GANITA, Vol.75(2), 2025, 87-97

GROWTH PROPERTIES OF COMPOSITE ANALYTIC FUNCTION IN THE UNIT POLYDISC BASED ON $(p, q, t)L$-TH RELATIVE GOL'DBERG ORDER

JYOTI RANI ✉ and ANUPMA RASTOGI

Abstract

This article is devoted to exploring the properties of the growth of analytic functions in the unit polydisc of the $(p, q, t)L$-th relative Gol'dberg order. We introduce some concepts related to the function $M_{f,D}(R)$, which is strictly increasing, and its inverse $M_{f,D}^{-1}$. We establish some new inequalities focusing on the $(p, q, t)L$-th relative Gol'dberg order and the $(p, q, t)L$-th relative Gol'dberg lower order.

2010 *Mathematics subject classification:* primary 30G30; secondary 30D05, 30H50 .

Keywords and phrases: Analytic functions, $(p, q, t)L$-th relative Gol'dberg order, $(p, q, t)L$-th relative Gol'dberg lower order, unit polydisc..

1. Introduction

([5],[6]) Let $\mathbb{C}^n$ and $\mathbb{R}^n$ respectively denote the complex and real n-spaces. Also let us indicate the point $(z_1, z_2, ..., z_n)$ and $(m_1, m_2, ..., m_n)$ of $\mathbb{C}^n$ or $\mathbb{I}^n$ by their corresponding unsuffixed symbols z and m respectively, where $\mathbb{I}$ denotes the set of non-negative integers. The modulus of z, denoted by $|z|$, is defined as

$$|z| = \left(|z_1|^2 + ... + |z_n|^2\right)^{\frac{1}{2}} .,$$

If the coordinates of the vector m are non-negative integers, then z^m will denote $z_1^{m_1} ... z_n^{m_n}$ and $\|m\| = m_1 + ... + m_n$.

If $D \subset \mathbb{C}^n$ (where $\mathbb{C}^n$ denotes the n-dimensional complex space) is a unit polydisc with center at the origin, then for any analytic function $f(z)$ of n complex variables and $R > 0$, $M_{f,D}(R)$ may be defined as $M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$, where a point $z \in D_R$ if and only if $\frac{z}{R} \in D$. If $f(z)$ is non-constant, then $M_{f,D}(R)$ is strictly increasing and its inverse $M_{f,D}^{-1} : (|f(0)|, 1) \to (0, 1)$ exists such that

$$\lim_{R \to 1} M_{f,D}^{-1}(R) = 1.$$

The authors express sincere gratitude to the referee for the thorough reading of the manuscript and for offering valuable comments and insightful suggestions, which significantly enhanced the quality and clarity of the paper.

Definition 1.1. ([2],[7],[8],[9]) Let $f(z)$ be an analytic function of n complex variables defined on the unit polydisc D centered at the origin in $\mathbb{C}^n$. Then the Gol'dberg order of $f(z)$ with respect to the unit polydisc D is given by

$$\rho_f^{(p,q)}(D) = \overline{\lim_{R\to 1}} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]}\left(\frac{1}{1-R}\right)}, \qquad \text{where } p \geq q \geq 1.$$

Similarly, the Gol'dberg lower order of $f(z)$ with respect to the unit polydisc D is defined as

$$\lambda_f^{(p,q)}(D) = \underline{\lim_{R\to 1}} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]}\left(\frac{1}{1-R}\right)}, \qquad \text{where } p \geq q \geq 1.$$

Here, the iterated logarithm is defined recursively by $\log^{[k]} R = \log(\log^{[k-1]} R)$ for $k = 1, 2, \ldots$, with $\log^{[0]} R = R$, and $exp^{[k]}R = exp\left(exp^{[k-1]}R\right)$ for $k = 1, 2, 3, ...; exp^{[0]}R = R$.

For a unit polydisc D, an analytic function of n complex variables for which the (p, q)-th Gol'dberg order and (p, q)-th Gol'dberg lower order are the same is said to be of regular growth. Functions which are not of regular growth are said to be of irregular growth. To compare the relative growth of analytic functions of n complex variables having the same non-zero finite Gol'dberg order.

Definition 1.2. ([2],[8],[12]) Let $f(z)$ and $g(z)$ be analytic functions of n complex variables defined on the unit polydisc D centered at the origin in $\mathbb{C}^n$. Then the relative Gol'dberg order of the analytic function $f(z)$ with respect to $g(z)$ in the unit polydisc D is defined as

$$\rho_{g,D}^{(p,q)} f = \overline{\lim_{R\to 1}} \frac{log^{[p]} M_{g,D}^{-1} M_{f,D}(R)}{log^{[q]}[\frac{1}{1-R}]}, \qquad where\ p \geq q \geq 1$$

Similarly, the relative Gol'dberg lower order of the analytic function $f(z)$ with respect to $g(z)$ in the unit polydisc D is given by

$$\lambda_{g,D}^{(p,q)} f = \underline{\lim_{R\to 1}} \frac{log^{[p]} M_{g,D}^{-1} M_{f,D}(R)}{log^{[q]}[\frac{1}{1-R}]}, \qquad where\ p \geq q \geq 1$$

In the context of relative Gol'dberg order, it is therefore reasonable to define a generalized concept known as the $(p, q, t)L$-th relative Gol'dberg order for analytic functions of n complex variables in the unit polydisc D centered at the origin in $\mathbb{C}^n$. In view of this, one may introduce the definition of the $(p, q, t)L$-th relative Gol'dberg order of an analytic function $f(z)$ with respect to another analytic function $g(z)$, where both $f(z)$ and $g(z)$ are analytic functions of n complex variables in the unit polydisc D centered at the origin in $\mathbb{C}^n$, in the framework of an index-pair.

Definition 1.3. ([2],[8]) Let $f(z)$ be an analytic function of n complex variables in the unit polydisc D centered at the origin in $\mathbb{C}^n$. Then the (p,q,t)-th Gol'dberg order of the function $f(z)$ with respect to D is defined as

$$\rho_D^{(p,q,t)}(f) = \overline{\lim_{R\to 1}} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]}\left[\frac{1}{1-R}\right] + \exp^{[t]}\left[\frac{1}{1-R}\right]}, \qquad \text{where } p \geq q \geq 1.$$

Similarly, the (p,q,t)-th Gol'dberg lower order of the function $f(z)$ with respect to D is defined as

$$\lambda_D^{(p,q,t)}(f) = \underline{\lim_{R\to 1}} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]}\left[\frac{1}{1-R}\right] + \exp^{[t]}\left[\frac{1}{1-R}\right]}, \qquad \text{where } p \geq q \geq 1.$$

Definition 1.4. [1] Let $L \equiv L(r)$ be a positive continuous function such that $L(ar) \sim L(r)$ as $r \to 1$, for every positive constant a. Following Somasundaram and Thamizharasi [1], one may define the $(p,q,t)L$-th Gol'dberg order and the $(p,q,t)L$-th Gol'dberg lower order for analytic functions of n complex variables in the unit polydisc as follows.

([2],[3],[8]) Let $f(z)$ and $g(z)$ be analytic functions of n complex variables, associated with index pairs (m,q) and (m,p) respectively, where p, q, and m are positive integers satisfying $m \geq q \geq 1$ and $m \geq p \geq 1$. Let D denote the unit polydisc centered at the origin in $\mathbb{C}^n$. Then the $(p,q,t)L$-th relative Gol'dberg order of $f(z)$ with respect to $g(z)$ is defined by

$$\rho_D^{(p,q,t)L}(f) = \overline{\lim_{R\to 1}} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]}\left[\frac{1}{1-R}\right] + \exp^{[t]} L\left[\frac{1}{1-R}\right]},$$

Similarly, the $(p,q,t)L$-th relative Gol'dberg lower order of $f(z)$ with respect to $g(z)$ is defined by

$$\lambda_D^{(p,q,t)L}(f) = \underline{\lim_{R\to 1}} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]}\left[\frac{1}{1-R}\right] + \exp^{[t]} L\left[\frac{1}{1-R}\right]}.$$

Definition 1.5. ([1],[3]) Let f be an analytic function of n complex variables in the unit polydisc D centered at the origin in $\mathbb{C}^n$. Then the $(p,q,t)L$-th relative Gol'dberg order of f is defined by

$$\rho_{g,D}^{(p,q,t)L}(f) = \overline{\lim_{R\to 1}} \frac{\log^{[p]} M_g^{-1} M_{f,D}(R)}{\log^{[q]}\left[\frac{1}{1-R}\right] + \exp^{[t]} L\left[\frac{1}{1-R}\right]},$$

and the $(p,q,t)L$-th relative Gol'dberg lower order of f is defined by

$$\lambda_{g,D}^{(p,q,t)L}(f) = \underline{\lim_{R\to 1}} \frac{\log^{[p]} M_g^{-1} M_{f,D}(R)}{\log^{[q]}\left[\frac{1}{1-R}\right] + \exp^{[t]} L\left[\frac{1}{1-R}\right]}.$$

During the past decades, several authors (see [2],[4],[9],[10],[13]) have made close investigations on the properties of the relative order of analytic functions of several complex variables using different growth indicators such as Gol'dberg order and (p,q)-th Gol'dberg order. In this paper, we aim to study some relative growth properties of analytic functions of n complex variables using the Gol'dberg order and the $(p,q,t)L$-th relative Gol'dberg order.

2. Main Results

THEOREM 2.1. *Let $f(z)$ and $g(z)$ be analytic functions of n-complex variables and D be the unit polydisc centered at the origin in $\mathbb{C}^n$. Also, let $0 < \lambda_D^{(p,q,t)L}(fog) < \rho_D^{(p,q,t)L}(fog) < \infty$ and $0 < \lambda_D^{(m,q,t)L}(g) < \rho_D^{(m,q,t)L}(g) < \infty$. Then*

$$\frac{\lambda_D^{(p,q,t)L}(fog)}{\rho_D^{(m,q,t)L}(g)} \leq \varliminf_{R\to 1} \frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \leq \min\left\{\frac{\lambda_D^{(p,q,t)L}(fog)}{\lambda_D^{(m,q,t)L}(g)}, \frac{\rho_D^{(p,q,t)L}(fog)}{\rho_D^{(m,q,t)L}(g)}\right\}$$

$$\leq \max\left\{\frac{\lambda_D^{(p,q,t)L}(fog)}{\lambda_D^{(m,q,t)L}(g)}, \frac{\rho_D^{(p,q,t)L}(fog)}{\rho_D^{(m,q,t)L}(g)}\right\} \leq \varlimsup_{R\to 1} \frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \leq \frac{\rho_D^{(p,q,t)L}(fog)}{\lambda_D^{(m,q,t)L}(g)}$$

PROOF. From the definition of $(p,q,t)L$-th relative Gol'dberg order and $(p,q,t)L$-th relative Gol'dberg lower order of the analytic composite function fog, for $\epsilon > 0$ and as $R \to 1$, we have:

$$\log^{[p,t]L} M_{fog,D}(R) \leq \left(\rho_D^{(p,q,t)L}(fog) + \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right) \tag{2.1}$$

and

$$\log^{[p,t]L} M_{fog,D}(R) \geq \left(\lambda_D^{(p,q,t)L}(fog) - \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right) \tag{2.2}$$

Again, for all sufficiently small values of $1-R$ (i.e., $R \to 1$),

$$\log^{[p,t]L} M_{fog,D}(R) \leq \left(\lambda_D^{(p,q,t)L}(fog) + \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right) \tag{2.3}$$

and

$$\log^{[p,t]L} M_{fog,D}(R) \geq \left(\rho_D^{(p,q,t)L}(fog) - \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right) \tag{2.4}$$

Now, from the definition of $(m,q,t)L$-th Gol'dberg order and $(m,q,t)L$-th lower Gol'dberg order of the analytic function $g(z)$, for $\epsilon > 0$ and sufficiently large values of R, we have:

$$\log^{[m,t]L} M_{g,D}(R) \leq \left(\rho_D^{(m,q,t)L}(g) + \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right) \tag{2.5}$$

and

$$\log^{[m,t]L} M_{g,D}(R) \geq \left(\lambda_D^{(m,q,t)L}(g) - \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right) \tag{2.6}$$

Also, for a sequence of values of $R \to 1$, we have:

$$\log^{[m,t]L} M_{g,D}(R) \leq \left(\lambda_D^{(m,q,t)L}(g) + \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right) \tag{2.7}$$

and

$$\log^{[m,t]L} M_{g,D}(R) \geq \left(\rho_D^{(m,q,t)L}(g) - \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right) \tag{2.8}$$

Now from (2.2) and (2.5), it follows for $R \to 1$ that

$$\frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \geq \frac{\left(\lambda_D^{(p,q,t)L}(fog) - \epsilon\right)}{\left(\rho_D^{(m,q,t)L}(g) + \epsilon\right)}.$$

As $\epsilon(> 0)$ is arbitrary, we obtain that

$$\varliminf_{R\to 1} \frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \geq \frac{\lambda_D^{(p,q,t)L}(fog)}{\rho_D^{(m,q,t)L}(g)}. \tag{2.9}$$

Again, from (2.3) and (2.6), it follows for $R \to 1$ that

$$\frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \leq \frac{\left(\lambda_D^{(p,q,t)L}(fog) + \epsilon\right)}{\left(\lambda_D^{(m,q,t)L}(g) - \epsilon\right)}.$$

As $\epsilon(> 0)$ is arbitrary, we obtain that

$$\varliminf_{R\to 1} \frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \leq \frac{\lambda_D^{(p,q,t)L}(fog)}{\lambda_D^{(m,q,t)L}(g)}. \tag{2.10}$$

Similarly, from (2.1) and (2.8), it follows for a sequence of values of $R \to 1$ that

$$\frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \leq \frac{\left(\rho_D^{(p,q,t)L}(fog) + \epsilon\right)}{\left(\rho_D^{(m,q,t)L}(g) - \epsilon\right)}.$$

As $\epsilon(> 0)$ is arbitrary, we obtain that

$$\varliminf_{R\to 1} \frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \leq \frac{\rho_D^{(p,q,t)L}(fog)}{\rho_D^{(m,q,t)L}(g)}. \tag{2.11}$$

Now combining (2.9), (2.10), and (2.11), we get

$$\frac{\lambda_D^{(p,q,t)L}(fog)}{\rho_D^{(m,q,t)L}(g)} \leq \varliminf_{R\to 1} \frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \leq \min\left\{\frac{\lambda_D^{(p,q,t)L}(fog)}{\lambda_D^{(m,q,t)L}(g)}, \frac{\rho_D^{(p,q,t)L}(fog)}{\rho_D^{(m,q,t)L}(g)}\right\}. \tag{2.12}$$

From (2.2) and (2.7), it follows for $R \to 1$ that

$$\frac{log^{[p,t]L} M_{fog,D}(R)}{log^{[m,t]L} M_{g,D}(R)} \geq \frac{\left(\lambda_D^{(p,q,t)L}(fog) - \epsilon\right)}{\left(\lambda_D^{(m,q,t)L}(g) + \epsilon\right)} .$$

As $\epsilon(> 0)$, we obtained that

$$\overline{\lim_{R\to 1}} \frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \geq \frac{\lambda_D^{(p,q,t)L}(fog)}{\lambda_D^{(m,q,t)L}(g)} . \tag{2.13}$$

From (2.1) and (2.6), it follows for $R \to 1$ that

$$\frac{log^{[p,t]L} M_{fog,D}(R)}{log^{[m,t]L} M_{g,D}(R)} \leq \frac{\left(\rho_D^{(p,q,t)L}(fog) + \epsilon\right)}{\left(\lambda_D^{(m,q,t)L}(g) - \epsilon\right)} .$$

As $\epsilon(> 0)$, we obtained that

$$\overline{\lim_{R\to 1}} \frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \leq \frac{\rho_D^{(p,q,t)L}(fog)}{\lambda_D^{(m,q,t)L}(g)} . \tag{2.14}$$

Similarly, from (2.4) and (2.5), it follows that

$$\frac{log^{[p,t]L} M_{fog,D}(R)}{log^{[m,t]L} M_{g,D}(R)} \geq \frac{\left(\rho_D^{(p,q,t)L}(fog) - \epsilon\right)}{\left(\rho_D^{(m,q,t)L}(g) + \epsilon\right)} .$$

As $\epsilon(> 0)$, we obtained that

$$\overline{\lim_{R\to 1}} \frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \geq \frac{\rho_D^{(p,q,t)L}(fog)}{\rho_D^{(m,q,t)L}(g)} . \tag{2.15}$$

Now combining (2.13), (2.14), and (2.15), we obtain

$$\max\left\{\frac{\lambda_D^{(p,q,t)L}(fog)}{\lambda_D^{(m,q,t)L}(g)}, \frac{\rho_D^{(p,q,t)L}(fog)}{\rho_D^{(m,q,t)L}(g)}\right\} \leq \overline{\lim_{R\to 1}} \frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \leq \frac{\rho_D^{(p,q,t)L}(fog)}{\lambda_D^{(m,q,t)L}(g)} . \tag{2.16}$$

Hence, the theorem follows from (2.12) and (2.16). □

Corollary 2.2. *Let us assume f and g be analytic in the unit polydisc $D \subset \mathbb{C}^n$. Let,*

$$\lambda_D^{(m,q,t)L}(g) = \rho_D^{(m,q,t)L}(g) = u, \qquad 0 < u < \infty.$$

So,

$$\frac{\lambda_D^{(p,q,t)L}(fog)}{u} \leq \lim_{R\to 1} \frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} \leq \frac{\rho_D^{(p,q,t)L}(fog)}{u}.$$

Additionally,

$$\lambda_D^{(p,q,t)L}(fog) = \rho_D^{(p,q,t)L}(fog) = v, \qquad 0 < v < \infty,$$

Therefore,

$$\lim_{R\to 1} \frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} = \frac{v}{u}.$$

Example: We assume that $g(z) = (1-z)^{-\alpha}$ and $f(z) = z^{\beta}$ with $\alpha > 0$ and $\beta > 0$. So, $fog(z) = (1-z)^{-\alpha\beta}$. For $0 < R < 1$, the maximum modulus of g on the unit polydisc D of radius R is

$$M_{g,D}(R) = \frac{1}{(1-R)^{\alpha}}, \qquad \log M_{g,D}(R) = \alpha \log \frac{1}{1-R}.$$

Likewise,

$$M_{fog,D}(R) = \frac{1}{(1-R)^{\alpha\beta}}, \qquad \log M_{fog,D}(R) = \alpha\beta \log \frac{1}{1-R}.$$

From here, the $(m, q, t)L$-th relative Gol'dberg order and lower order of g are

$$\lambda_D^{(m,q,t)L}(g) = \rho_D^{(m,q,t)L}(g) = \alpha,$$

as well as the $(p, q, t)L$-th relative Gol'dberg order and lower order of the composition fog are

$$\lambda_D^{(p,q,t)L}(fog) = \rho_D^{(p,q,t)L}(fog) = \alpha\beta..$$

Consequently,

$$\frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} = \frac{\alpha\beta \log \frac{1}{1-R}}{\alpha \log \frac{1}{1-R}} = \beta.$$

and from here,

$$\lim_{R\to 1} \frac{\log^{[p,t]L} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}(R)} = \beta = \frac{\rho_D^{(p,q,t)L}(fog)}{\rho^D_{(m,q,t)L}(g)} = \frac{\lambda_D^{(p,q,t)L}(fog)}{\lambda_D^{(m,q,t)L}(g)}..$$

So, this example verifies the Theorem 2.1

Theorem 2.3. *Let $f(z)$, $g(z)$, and $h(z)$ be three analytic functions, and let D be the unit polydisc centered at the origin in $\mathbb{C}^n$.*
Also, let $0 < \rho_{h,D}^{(p,q,t)L}(fog) < 1$ and $0 < \rho_{g,D}^{(m,q,t)L}(f) < 1$. Then,

$$\varliminf_{R\to 1} \frac{\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R)} \leq \frac{\rho_{h,D}^{(p,q,t)L}(fog)}{\rho_{g,D}^{(m,q,t)L}(f)} \leq \varlimsup_{R\to 1} \frac{\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R)},$$

where p, q, and m are positive integers such that $p > q$ and $m > q$, that is, $q < \min\{p, m\}$.

Proof. From the definition of the $(p,q,t)L$-th relative Gol'dberg order with respect to another analytic function, we get, for a sequence of values of $R \to 1$,

$$\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R) \leq \left(\rho_{h,D}^{(p,q,t)L}(fog) + \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right), \tag{2.17}$$

and

$$\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R) \geq \left(\rho_{h,D}^{(p,q,t)L}(fog) - \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right). \tag{2.18}$$

Also,

$$\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R) \leq \left(\rho_{g,D}^{(m,q,t)L}(f) + \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right), \tag{2.19}$$

and

$$\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R) \geq \left(\rho_{g,D}^{(m,q,t)L}(f) - \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right). \tag{2.20}$$

From (2.18) and (2.19), it follows for a sequence of values of $R \to 1$ that

$$\frac{\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R)} \geq \frac{\left(\rho_{h,D}^{(p,q,t)L}(fog) - \epsilon\right)}{\left(\rho_{g,D}^{(m,q,t)L}(f) + \epsilon\right)}.$$

As $\epsilon > 0$ is arbitrary, we obtain that

$$\overline{\lim_{R\to 1}} \frac{\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R)} \geq \frac{\rho_{h,D}^{(p,q,t)L}(fog)}{\rho_{g,D}^{(m,q,t)L}(f)}. \tag{2.21}$$

Again, from (2.17) and (2.20), it follows for a sequence of values of $R \to 1$ that

$$\frac{\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R)} \leq \frac{\left(\rho_{h,D}^{(p,q,t)L}(fog) + \epsilon\right)}{\left(\rho_{g,D}^{(m,q,t)L}(f) - \epsilon\right)}.$$

As $\epsilon > 0$ is arbitrary, we obtain that

$$\underline{\lim_{R\to 1}} \frac{\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R)} \leq \frac{\rho_{h,D}^{(p,q,t)L}(fog)}{\rho_{g,D}^{(m,q,t)L}(f)}. \tag{2.22}$$

Hence, the theorem follows from (2.21) and (2.22).

□

Theorem 2.4. *Let $f(z)$, $g(z)$, and $h(z)$ be three analytic functions, and let D be the unit polydisc centered at the origin in $\mathbb{C}^n$.*
Also, let $0 < \lambda_{h,D}^{(p,q,t)L}(fog) < \infty$ and $0 < \lambda_{g,D}^{(m,q,t)L}(f) < \infty$. Then,

$$\underline{\lim_{R\to 1}} \frac{\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R)} \leq \frac{\lambda_{h,D}^{(p,q,t)L}(fog)}{\lambda_{g,D}^{(m,q,t)L}(f)} \leq \overline{\lim_{R\to 1}} \frac{\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R)}.$$

Here, p, q, and m are positive integers such that $p > q$ and $m > q$, that is, $q < \min\{p, m\}$.

Proof. From the definition of the $(p, q, t)L$-th relative lower Gol'dberg order with respect to another analytic function, we get, for a sequence of large values of R,

$$\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R) \geq \left(\lambda_{h,D}^{(p,q,t)L}(fog) - \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right), \tag{2.23}$$

and

$$\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R) \leq \left(\lambda_{h,D}^{(p,q,t)L}(fog) + \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right). \tag{2.24}$$

Also, for a sequence of values of $R \to 1$, we have

$$\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R) \geq \left(\lambda_{g,D}^{(m,q,t)L}(f) - \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right), \tag{2.25}$$

and

$$\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R) \leq \left(\lambda_{g,D}^{(m,q,t)L}(f) + \epsilon\right)\left(\log^{[q]} \frac{1}{1-R} + \exp^{[t]} L\left(\frac{1}{1-R}\right)\right). \tag{2.26}$$

Now, from (2.23) and (2.26), it follows for a sequence of values of $R \to 1$ that

$$\frac{\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R)} \geq \frac{\left(\lambda_{h,D}^{(p,q,t)L}(fog) - \epsilon\right)}{\left(\lambda_{g,D}^{(m,q,t)L}(f) + \epsilon\right)}.$$

As $\epsilon > 0$ is arbitrary, we obtain that

$$\varlimsup_{R\to 1} \frac{\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R)} \geq \frac{\lambda_{h,D}^{(p,q,t)L}(fog)}{\lambda_{g,D}^{(m,q,t)L}(f)}. \tag{2.27}$$

Now, from (2.24) and (2.25), it follows for a sequence of values of $R \to 1$ that

$$\frac{\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R)} \leq \frac{\left(\lambda_{h,D}^{(p,q,t)L}(fog) + \epsilon\right)}{\left(\lambda_{g,D}^{(m,q,t)L}(f) - \epsilon\right)}.$$

As $\epsilon > 0$ is arbitrary, we obtain that

$$\varliminf_{R\to 1} \frac{\log^{[p,t]L} M_{h,D}^{-1} M_{fog,D}(R)}{\log^{[m,t]L} M_{g,D}^{-1} M_{f,D}(R)} \leq \frac{\lambda_{h,D}^{(p,q,t)L}(fog)}{\lambda_{g,D}^{(m,q,t)L}(f)}. \tag{2.28}$$

Hence, the theorem follows from (2.27) and (2.28).

□

Acknowledgement

The authors express sincere gratitude to the referee for the thorough reading of the manuscript and for offering valuable comments and insightful suggestions, which significantly enhanced the quality and clarity of the paper.

3. conclusion

The primary objective of this paper has been to extend and refine the concept of the (p,q)-th relative Gol'dberg order to the more general framework of the $(p,q,t)L$-th relative Gol'dberg order for analytic functions, and to establish its existence. The results obtained herein provide a foundational base that may be further generalized by employing the notions of $(p,q,t)L$-th relative Gol'dberg order and $(p,q,t)L$-th relative Gol'dberg lower order of analytic functions in the unit polydisc. These avenues are left open for interested researchers or contributing authors to explore in future investigations within this line of study.

References

[1] Tanmay Biswas, *Some Results Relating to Sum and Product Theorem of Relative $(p,q,t)L$-th Order and Relative $(p,q,t)L$-th Type of Entire Functions.*, Korean J. Math. **26** (2018) pp. 215-269.,

[2] Tanmay Biswas, *Some Results relating to (p,q)-th relative Gol'dberg order and (p,q)-th relative Gol'dberg type of entire functions of several variables*, Journal of Fractional Calculus and Applications **10(2)** (2019) pp. 249-272.,

[3] Tanmay Biswas, *Relative $(p,q,t)L$-th order and Relative $(p,q,t)L$-th lower order oriented growth properties of composite entire functions*, Journal of Fractional calculus and Applications **13(1)** (2022) pp. 95-105.,

[4] S.K. Datta and A.R. Maji, *Study of growth properties on the basis of generalized Gol'dberg order of composite entire functions of several complex variables*, International Journal of Math. Sci. and Engg. Appls **5** (2011) pp. 297-311.,

[5] S. Datta and A. Mallik, *Study of Growth Properties on the Basis of Gol'dberg Order of Composition of Entire Functions of Several Complex Variables*, International Journal of Mathematical Analysis **5** (2011) pp. 1075-1087.,

[6] S. Datta, S. Biswas and D. Deb, *Growth Analysis of Functions Analytic in the Unit Polydisc*, International Journal of Analysis and Applications **1** (2013) pp. 123-131.,

[7] B. A. Fuks, *Introduction to the theory of analytic function of several complex variables*, American mathematical Society **8** (1963) pp. 17-28.,

[8] D. K. Gautam and A. Rastogi, *Growth analysis of $(p,q)^{th}$ relative Gol'dberg order of composite entire functions*, GANITA **69(2)** (2019) pp. 67-75.,

[9] A. A. Gol'dberg, *Elementary remarks on the formulas defining order and type of function of several complex variables*, S.S.R. Dokl. **29** (1959) pp. 145-151.,

[10] S. Halvarsson, *Growth properties of entire functions depending on a parameter*, Annales Polonici Mathe-matici **14(1)** (1996) pp. 71-96.,

[11] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, *On the (p,q)-order and lower (p,q)-order of an entire function*, J. Reine Angew. Math. **282** (1976) pp. 53-67.,

[12] B. C. Mondal and C. Roy, *Relative Gol'dberg order of an entire function of several variables*, Bull. cal. Math. Society **102** (2010) pp. 371-380.,

[13] G. Valiron, *lecture on the general theory of intgral functions*, Chelsea publishing company **1** (1949) pp. 16-21.,

Jyoti Rani,
Department of Mathematics and Astronomy,
University of Lucknow, Lucknow (U.P)
e-mail: gaytriarya100@gmail.com

Anupma Rastogi,
Department of Mathematics and Astronomy,
University of Lucknow, Lucknow (U.P)
e-mail: anupmarastogi13121993@gmail.com