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A PÁL-TYPE INTERPOLATION ON THE ROOTS OF THE INTEGRATED LEGENDRE POLYNOMIAL

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Abstract

The purpose of this paper is to study an interpolation process on the roots of polynomial $\pi_n(x)$and it's derivative $\pi_n'(x)$ with an additional conditional point $x_0 = 0$. Here, we have two sets of nodes $\{x_i\}_{i=1}^n$ and $\{x_i^*\}_{i=1}^{n-1}$, which are the roots of polynomials $\pi_n(x)$ and $\pi_n'(x)$, respectively. Further, we study the existence, uniqueness, explicit representation and order of convergence of interpolatory polynomial.

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1. Introduction

Pál [14], Mathur P. and Datta S. [12] and many other authors [1], [4], [7], [11] have discussed various kinds of interpolation problem. In 1975, Pál [8] proved that when the values are fixed on one set of n points and derivative values on other set of $n-1$ points, then there exist no unique polynomial $\leq 2n-2$, but prescribing function value at one more point not belonging to above set of n points there exists a unique polynomial of degree $\leq 2n-1$. In [16], Eneduanya investigated special case when

$$\pi_n(x) = -n(n-1)\int_1^x P_{n-1}(x)dx = (1-x^2)P_{n-1}'(x), \tag{1.1}$$

where $P_{n-1}(x)$ is the $(n-1)th$ the Legendre polynomial with the usual normalization $\max\{|P_{n-1}(x)| : x \in [-1,1]\} = 1$. For the uniqueness, Eneduanya used also the additional condition nodal points $x_n^* = -1$. Szili [10] investigated the Pál-type interpolation on the roots of the Hermite-polynomials with the additional conditional point $x_0 = 0$. Both Szili and Eneduanya gave explicit formula and proved approximation theorems. Joó and Szabó [3] gave a common generalization of the classical Fejér interpolation and Pál interpolation. Szili[9] studied the inverse Pál interpolation problem on the roots of integrated Legendre polynomials. Later, R.Srivastava and Yamini Singh [15] studied an interpolation process on the roots of ultraspherical polynomials.

In this paper, we have studied an interpolation on the roots of polynomials $\pi_n(x)$. Let $x_{0,n}, x_{1,n}, x_{2,n}, \ldots, x_{n,n}$ be the roots of the polynomial $\pi_n(x)$ and $x^*_{1,n}, x^*_{2,n}, x^*_{3,n}, ..., x^*_{n-1,n}$ be the roots of polynomial $\pi'_n(x)$. Let

$$-1 = x_{n,n} < x^*_{n-1,n} < x_{n-1,n} < ... < x_{2,n} < x^*_{1,n} < x_{1,n} = 1. \tag{1.2}$$

Further, we investigate the following problem by assuming a polynomial $R_n(x)$ of lowest possible degree satisfying the conditions,

$$R_n(x_{i,n}) = y_{i,n} \quad (i = 1, ...n),\ R_n(x_{0,n}) = 0,\ \textit{and}\ R'_n(x^*_{i,n}) = y'_{i,n} \quad (i = 1, 2, ..., n-1), \tag{1.3}$$

where, $y_{i,n}$ and $y'_{i,n}$ are arbitrary given real numbers. Moreover, we prove the existence, uniqueness, explicit representation and order of convergence of interpolatory polynomials.

2. Preliminaries

According to [7], the following relationships for Legendre polynomials are observed as:

1. $P'_{n-1}(1) = \frac{1}{2}n(n-1) = (-1)^n P'_{(n-1)}(-1)$
2. $P''_{n-1}(1) = \frac{1}{2}n(n-1)(n+1)P'_{n-1}(1)$
3. $\pi'_n(1) = (-1)^{n-1}\pi'_n(-1) = -n(n-1)$
4. $\pi''_n(1) = -\frac{1}{2}n^2(n-1)^2.$

Furthermore, $P_{n-1}(x)$ and $\pi_n(x)$ satisfy the following differential equation, respectively

$$\begin{aligned}(1-x^2)P''_{n-1}(x) - 2xP'_{n-1}(x) + n(n-1)P_{n-1}(x) = 0,\\ 1-x^2)\pi''_n(x) + n(n-1)\pi_n(x) = 0.\end{aligned} \tag{2.1}$$

3. Explicit Representation of Interpolatory Polynomial

Let us consider the following polynomials for even values of n,

$$A_{0,n}(x) = \frac{\pi_n(x)}{\pi_n(0)}, \tag{3.1}$$

$$\begin{aligned} A_{i,n}(x) = {} & \frac{x\pi'_n(x)\pi_n(x)}{x_{i,n}\left(\pi'_n(x_{i,n})\right)^2(x-x_{i,n})} - \frac{\pi_n(x)}{x_{i,n}\left(\pi'_n(x_{i,n})\right)^3} \\ & \times \left\{ \int_0^x \frac{t\pi''_n(t)\pi'_n(x_{i,n}) - x_{i,n}\pi''_n(x_{i,n})\pi'_n(t)}{(t-x_{i,n})}\,dt \right\}, \\ & (i = 1, 2, \ldots, n) \end{aligned} \tag{3.2}$$

and

$$B_{i,n}(x) = \frac{\pi_n(x)(1 - x_{i,n}^{*2})}{n(n-1)\pi_n^2(x_{i,n}^*)} \int_0^x \frac{\pi_n'(t)}{(t - x_{i,n}^*)} dt, \qquad (i = 1, 2, \ldots, n-1). \tag{3.3}$$

The polynomials $A_{i,n}(x)$ and $B_{i,n}(x)$ are uniquely determined by following conditions:

$$\begin{aligned} A_{i,n}(x_{j,n}) &= \delta_{ij} \quad (i = 0, 1, 2, 3, \ldots, n; j = 0, 1, 2, 3, \ldots, n), \\ A'_{i,n}(x^*_{j,n}) &= 0 \quad (i = 0, 1, 2, 3, \ldots, n; j = 1, 2, 3, \ldots, n-1), \\ B_{i,n}(x_{j,n}) &= 0 \quad (i = 1, 2, 3, \ldots, n-1; j = 0, 1, 2, 3, \ldots, n), \\ B'_{i,n}(x^*_{j,n}) &= \delta_{ij} \quad (i = 1, 2, 3, \ldots, n-1; j = 1, 2, 3, \ldots, n-1), \end{aligned}$$

where $\delta_{i,j}$ is the kronecker symbol. Now, let $f : [-1, 1] \to \mathbb{R}$ be a differentiable function. If n is even, then we get that

$$R_n(f, x) = \sum_{i=0}^{n} f(x_{i,n}) A_{i,n}(x) + \sum_{i=1}^{n-1} f'(x^*_{i,n}) B_{i,n}(x), \tag{3.4}$$

is the uniquely determined polynomial of degree $\leq 2n - 1$ satisfying the condition

$$\begin{aligned} R_n(f; x_{i,n}) &= f(x_{i,n}) \quad (i = 1, 2, 3, ..., n), \\ R'_n(f; x^*_{i,n}) &= f'(x^*_{i,n}) \quad (i = 1, 2, 3, ..., n-1), \\ R_n(f; x_{0,n}) &= 0. \end{aligned}$$

Note: For conciseness, we use subscript (i), in place of subscript (i,n).

Lemma 3.1. *The following estimates hold:*

$$\left|\pi_n(x_i^*)\right| \geq \left[\frac{(1 - x_i^{*2}) n(n-1)}{8\pi(i+1)}\right]^{\frac{1}{2}} \qquad (i = 1, 2, 3, ..., (n-2)/2), \tag{3.5}$$

$$\left|\pi_n(x^*_{n/2})\right| = |\pi_n(0)| > \frac{n^{1/2}}{3}, \tag{3.6}$$

$$\left|\pi_n(x_i^*)\right| \geq \left[\frac{(1 - x_i^{*2}) n(n-1)}{8\pi(i+1)}\right]^{\frac{1}{2}} \qquad (i = 1, 2, 3, ..., (n+2)/2). \tag{3.7}$$

The proof of this lemma can be found in [9].

Lemma 3.2. *For the Lebesgue function of the fundamental polynomials*

$$\sum_{i=0}^{n} |A_i(x)| = O(n^{3/2}) \qquad (x \in [-1, 1],\ n = 2, 4, ...),$$

where O does not depend on x.

Proof. We have

$$A_i(x) = \frac{x\pi_n'(x)\pi_n(x)}{x_i(\pi_n'(x_i))^2(x-x_i)} - \frac{\pi_n(x)}{x_i(\pi_n'(x_i))^3}\left\{\int_0^x \frac{t\pi_n''(t)\pi_n'(x_i) - x_i\pi_n''(x_i)\pi_n'(t)}{t-x_i}\,dt\right\}$$

$$\sum_{i=0}^{n}|A_i(x)| = \sum_{i=0}^{n}\left|\frac{x\pi_n'(x)\pi_n(x)}{x_i(\pi_n'(x_i))^2(x-x_i)}\right| - \sum_{i=0}^{n}\left|\frac{\pi_n(x)}{x_i(\pi_n'(x_i))^3}\left\{\int_0^x \frac{t\pi_n''(t)\pi_n'(x_i) - x_i\pi_n''(x_i)\pi_n'(t)}{t-x_i}\,dt\right\}\right|$$

since $|\pi_n(x)| = O(n^{1/2})$ and $|P_{n-1}(x)| \le 1 \qquad x \in [-1,1]$ (from [5, 2.3.4]).

$$\sum_{i=0}^{n}|A_i(x)| = \sum_{i=0}^{n}\frac{|x|\left|\pi_n'(x)\right||\pi_n(x)|}{x_i\left|(\pi_n'(x_i))^2\right||(x-x_i)|} - \sum_{i=0}^{n}\frac{|\pi_n(x)|}{x_i\left|(\pi_n'(x_i))^3\right|}\left|\int_0^x \frac{t\pi_n''(t)\pi_n'(x_i) - x_i\pi_n''(x_i)\pi_n'(t)}{t-x_i}\,dt\right|$$

$$\sum_{i=0}^{n}|A_i(x)| = \sum_{i=0}^{n}\frac{|x|\left|\pi_n'(x)\right||\pi_n(x)|}{x_i\left|(\pi_n'(x_i))^2\right||(x-x_i)|} - \sum_{i=0}^{n}\frac{|\pi_n(x)|}{x_i\left|(\pi_n'(x_i))^3\right|}\left|\int_0^x \frac{t\pi_n''(t)\pi_n'(x_i) - x_i\pi_n''(x_i)\pi_n'(t)}{t-x_i}\,dt\right|$$

$$\sum_{i=0}^{n}|A_i(x)| = D_1 + D_2.$$

For next estimation, we use following relations

$$\int_{-1}^{1}\frac{P_{n-1}(t)}{t-x_i^*}dt = \frac{2}{(1-x_i^{*2})\left|P_{n-1}'(x_i^*)\right|} \qquad (i = 1,2,\ldots,n-1), \tag{3.8}$$

(from ([2],(3.4.3)*and* (15.3.1)),

$$|P_{n-1}'(x_i^*)| \sim i^{\frac{-3}{2}}n^2 \quad (i = 1,2,3,........n/2), \tag{3.9}$$

(from ([2], (8.9.2)]),

$$(1-x_i^{*2}) \sim (i/n)^2, \tag{3.10}$$

(from ([2], (6.3.7)),

$$|P_{n-1}(x_i)| = (8\pi i)^{-1/2}, \tag{3.11}$$

(from ([5],Lemma 2.1).
Now,

$$D_1 = \sum_{i=0}^{n}\frac{|x|\left|\pi_n'(x)\right||\pi_n(x)|}{x_i\left|(\pi_n'(x_i))^2\right||(x-x_i)|},$$

using equation (1.1)

$$D_1 = O(n^{1/2})\sum_{i=0}^{n}\frac{|x|\,n(n-1)\,|P_{n-1}(x)|}{x_i n^2(n-1)^2\left|P_{n-1}^2(x_i)\right||(x-x_i)|} = O(n^{3/2}),$$

$$D_2 = \sum_{i=0}^{n} \frac{|\pi_n(x)|}{x_i \left|(\pi_n'(x_i))^3\right|} \left|\int_0^x \frac{t\pi_n''(t)\pi_n'(x_i) - x_i\pi_n''(x_i)\pi_n'(t)}{t - x_i}\, dt\right|$$

.

$$D_2 = O(n^{1/2}) \sum_{i=0}^{n} \frac{1}{x_i\, n^3(n-1)^3 \left|P_{n-1}(x_i)^3\right|} \times \left|\int_0^x \frac{t\, n^2(n-1)^2 P_{n-1}'(t)P_{n-1}(x_i) - x_i\, n^2(n-1)^2 P_{n-1}'(x_i)P_{n-1}(t)}{t - x_i}\, dt\right|$$

$$D_2 = O(n^{1/2}) \sum_{i=0}^{n} \frac{n^2(n-1)^2}{x_i\, n^3(n-1)^3\, |P_{n-1}(x_i)^3|} \left|\int_0^x \frac{t\, P_{n-1}'(t)P_{n-1}(x_i) - x_i\, P_{n-1}'(x_i)P_{n-1}(t)}{t - x_i}\, dt\right|$$
$$= O(n^{-1/2}).$$

$$\sum_{i=0}^{n} |A_i(x)| = O(n^{3/2}).$$

Thus, the proof of Lemma 3.2 is completed.

□

LEMMA 3.3. *For the Lebesgue function of the fundamental polynomials $B_{i,n}$ the following estimate holds:*

$$\sum_{i=1}^{n-1} |B_i(x)| = O(n^{-1}) \quad (x \in [-1, 1],\ n = 2, 4, \dots),$$

where O does not depend on x.

PROOF. We have

$$B_i(x) = \frac{\pi_n(x)(1 - x_i^{*2})}{n(n-1)\pi_n^2(x_i^*)} \int_0^x \frac{\pi_n'(t)}{t - x_i^*}\, dt$$

$$\sum_{i=1}^{n-1} |B_i(x)| = \sum_{i=1}^{n-1} \left|\frac{\pi_n(x)(1 - x_i^{*2})}{n(n-1)\pi_n^2(x_i^*)} \int_0^x \frac{\pi_n'(t)}{t - x_i^*}\, dt\right|$$

$$\sum_{i=1}^{n-1} |B_i(x)| = \sum_{i=1}^{n-1} \frac{|\pi_n(x)| \left|1 - x_i^{*2}\right|}{\left|\pi_n^2(x_i^*)\right| \cdot |n(n-1)|} \left|\int_{-1}^x \frac{\pi_n'(t)}{t - x_i^*}\, dt\right|$$

$$\sum_{i=1}^{n-1}|B_i(x)| = O(n^{1/2})\sum_{i=1}^{n-1}\frac{\left|(1-x_i^{*2})\right|}{\left|(1-x_i^{*2})^2\right|\left|(P'_{(n-1)}(x_i^*))^2\right|n(n-1)}\left|\int_0^x \frac{n(n-1)P_{n-1}(t)}{(t-x_i^*)}dt\right|$$

$$\sum_{i=1}^{n-1}|B_i(x)| = O(n^{1/2})\sum_{i=1}^{n-1}\frac{1}{\left|(1-x_i^{*2})\right| \quad |(P'_{(n-1)}(x_i^*))^2|}\left|\int_0^x \frac{P_{n-1}(t)}{(t-x_i^*)}dt\right| = O(n^{-1}).$$

Hence, the Lemma 3.3 proved . □

4. Theorem

THEOREM 4.1. *Let $f : [-1,1] \to \mathbb{R}$ be continuously differentiable function, then the sequence of the interpolation polynomials $R_n(f;x)\,(n = 2,4,6,......)$ given by (3.4) satisfy the following:*

$$|R_n(f;x) - f(x)| = O\left(n^{1/2}w\left(f';\frac{1}{n}\right)\right) \quad x \in [-1,1], \tag{4.1}$$

where $w(f',\delta)$ is the modulus of continuity of f' and O does not depend on x.

PROOF. If $Q_n(x)$ is an arbitrary polynomial of degree $\leq 2n-1$ then by uniqueness of the polynomial R_n, we have

$$Q_n(x) = \sum_{i=0}^{n} Q_n(x_i)A_i(x) + \sum_{i=1}^{n-1} Q'_n(x_i^*)B_i(x). \tag{4.2}$$

Let $f : [-1,1] \to \mathbb{R}$ be a continuously differentiable function. It is well known from, e.g.([2],Theorem1.3.3) that there exist a polynomial $Q_n(x)$ of degree at most (2n-1) such that

$$|f(x) - Q_n(x)| = O\left(n^{-1}w\left(f';\frac{1}{n}\right)\right)$$

and

$$\left|f'(x) - Q'_n(x)\right| = O\left(w\left(f';\frac{1}{n}\right)\right), \quad x \in [-1,1],$$

then by equation (4.2), we get

$$|f(x) - R_n(f;x)| \leq |f(x) - Q_n(x)| + \left|\sum_{i=0}^{n}(Q_n(x_i) - f(x_i))\,A_i(x)\right|$$
$$+ \left|\sum_{i=1}^{n-1}(Q'_n(x_i^*) - f'(x_i^*))\,B_i(x)\right|.$$

Based on Lemmas (3.2) and (3.3), it follows that

$$|f(x) - R_n(f;x)| = O\left(n^{-1}w\left(f';\frac{1}{n}\right)\right) + O\left(n^{1/2}w\left(f';\frac{1}{n}\right)\right) + O\left(n^{-1}w\left(f';\frac{1}{n}\right)\right),$$

which completes the proof of theorem 4.1. □

5. Conclusion

In this paper, we have proved the existence, uniqueness, explicit representation, and order of convergence of the given interpolatory problem, when $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$ are the roots of polynomials $\pi_n(x)$ and $\pi'_n(x)$ respectively, with additional conditional point. If $f : [-1, 1] \to \mathbb{R}$ be continuously differentiable function, then the sequence of the interpolation polynomials $R_n(f;x)$ and $R'_n(f;x)$ uniformaly converge to $f(x)$ and $f'(x)$ respectively on [-1,1] as $n \to \infty$.

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Conflicts of Interest:
The authors declare no conflict of interest.

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