m_4\frac{t_1^2}{T^*}, \tag{4.13}$$

where, $m_1 > 0$, $m_2 > 0$, $m_3 > 0$ and $m_4 > 0$, to be chosen properly.
Differentiating V taken in equation (4.13) with respect to t yields,

$$\frac{dV}{dt} = \frac{m_1b}{B^*}\frac{db}{dt} + m_2l_1\frac{dl_1}{dt} + m_3l_2\frac{dl_2}{dt} + \frac{m_4t_1}{T^*}\frac{dt_1}{dt}, \tag{4.14}$$

using the model equations in (4.14) and after simplification we get,

$$\frac{dV}{dt} = -m_1[\frac{s}{L} - s_2L_2^*]b^2 - m_2[\delta_0 + \delta_1T^*]l_1^2 - m_3\theta_0 l_2^2 - m_4\mu_0 t_1^2 + [m_1(s_1 + s_2B^*)]bl_2 + [m_4\mu - m_2\delta_1L_1^*]l_1t_1 + m_3[\theta\delta_1L_1^*]l_2t_1 + m_3[\theta\delta_1T^*]l_1l_2. \quad (4.15)$$

$\frac{dV}{dt}$ comes out to be negative definite if, the some constraints are satisfied, which are as follows:

$$[m_1(s_1 + s_2B^*)]^2 < \frac{4}{3}m_1m_3\theta_0(\frac{s}{L} - s_2L_2^*) \quad (4.16)$$

$$[m_4\mu - m_2\delta_1L_1^*]^2 < m_2m_4\mu_0(\delta_0 + \delta_1T^*) \quad (4.17)$$

$$m_3(\theta\delta_1L_1^*)^2 < \frac{2}{3}m_4\theta_0\mu_0 \quad (4.18)$$

$$m_3(\theta\delta_1T^*)^2 < \frac{2}{3}m_2\theta_0(\delta_0 + \delta_1T^*) \quad (4.19)$$

choosing, $m_2 = 1$, $m_4 = \frac{\delta_1L_1^*}{\mu}$, we get,

$$m_1 < \frac{8}{9}\frac{(\frac{s}{L} - s_2L_2^*)\theta_0^2}{(s_1 + s_2B^*)^2\theta^2\delta_1}Min\{\frac{\mu_0}{\mu L_1^*}, \frac{(\delta_0 + \delta_1T^*)}{\delta_1T^{*2}}\}. \quad (4.20)$$

Thus, $\frac{dV}{dt}$ is negative definite and there is not any constraint required for the interior (non-trivial) equilibrium point E^* to be locally stable.

□

4.4.2. Global stability The summarization of global stability behavior of E^* is provided in the following theorem.

THEOREM 4.3 (Global stability). *There is not any constraint required for the interior (non-trivial) equilibrium point E^* to be globally stable inside the region of attraction Δ.*

PROOF. Take a positive definite function

$$W = k_1(B - B^* - B^*\ln\frac{B}{B^*}) + \frac{k_2}{2}(L_1 - L_1^*)^2 + \frac{k_3}{2}(L_2 - L_2^*)^2 + k_4(T - T^* - T^*\ln\frac{T}{T^*}), \quad (4.21)$$

where, $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, $k_4 > 0$, to be chosen properly. The function W is considered after checking that it is zero at equilibrium $E^*(B^*, L_1^*, L_2^*, T^*)$ and it is positive for all different positive values of B, L_1, L_2 and T.
Now, differentiating W taken in equation (4.21) with respect to t we get,

$$\begin{aligned}\dot{W} = &-k_1(\frac{s}{L} - s_2L_2^*)(B - B^*)^2 - k_2(\delta_0 + \delta_1T^*)(L_1 - L_1^*)^2\\ &- k_3\theta_0(L_2 - L_2^*)^2 - k_4\mu_0(T - T^*)^2 + [k_1(s_1 + s_2B)](B - B^*)(L_2 - L_2^*)\\ &+ [k_4\mu - k_2\delta_1L_1](L_1 - L_1^*)(T - T^*) + k_3\theta\delta_1L_1(T - T^*)(L_2 - L_2^*)\\ &+ k_3\theta\delta_1T^*(L_1 - L_1^*)(L_2 - L_2^*). \end{aligned} \tag{4.22}$$

$\dot{W}$ comes out to be positive definite [21] in the region of attraction Δ if, the following constraints are satisfied:

$$[(s_1 + s_2B_{max})k_1]^2 < \frac{4}{3}k_1k_3\theta_0(\frac{s}{L} - s_2L_2^*) \tag{4.23}$$

$$[k_4\mu - k_2\delta_1\frac{Q}{\delta_0}]^2 < k_2k_4\mu_0(\delta_0 + \delta_1T^*) \tag{4.24}$$

$$k_3(\theta\delta_1\frac{Q}{\delta_0})^2 < \frac{2}{3}k_4\mu_0\theta_0 \tag{4.25}$$

$$k_3(\theta\delta_1T^*)^2 < \frac{2}{3}k_2\theta_0(\delta_0 + \delta_1T^*) \tag{4.26}$$

Choosing, $k_2 = 1$, $k_4 = \dfrac{\delta_1Q}{\mu\delta_0}$, we get,

$$k_1 < \frac{8}{9}\frac{\theta_0^2(\frac{s}{L} - s_2L_2^*)}{(s_1 + s_2B_{max})^2(\theta^2\delta_1)}Min\{\frac{\mu_0\delta_0}{Q\mu}, \frac{(\delta_0 + \delta_1T^*)}{\delta_1T^{*2}}\}. \tag{4.27}$$

Thus, $\dfrac{dW}{dt}$ is negative definite and there is not any constraint required for the interior (non-trivial) equilibrium point E^* to be globally stable inside the region of attraction Δ. □

4.5. Model's Persistence The persistence of the system (4.1) - (4.4) is provided in the following theorem:

Theorem 4.4 (Persistence). *The uniform persistence of the system* (4.1) - (4.4) *for increasing biomass density by improving degraded land is hold if* $\frac{s}{L} - s_2L_{2_{max}} > 0$ *where,* $L_{2_{max}}$ *is the upper bound of the fertile land.*

We skip the proof of theorem 4.4 since it is similar to the proof of theorem 3.5 given in 3.5 .

4.6. Numerical Simulation The feasibility of analytical results for the system (4.1) - (4.4) is illustrated using numerical simulation in this section. For the model, we consider the following suitable set of parameter values,
$s = 0.05$, $L = 10000$, $s_0 = 0.001$, $s_1 = 0.00001$, $s_2 = 0.0000000002$, $Q = 10$, $\delta_0 = 0.05$, $\delta_1 = 0.00001$, $\theta = 0.2$, $\theta_0 = 0.00001$, $\mu = 0.01$, $\mu_0 = 0.005$.
Considering the aforemention set of parameters, the values of variables B, L_1, L_2, T in E^* (non-trivial equilibrium point) find out as $B^* = 84193.10711$, $L_1^* = 186.1406$, $L_2^* = 13859.3384$, $T^* = 372.2813$. To understand the variation in variables like B, L_1 & L_2 with time for distinct values of parameters, plots of these variables with time are shown in fig. 10 to fig. 14. In fig. 10, there is an increase in biomass density as the parameter μ increases considering other parameters fixed. In fig. 11, the fertile land is seen to increase as the parameter θ increases while the other parameters are kept fixed. In fig. 12, we have shown an increase in biomass density as θ increases taking other parameters fixed. In fig. 13, we see that as we increase the parameter μ and take the remaining parameters fixed, there is a decline in the degraded land. From fig. 14, we conclude that fertile land increases as the μ increases taking other parameters fixed.
In figs. 15 and 16, plots of the solutions commencing inside the region of attraction Δ are shown. All the solution curves tend toward the equilibrium value showing the global stability in $L_1 - T$-space and $L_1 - L_2 - T$-space.

4.7. Sensitivity Analysis As we have done in the subsection 3.7, the semi-relative sensitivity system for state variables of model equations (4.1) - (4.4) concerning parameters θ, δ_1 and μ can be obtained. The basic differential sensitivity solution of four state variables B, L_1, L_2, T are plotted relative to the parameters θ, δ_1, μ. These plots are illustrated in the fig. 17. From plot 17a of fig. 17, we discover an increase in the biomass density due to the doubling of θ, δ_1, μ. Here, we notice that biomass density increases by 74.7 kg/m^3, 67.6 kg/m^3, 67.6 kg/m^3 for θ, δ_1, μ respectively, in 50 years. We notice from plot 17b of fig. 17 that there is a negative impact on the area of degraded land due to the doubling of θ, δ_1, μ, this is because fertile land obtains from degraded land. From plot 17b, we note that L_1 decreases by 9.6 km^2 due to doubling of δ_1 and μ in 50 years. In plot 17c, we observe that L_2 increases by 6.94 km^2, 6.27 km^2 and 6.27 km^2 due to θ, δ_1, μ in 50 years. From the last plot 17d, we detect that δ_1 and θ decrease the magnitude of technology because degraded land is converted into fertile land by using technology while μ increases the magnitude of technology by 349.6 kg/m^2 in 50 years.

5. Conclusion

The world's population is growing swiftly. In the present circumstances, it is a huge challenge to meet the elementary needs of the growing population. To prevail over this situation, we need to adopt various methods or techniques to increase biomass production. Based on this thought we have proposed and analysed two non-linear mathematical models for increasing the biomass density by re-obtaining fertile land from degraded land using technology on degraded land. In these models, four variables

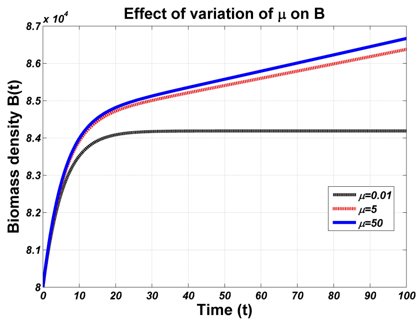


FIGURE 10: Variation in biomass density B for distinct values of μ.

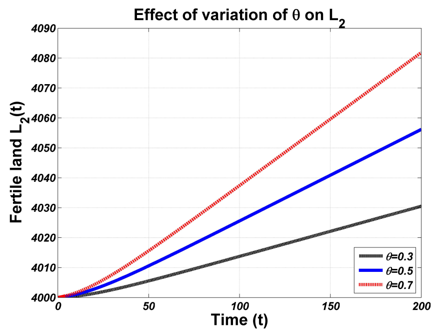


FIGURE 11: Variation in fertile land L_2 for distinct values of θ.

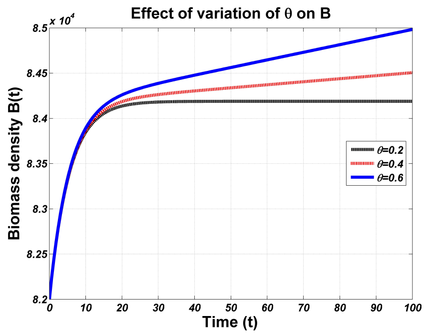


FIGURE 12: Variation in biomass density B for distinct values of θ.

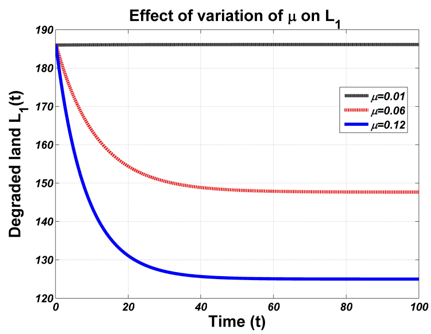


FIGURE 13: Variation in degraded land L_1 for distinct values of μ.

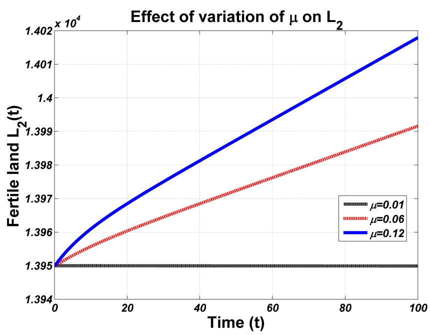


FIGURE 14: Variation in fertile land L_2 for distinct values of μ.

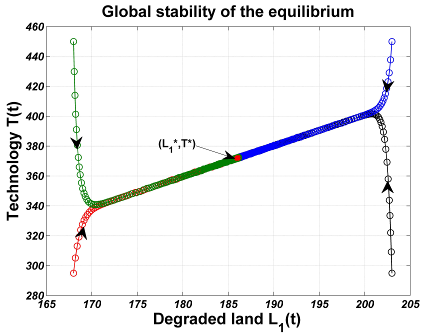


FIGURE 15: Global stability of L_1^*-T^* in $L_1 - T$-space.

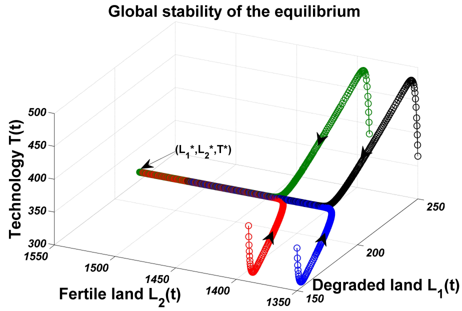
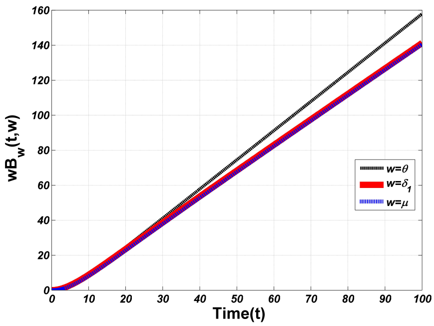
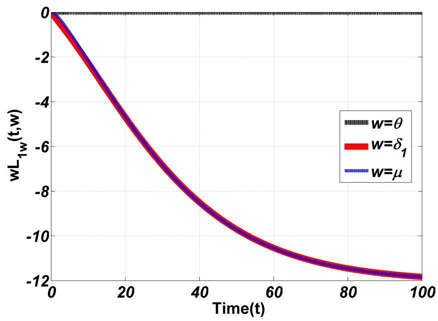


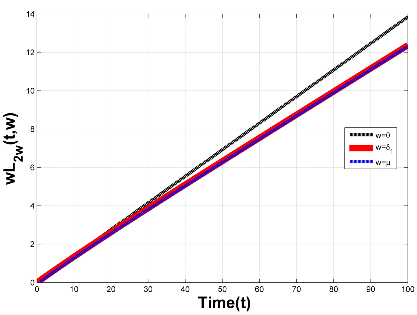
FIGURE 16: Global stability of L_1^*-L_2^*-T^* in $L_1 - L_2 - T$-space.



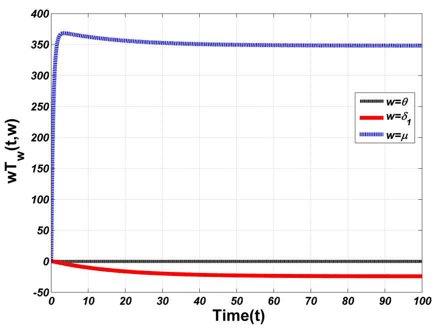
(a) Effect of doubling of parameters on $B(t)$



(b) Effect of doubling of parameters on $L_1(t)$



(c) Effect of doubling of parameters on $L_2(t)$



(d) Effect of doubling of parameters on $T(t)$

FIGURE 17: plots of semi-relative basic differential sensitivity taking parameters θ, δ_1, μ for different state variables

are considered. These variables are the biomass density, the area of the degraded land, the area of the fertile land and the technology. The following two cases for use of technology are considered:

Scenario I- Technology linearly varying with degraded land.

Scenario II- Technology logistically (non linearly) varying with degraded land.

The model is analysed by using equilibrium analysis & the stability theory (local and global stability), persistence of system and sensitivity analysis of differential equation. In scenario I, two equilibrium points are obtained. One is stable and another is unstable. In scenario II, we have found four equilibrium points. Only one is stable and remaining three are unstable. The effect of various parameters on equilibrium points is shown in the numerical simulation. All the solution trajectories converge to the equilibrium point with increase in time. we have examined the effect of doubling vital parameters on the state variables in the sensitivity analysis. In both the cases, key parameters increases the biomass density positively and there is a negative effect on degraded land because it is used to obtain fertile land by applying technology. In both cases, analysis of models reveals that by using technology, degraded land can be converted into fertile, leading to an increase in the biomass density.

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Methodology: Deepika Marwar
Writing-Original Draft: Deepika Marwar.
All authors have equal contribution in manuscript formation.

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