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CENTRAL INDEX ORIENTED GROWTH ANALYSIS OF COMPOSITE ENTIRE FUNCTIONS IN TERMS OF (α,β)-ORDER

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Abstract

Belaïdi et al. [1] have introduced the definitions of (α,β)-order and (α,β)-lower order of entire function using central index. In this paper, we have discussed on central index oriented some growth properties of composite entire functions on the basis of their (α,β)-order and (α,β)-lower order, and have generalized some previous works in this line.

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1. Introduction

Let $f = \sum_{n=0}^{+\infty} a_n z^n$ be an entire function defined on the finite complex plane $\mathbb{C}$. Also, let $M_f(r)$ and $\mu_f(r)$ respectively denote the maximum modulus function and the maximum term function of f, which are defined as $M_f = \max_{|z|=r} |f(z)|$ and $\mu_f = \max_{n\geq 0} (|a_n| r^n)$. The central index $\nu_f(r)$ of an entire function f is the greatest exponent n for which $|a_n| r^n = \mu_f(r)$. Clearly, like $M_f(r)$ and $\mu_f(r)$, $\nu_f(r)$ is also real and increasing function of r. Though $\nu_f(r)$ is much weaker than $M_f(r)$ and $\mu_f(r)$ in some sense, from another angle of view $\frac{\nu_f(r)}{\nu_h(r)}$ is called the growth of f with respect to h in terms of the central index. Order and lower order are classical growth indicators of entire function in complex analysis. Several authors have made the close investigations on the growth properties of entire functions in different directions using the concepts of order, iterated p-order [9, 10], (p,q)-th order [8], (p,q)-φ order [11], generalized (α,β) order [4, 5] using maximum modulus functions and the maximum term functions and achieved many valuable results. In this paper, we have studied some growth properties relating to the composition of entire functions on the basis of (α,β)-order and (α,β)-lower order in terms of central index. In fact some works in this area have also been explored in [2, 3, 6]. The standard notations and definitions of the theory of entire functions are available in [12, 13] and therefore we do not explain those in details. Now to start our paper, we just recall the following definition:

Definition 1.1. The order ρ_f and the lower order λ_f of an entire function f are defined as:

$$\rho_f = \limsup_{r\to+\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to+\infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

He and Xiao [7] gave the alternative definitions of order and lower order of entire function in terms of its central index which are as follows:

Definition 1.2. [7] The order ρ_f and the lower order λ_f of an entire function f in terms of its central index are defined as:

$$\rho_f = \limsup_{r\to+\infty} \frac{\log \nu_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to+\infty} \frac{\log \nu_f(r)}{\log r}.$$

Now let L be a class of continuous non-negative functions α on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$.

Further we assume that throughout the present paper α, α_1, α_2, β, β_1 and β_2 always denote the functions belonging to L^0. Now considering this, Belaïdi et al. [1] have introduced the definitions of (α,β)-order and (α,β)-lower order of an entire function f in terms of its central index which are as follows:

Definition 1.3. [1] The (α,β)-order denoted by $\rho_{(\alpha,\beta)}[f]$ and (α,β)-lower order denoted by $\lambda_{(\alpha,\beta)}[f]$ of an entire function f are defined as:

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log(\nu_f(r)))}{\beta(\log r)}$$
$$\text{and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\log(\nu_f(r)))}{\beta(\log r)}.$$

Remark 1.4. *Let $\alpha(r) = \beta(r) = r$, the Definition 1.3 coincides with the Definition 1.2.*

2. Main results

In this section, we present the main results of the paper.

Theorem 2.1. *Let f and h are entire functions such that* $0 < \lambda_{(\alpha,\beta)}[f] \leq \rho_{(\alpha,\beta)}[f] < +\infty$ *and* $\lambda_{(\alpha,\beta)}[h \circ f] = +\infty$. *Then*

$$\lim_{r\to+\infty} \frac{\alpha(\log(\nu_{h\circ f}(r)))}{\alpha(\log(\nu_f(r)))} = +\infty.$$

Proof. If possible, let the conclusion of the theorem does not hold. Then we can find a constant $K > 0$ such that for a sequence of values of r tending to infinity

$$\alpha(\log(\nu_{h\circ f}(r))) \leq K \cdot \alpha(\log(\nu_f(r))). \tag{2.1}$$

Now from the definition of $\rho_{(\alpha,\beta)}[f]$, it follows for all sufficiently large values of r that

$$\alpha(\log(\nu_f(r))) \leq (\rho_{(\alpha,\beta)}[f] + \epsilon)\beta(\log r). \tag{2.2}$$

From (2.1) and (2.2), for a sequence of values of r tending to $+\infty$, we have

$$\alpha(\log(\nu_{h\circ f}(r))) \leq K(\rho_{(\alpha,\beta)}[f] + \epsilon)\beta(\log r),$$

$$i.e., \ \frac{\alpha(\log(\nu_{h\circ f}(r)))}{\beta(\log r)} \leq K(\rho_{(\alpha,\beta)}[f] + \epsilon),$$

$$i.e., \ \liminf_{r\to+\infty} \frac{\alpha(\log(\nu_{h\circ f}(r)))}{\beta(\log r)} < +\infty,$$

$$i.e., \ \lambda_{(\alpha,\beta)}[h \circ f] < +\infty.$$

This is a contradiction.

Thus the theorem follows. □

Remark 2.2. *Theorem 2.1 is also valid with "limit superior" instead of "limit" if "$\lambda_{(\alpha,\beta)}[h \circ f] = +\infty$" is replaced by "$\rho_{(\alpha,\beta)}[h \circ f] = +\infty$" and the other conditions remain the same.*

Theorem 2.3. *Let f and h are entire functions such that $0 < \lambda_{(\alpha,\beta)}[h\circ f] \leq \rho_{(\alpha,\beta)}[h\circ f] < +\infty$ and $0 < \lambda_{(\alpha,\beta)}[f] \leq \rho_{(\alpha,\beta)}[f] < +\infty$. Then*

$$\frac{\lambda_{(\alpha,\beta)}[f]}{\rho_{(\alpha,\beta)}[h \circ f]} \leq \liminf_{r\to+\infty} \frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)} \leq \min\left\{\frac{\lambda_{(\alpha,\beta)}[f]}{\lambda_{(\alpha,\beta)}[h \circ f]}, \frac{\rho_{(\alpha,\beta)}[f]}{\rho_{(\alpha,\beta)}[h \circ f]}\right\}.$$

Proof. From the definitions of $\lambda_{(\alpha,\beta)}[h\circ f]$, $\rho_{(\alpha,\beta)}[h\circ f]$, $\lambda_{(\alpha,\beta)}[f]$ and $\rho_{(\alpha,\beta)}[f]$, we have for arbitrary positive ε and for all sufficiently large values of r that

$$\alpha\left(\log(\nu_{h\circ f}(r))\right) \geqslant \left(\lambda_{(\alpha,\beta)}[h \circ f] - \varepsilon\right)\beta(\log r), \tag{2.3}$$

$$\alpha\left(\log(\nu_{h\circ f}(r))\right) \leq \left(\rho_{(\alpha,\beta)}[h \circ f] + \varepsilon\right)\beta(\log r), \tag{2.4}$$

$$\alpha\left(\log(\nu_f(r))\right) \geqslant \left(\lambda_{(\alpha,\beta)}[f] - \varepsilon\right)\beta(\log r) \tag{2.5}$$

$$\text{and } \alpha\left(\log(\nu_f(r))\right) \leq \left(\rho_{(\alpha,\beta)}[f] + \varepsilon\right)\beta(\log r). \tag{2.6}$$

Again for a sequence of values of r tending to infinity,

$$\alpha\left(\log(\nu_{h\circ f}(r))\right) \geqslant \left(\rho_{(\alpha,\beta)}[h \circ f] - \varepsilon\right)\beta(\log r), \tag{2.7}$$

$$\text{and } \alpha\left(\log(\nu_f(r))\right) \leq \left(\lambda_{(\alpha,\beta)}[f] + \varepsilon\right)\beta(\log r). \tag{2.8}$$

Now from (2.4) and (2.5) it follows for all sufficiently large values of r that

$$\frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)} \geqslant \frac{\lambda_{(\alpha,\beta)}[f] - \varepsilon}{\rho_{(\alpha,\beta)}[h \circ f] + \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)}\geqslant\frac{\lambda_{(\alpha,\beta)}[f]}{\rho_{(\alpha,\beta)}[h\circ f]}. \tag{2.9}$$

Combining (2.3) and (2.8) , we have for a sequence of values of r tending to infinity that

$$\frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)}\leq\frac{\lambda_{(\alpha,\beta)}[f]+\varepsilon}{\lambda_{(\alpha,\beta)}[h\circ f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\liminf_{r\to+\infty}\frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)}\leq\frac{\lambda_{(\alpha,\beta)}[f]}{\lambda_{(\alpha,\beta)}[h\circ f]}. \tag{2.10}$$

Now from (2.6) and (2.7) , it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)}\leq\frac{\rho_{(\alpha,\beta)}[f]+\varepsilon}{\rho_{(\alpha,\beta)}[h\circ f]-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)}\leq\frac{\rho_{(\alpha,\beta)}[f]}{\rho_{(\alpha,\beta)}[h\circ f]}. \tag{2.11}$$

Thus the theorem follows from (2.9), (2.10) and (2.11). □

Theorem 2.4. *Let f and h are entire functions such that* $0<\lambda_{(\alpha,\beta)}[h\circ f]\leq\rho_{(\alpha,\beta)}[h\circ f]<+\infty$ *and* $0<\lambda_{(\alpha,\beta)}[f]\leq\rho_{(\alpha,\beta)}[f]<+\infty$. *Then*

$$\max\left\{\frac{\lambda_{(\alpha,\beta)}[f]}{\lambda_{(\alpha,\beta)}[h\circ f]},\frac{\rho_{(\alpha,\beta)}[f]}{\rho_{(\alpha,\beta)}[h\circ f]}\right\}\leq\limsup_{r\to+\infty}\frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)}\leq\frac{\rho_{(\alpha,\beta)}[f]}{\lambda_{(\alpha,\beta)}[h\circ f]}.$$

Proof. From the definitions of $\lambda_{(\alpha,\beta)}[h\circ f]$ and $\rho_{(\alpha,\beta)}[f]$, we have for a sequence of values of r tending to infinity,

$$\alpha\left(\log(\nu_{h\circ f}(r))\right)\leq\left(\lambda_{(\alpha,\beta)}[h\circ f]+\varepsilon\right)\beta(\log r), \tag{2.12}$$

$$\text{and } \alpha\left(\log(\nu_f(r))\right)\geqslant\left(\rho_{(\alpha,\beta)}[f]-\varepsilon\right)\beta(\log r). \tag{2.13}$$

Now from (2.5) and (2.12), for a sequence of values of r tending to infinity, we get

$$\frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)}\geq\frac{\lambda_{(\alpha,\beta)}[f]-\varepsilon}{\lambda_{(\alpha,\beta)}[h\circ f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)} \geq \frac{\lambda_{(\alpha,\beta)}[f]}{\lambda_{(\alpha,\beta)}[h\circ f]}. \tag{2.14}$$

Now, it follows from (2.3) and (2.6) , for all sufficiently large values of r that

$$\frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)} \leq \frac{\rho_{(\alpha,\beta)}[f]+\varepsilon}{\lambda_{(\alpha,\beta)}[h\circ f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)} \leq \frac{\rho_{(\alpha,\beta)}[f]}{\lambda_{(\alpha,\beta)}[h\circ f]}. \tag{2.15}$$

So combining (2.4) and (2.13) , we get for a sequence of values of r tending to infinity that

$$\frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)} \geqslant \frac{\rho_{(\alpha,\beta)}[f]-\varepsilon}{\rho_{(\alpha,\beta)}[h\circ f]+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\alpha\left(\log(\nu_f(r))\right)}{\alpha\left(\log(\nu_{h\circ f}(r))\right)} \geqslant \frac{\rho_{(\alpha,\beta)}[f]}{\rho_{(\alpha,\beta)}[h\circ f]}. \tag{2.16}$$

Thus the theorem follows from (2.14), (2.15) and (2.16). □

Remark 2.5. *If we take "$0<\lambda_{(\alpha,\beta)}[h]\leq\rho_{(\alpha,\beta)}[h]<+\infty$" instead of "$0<\lambda_{(\alpha,\beta)}[f]\leq\rho_{(\alpha,\beta)}[f]<+\infty$" and other conditions remain same, the conclusion of Theorem 2.3 and Theorem 2.4 remain true with "$\lambda_{(\alpha,\beta)}[h]$", "$\rho_{(\alpha,\beta)}[h]$" and "$\alpha\left(\log(\nu_h(r))\right)$" in place of "$\lambda_{(\alpha,\beta)}[f]$", "$\rho_{(\alpha,\beta)}[f]$" and "$\alpha\left(\log(\nu_f(r))\right)$" respectively in the numerators.*

3. Conclusion

The main aim of this paper is to establish some results related to the growth of composite entire functions using their central index. The study will provide a scope for further research in different growth measurements. The interested researchers may be motivated from this idea and they can try to investigate the growth results of entire functions regarding the generalized order and generalized type using the central index.

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