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SOME RESULTS ON $(p, q, t)L$-TH ORDER & $(p, q, t)L$-TH LOWER ORDER OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES ON THE BASIS OF CENTRAL INDEX

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Abstract

The central index denoted by $\nu_f(r)$ of entire function f(z)=$\sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined as max $\{m : \mu_f(r) = |a_m| r^m\}$. In this paper, we discuss about the growth rates of central indices of composition of entire functions of several complex variables with their $(p, q, t)L$-th order & $(p, q, t)L$-th lower order.

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1. Introduction

We denote complex n- space by $\mathbb{C}^n$ and indicate its elements $(z_1, z_2, ..., z_n)$ by z.
Here a nonempty open complete n-circular region in $\mathbb{C}^n$ with centre at $(0, 0, ..., 0)$ is denoted by $\Omega_n = \Omega$
We define

$$|\Omega| = \{r : r = |z| \text{ for some } z \in \Omega\}$$

and

$$\Omega^+ = \{r : r \in |\Omega|, \text{ no } r_j = 0\}$$

and regard these as subsets of the n-dimensional Euclidean space $\mathbb{C}^n$.

A function $f(z)$ is analytic at a point $\eta \in \mathbb{C}^n$ if it can be extended as an absolutely convergent power series in some neighbourhood of η.
If we assume $\eta = (0, 0, ..., 0)$, then $f(z)$ has representation ([1] and [2])

$$f(z) = \sum_{k=(0,0,...,0)}^{\infty} a_{k_1,k_2,...,k_n} z_1^{k_1} z_2^{k_2} ... z_n^{k_n} = \sum_{|k|=0}^{\infty} a_k z^k,$$

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where a_k are complex coefficients such that $k = (k_1, k_2, ..., k_n)$ belongs to $S = \{k : k \in \mathbb{C}^n$, each k_i is non-negative integer$\}$ and $|k| = k_1 + k_2 + ... + k_n$.

Now, Corresponding to an entire function $f(z)$ on $|\Omega|$ ([1] and [3]):
The maximum modulus

$$M(r) = M_f(r)$$

defined as

$$M(r) = M_f(r) = \max_{|z|=r}\{|f(z)|\}$$

The maximum term

$$\mu(r) = \mu_f(r)$$

defined as

$$\mu(r) = \mu_f(r) = \max_{k\in S}\{|a_k|r^k\}$$

and the central index

$$\nu(r) = \nu_f(r) = (\nu_1(r, f), \nu_2(r, f), ..., \nu_n(r, f))$$

defined as

$$\nu_i(r) = \nu_i(r, f) = \begin{cases} \max\{k_i : |a_k|r^k = \mu(r)\}, & \text{if } \mu(r) > 0 \\ 0, & \text{if } \mu(r) = 0, \text{ for } 1 \le i \le n. \end{cases}$$

Also, the central index $\nu_f(r)$ for which the maximum term is achieved

$$|\nu_f(r)| = \nu_1(r, f) + \nu_2(r, f) + ... + \nu_n(r, f)$$

Let p, q, m and n be positive integers and $t \in \mathbb{N} \cup \{-1, 0\}$.
Now we recall that Shen et al.[4] defined the (m, n)-φ order and (m, n)-φ lower order of entire function $f(z)$ which are as follows:

Definition 1.1. [4] Let φ:[0,∞] → (0,∞) be a non-decreasing unbounded function and $m \ge n$. The (m, n)-φ order $\rho^{(m,n)}(f, \varphi)$ and (m, n)-φ lower order $\lambda^{(m,n)}(f, \varphi)$ of entire function $f(z)$ are defined as:

$$\begin{aligned} \rho^{(m,n)}(f, \varphi) &= \limsup_{r\to\infty} \frac{\log^{[m+1]} M_f(r)}{\log^{[n]} \varphi(r)} \\ \textit{and} \quad \lambda^{(m,n)}(f, \varphi) &= \liminf_{r\to\infty} \frac{\log^{[m+1]} M_f(r)}{\log^{[n]} \varphi(r)} \end{aligned}$$

where

$$\log^{[k]} x = \log(\log^{[k-1]} x) \text{ for } k = 1, 2, 3, ... \text{ and } \log^{[0]} x = x$$

If we take $m = 1, n = 1$ and $\varphi(r) = r$, then we denote $\rho^{(2,1)}(f,r)$ and $\lambda^{(2,1)}(f,r)$ by ρ_f and λ_f respectively, which are the order and lower order of entire function $f(z)$. Now, He and Xiao [5] introduced an alternative definition of order and lower order of an entire function $f(z)$ in terms of its central index in the following way:

$$\begin{aligned} \rho(f) &= \limsup_{r\to\infty} \frac{\log \nu_f(r)}{\log r} \\ \textit{and} \quad \lambda(f) &= \liminf_{r\to\infty} \frac{\log \nu_f(r)}{\log r} \end{aligned}$$

Let $L \equiv L(r)$ is a positive continuous function increasing slowly i.e. $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a' i.e., $\lim_{r\to+\infty} \frac{L(ar)}{L(r)} = 1$. If we take $m = p, n = 1$ and $\varphi(r) = \log^{[q-1]} r . \exp^{[t+1]} L(r)$, then Definition 1.1 turns into the definitions of $(p,q,t)L$-th order and $(p,q,t)L$-th lower order of an entire function $f(z)$ ([6]) which are as follows:

$$\begin{aligned} \rho^{(p,q,t)L}(f) &= \limsup_{r\to\infty} \frac{\log^{[p+1]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)} \\ \textit{and} \quad \lambda^{(p,q,t)L}(f) &= \liminf_{r\to\infty} \frac{\log^{[p+1]} M_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)} \end{aligned}$$

Shen et al. [4] also established the equivalence of the definition of $(m,n) - \varphi$ order of entire functions in terms of maximum modulus and central index under some conditions. If we take $m = p, n = 1$ and $\varphi(r) = \log^{[q-1]} r . \exp^{[t+1]} L(r)$, then in view of Lemma 3.4 of [4] and Definition 1.1, one can write the following definition:

Definition 1.2. Let $f(z)$ be an entire function and $\nu_f(r)$ be the central index of $f(z)$, then

$$\begin{aligned} \rho^{(p,q,t)L}(f) &= \limsup_{r\to\infty} \frac{\log^{[p]} |\nu_f(r)|}{\log^{[q]} r + \exp^{[t]} L(r)} \\ \textit{and} \quad \lambda^{(p,q,t)L}(f) &= \liminf_{r\to\infty} \frac{\log^{[p]} |\nu_f(r)|}{\log^{[q]} r + \exp^{[t]} L(r)} \end{aligned}$$

In order to compare the relative growth of two entire functions having same non zero finite $(p,q,t)L$-th order, one may introduce the definitions of $(p,q,t)L$-th type (also $(p,q,t)L$-th lower type) of entire functions having finite positive $(p,q,t)L$-th order in the following manner:

Definition 1.3. [6] Let $f(z)$ be an entire function with non-zero finite $(p,q,t)L$-th order. The $(p,q,t)L$-th type denoted by $\sigma_f^{(p,q,t)L}$ and $(p,q,t)L$-th lower type denoted by $\bar{\sigma}_f^{(p,q,t)L}$ are respectively defined as follows:

$$\sigma^{(p,q,t)L}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{[\log^{[q-1]} r.\exp^{[t+1]} L(r)]^{\rho_f^{(p,q,t)L}}}$$

$$and \quad \bar{\sigma}^{(p,q,t)L}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{[\log^{[q-1]} r.\exp^{[t+1]} L(r)]^{\rho_f^{(p,q,t)L}}}$$

Now, in the line of Juneja, Kapoor and Bajpai [7], (p,q)-th order and (p,q)-th lower order of an entire function $f(z)$ of n-complex variables are respectively defined as follows:

Definition 1.4. [7] The (p,q)-order $\rho^{(p,q)}(f)$ and (p,q)-lower order $\lambda^{(p,q)}(f)$ of entire function $f(z)$ of n-complex variables are:

$$\rho^{(p,q)}(f) = \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p+1]} M_f(r_1,r_2,\ldots,r_n)}{\log^{[q]}(r_1 r_2 \ldots r_n)}$$

$$and \quad \lambda^{(p,q)}(f) = \liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p+1]} M_f(r_1,r_2,\ldots,r_n)}{\log^{[q]}(r_1 r_2 \ldots r_n)}$$

where p and q are any two positive integers with $p \geq q$.

Definition 1.5. The $(p,q,t)L$-th order $\rho^{(p,q,t)L}(f)$ and $(p,q,t)L$-th lower order $\lambda^{(p,q,t)L}(f)$ of entire function $f(z)$ of n-complex variables are respectively defined as follows:

$$\rho^{(p,q,t)L}(f) = \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p+1]} M_f(r_1,r_2,\ldots,r_n)}{\log^{[q]}(r_1 r_2 \ldots r_n) + \exp^{[t]} L(r_1 r_2 \ldots r_n)} \tag{1.1}$$

and

$$\lambda^{(p,q,t)L}(f) = \liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p+1]} M_f(r_1,r_2,\ldots,r_n)}{\log^{[q]}(r_1 r_2 \ldots r_n) + \exp^{[t]} L(r_1 r_2 \ldots r_n)} \tag{1.2}$$

Also, let $f(z)$ be an entire function of n-complex variables with non-zero finite $(p,q,t)L$-th order. The $(p,q,t)L$-th type denoted by $\sigma_f^{(p,q,t)L}$ and $(p,q,t)L$-th lower type denoted by $\bar{\sigma}_f^{(p,q,t)L}$ are respectively defined as follows:

$$\sigma^{(p,q,t)L}(f) = \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]} M_f(r_1,r_2,\ldots,r_n)}{[\log^{[q-1]}(r_1,r_2,\ldots,r_n).\exp^{[t+1]} L(r_1,r_2,\ldots,r_n)]^{\rho_f^{(p,q,t)L}}}$$

$$and \quad \bar{\sigma}^{(p,q,t)L}(f) = \liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]} M_f(r_1,r_2,\ldots,r_n)}{[\log^{[q-1]}(r_1,r_2,\ldots,r_n).\exp^{[t+1]} L(r_1,r_2,\ldots,r_n)]^{\rho_f^{(p,q,t)L}}}$$

Recently, many authors ([5],[8] and [9]) have investigated the growth of entire functions of several complex variables in terms of their central index.

In this paper we would like to establish $(p,q,t)L$-th order $\rho^{(p,q,t)L}(f)$ and $(p,q,t)L$-th lower order $\lambda^{(p,q,t)L}(f)$ of entire function $f(z)$ of n-complex variables in terms of its central index. The main purpose of this paper is to investigate the growth estimates of the central indices of composite entire functions of n-complex variables as compared to their left or right factor based on $(p,q,t)L$-th order ($(p,q,t)L$-th lower order), where p and q are any two positive integers with $p \geq q$ and $t \in \mathbb{N} \cup \{-1,0\}$.

2. Lemmas

In this section, we present some lemmas that will be needed in the sequel.

Lemma 2.1. [1] *Let $p,r \in |\Omega|$ and let $\mu(p)$ and $\mu(r)$ be both positive. Then the line integral,*

$$I = \int_p^r \sum_{j=1}^n \frac{\nu_j(x)}{x_j} dx_j$$

taken over any connected polygon in $|\Omega|$ with sides parallel to the axes and from p to r,

1. *exists,*
2. *is independent of the polygon and*
3. *is such that $log\mu(r) = log\mu(p) + I$*

Lemma 2.2. [1] *Let $r \in |\Omega|$. Let $p \in \mathbb{C}^n$ and be such that $p >> (1,1,...,1)$, while $pr = (p_1r_1, p_2r_2, ..., p_nr_n) \in |\Omega|$.*

Let

$$N_j = \max_{r \leq t \leq pr} \nu_j(t), \quad for \ \ 1 \leq j \leq n.$$

Then

1. $\mu(r) \leq M(r) \leq \mu(r) \prod_{j=1}^n [N_j + \frac{p_j}{p_j - 1}]$ *;*
2. $\mu(r) = M(r)$*, if and only if the series* $\sum_{|k|=0}^{\infty} a_k r^k$ *has at most one non vanishing term;*
3. *the last relation in* 1 *is an equality if and only if* $\mu(r) = 0$

3. Main Results

In this section, we present the main results of this paper.

Theorem 3.1. *Let $f(z)$ be an entire function of n- complex variables with $(p,q,t)L$-th order $\rho^{(p,q,t)L}(f)$ and $(p,q,t)L$-th lower order $\lambda^{(p,q,t)L}(f)$. If $\nu_f(r) = \nu_f(r_1, r_2, ..., r_n)$ be the central index of f, then*

$$\rho^{(p,q,t)L}(f) = \limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log^{[p]} |\nu_f(r_1, r_2, ..., r_n)|}{\log^{[q]}(r_1 r_2 ... r_n) + exp^{[t]} L(r_1 r_2 ... r_n)} \tag{3.1}$$

$$\lambda^{(p,q,t)L}(f) = \liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[p]}|\nu_f(r_1,r_2,...,r_n)|}{\log^{[q]}(r_1r_2...r_n) + exp^{[t]}L(r_1r_2...r_n)} \tag{3.2}$$

Proof. By Lemma 2.1, we see that the maximum term $\mu(r)$ of f satisfies

$$\log\mu(r) = \log\mu(p) + \int_p^r \sum_{j=1}^{n} \frac{\nu_j(x)}{x_j}dx_j \tag{3.3}$$

Since Krishna, J.G.([1],Corollary 2.9) proved that $\nu_j(r)$ is increasing and right continuous in j-th variable for $1 \le j \le n$. Therefore, for any $p, r \in |\Omega|$ such that $\mu(r) > 0$ and $p >> (1, 1, ..., 1)$, we get for $1 \le j \le n$,

$$\nu_j(r) \le \frac{1}{\log(p_j)} \int_p^r \nu_j(r_1, .., r_{j-1}, .., r_n)\frac{dx_j}{x_j} \tag{3.4}$$

From (3.3) and (3.4), we get

$$\log\mu(r) \ge \log\mu(p) + \sum_{j=1}^{n} \nu_j(r)\log(p_j) \tag{3.5}$$

By Lemma 2.2, we have

$$\mu_f(r) \le M_f(r) \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\sum_{j=1}^{n} \nu_j(r)\log(p_j) \le \log M_f(r) + c_1 \tag{3.7}$$

As $p >> (1, 1..., 1)$, choosing $p_j = 2$ for $1 \le j \le n$, we get

$$\begin{aligned} &\textstyle\sum_{j=1}^{n} \quad \nu_j(r)\log 2 \le \log M_f(r) + c_1 \\ &\Longrightarrow \quad |\nu_f(r)|\log 2 \le \log M_f(r) + c_1 \\ &\Longrightarrow \quad \log^{[p]}|\nu_f(r)| + \log^{[p+1]} 2 \le \log^{[p+1]} M_f(r) + c_2 \end{aligned}$$

where $c_i (i = 1, 2)$ are constants. By this and (1.1), we have

$$\rho^{(p,q,t)L}(f) \ge \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[p]}|\nu_f(r_1,r_2,...,r_n)|}{\log^{[q]}(r_1r_2...r_n) + exp^{[t]}L(r_1r_2...r_n)} \tag{3.8}$$

Now by letting $p_j = 2$ for $1 \le j \le n$ in (1) of Lemma (2.2), we have

$$\begin{aligned} M_f(r) \quad &\le \mu_f(r)\prod_{j=1}^{n} |N_j + 2|, \\ where \quad N_j &= \max_{r\le t\le pr} \nu_j(t), \quad for \;\; 1 \le j \le n. \end{aligned}$$

$$\Longrightarrow M_f(r) \leq |a_{\nu_f(r)}| r^{\nu_f(r)} \prod_{j=1}^{n} |N_j + 2| \tag{3.9}$$

Since $|a_k|$ is bounded, from (3.9) we get

$$\begin{aligned}
\log M_f(r) &\leq \sum_{j=1}^{n} \nu_j(r) \log r_j + \sum_{j=1}^{n} \log N_j + c_3 \\
&\leq \sum_{j=1}^{n} \nu_f(r) \log r_j + \sum_{j=1}^{n} \log N_j + c_3 \\
&\leq |\nu_f(r)| \log(r_1 r_2 ... r_n) + \log(N_1 N_2 ... N_n) + c_3 \\
&\Longrightarrow \log^{[p+1]} M_f(r) \leq \log^{[p]} |\nu_f(r)| + \log^{[p+1]}(r_1 r_2 ... r_n) \\
&+ \log^{[p+1]}(N_1 N_2 ... N_n) + c_4
\end{aligned}$$

where $c_i (i > 0)(i = 3, 4)$ are constants. By this and (1.1), we get

$$\rho^{(p,q,t)L}(f) \leq \limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log^{[p]} |\nu_f(r_1, r_2, ..., r_n)|}{\log^{[q]}(r_1 r_2 ... r_n) + \exp^{[t]} L(r_1 r_2 ... r_n)} \tag{3.10}$$

By (3.8) and (3.10), it follows that

$$\rho^{(p,q,t)L}(f) = \limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log^{[p]} |\nu_f(r_1, r_2, ..., r_n)|}{\log^{[q]}(r_1 r_2 ... r_n) + \exp^{[t]} L(r_1 r_2 ... r_n)}$$

Similarly, we can prove for the lower order

$$\lambda^{(p,q,t)L}(f) = \liminf_{r_1, r_2, ..., r_n \to \infty} \frac{\log^{[p]} |\nu_f(r_1, r_2, ..., r_n)|}{\log^{[q]}(r_1 r_2 ... r_n) + \exp^{[t]} L(r_1 r_2 ... r_n)}$$

□

Theorem 3.2. *If f and g are entire functions of n-complex variables such that $0 < \lambda^{(p,q,t)L}(fog) \leq \rho^{(p,q,t)L}(fog) < \infty$ and $0 < \lambda^{(m,q,t)L}(g) \leq \rho^{(m,q,t)L}(g) < \infty$, then*

$$\begin{aligned}
\frac{\lambda^{(p,q,t)L}(fog)}{\rho^{(m,q,t)L}(g)} &\leq \liminf_{r_1, r_2, ..., r_n \to \infty} \frac{\log^{[p]} |\nu_{fog}(r_1, r_2, ..., r_n)|}{\log^{[m]} |\nu_g(r_1, r_2, ..., r_n)|} \leq \frac{\lambda^{(p,q,t)L}(fog)}{\lambda^{(m,q,t)L}(g)} \\
&\leq \limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log^{[p]} |\nu_{fog}(r_1, r_2, ..., r_n)|}{\log^{[m]} |\nu_g(r_1, r_2, ..., r_n)|} \leq \frac{\rho^{(p,q,t)L}(fog)}{\lambda^{(m,q,t)L}(g)}
\end{aligned}$$

where p, q and m are positive integers with $p \geq q \geq m$.

Proof. Using Theorem 3.1 for the entire functions fog and g respectively, we have for arbitrary positive ϵ and for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\log^{[p]} |\nu_{fog}(r_1, r_2, ..., r_n)| \geq (\lambda^{(p,q,t)L}(fog) - \epsilon)(\log^{[q]}(r_1 r_2 ... r_n) + \exp^{[t]} L(r_1 r_2 ... r_n)) \tag{3.11}$$

and

$$\log^{[m]}|\nu_g(r_1,r_2,...,r_n)| \geq (\rho^{(m,q,t)L}(g)+\epsilon)(\log^{[q]}(r_1r_2...r_n)+\exp^{[t]}L(r_1r_2...r_n)) \quad (3.12)$$

Now, from (3.11) and (3.12) it follows for all sufficiently large values of $r_1,r_2,...,r_n$,

$$\frac{\log^{[p]}|\nu_{fog}(r_1,r_2,...,r_n)|}{\log^{[m]}|\nu_g(r_1,r_2,...,r_n)|} \geq \frac{\lambda^{(p,q,t)L}(fog)-\epsilon}{\rho^{(m,q,t)L}(g)+\epsilon}$$

As $\epsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[p]}|\nu_{fog}(r_1,r_2,...,r_n)|}{\log^{[m]}|\nu_g(r_1,r_2,...,r_n)|} \geq \frac{\lambda^{(p,q,t)L}(fog)}{\rho^{(m,q,t)L}(g)} \quad (3.13)$$

Again, for a sequence of values of $(r_1,r_2,...,r_n)\to\infty$

$$\log^{[p]}|\nu_{fog}(r_1,r_2,...,r_n)| \leq (\lambda^{(p,q,t)L}(fog)+\epsilon)(\log^{[q]}(r_1r_2...r_n)+\exp^{[t]}L(r_1r_2...r_n)) \quad (3.14)$$

and for all large values of $r_1,r_2,...,r_n$

$$\log^{[m]}|\nu_g(r_1,r_2,...,r_n)| \geq (\lambda^{(m,q,t)L}(g)-\epsilon)(\log^{[q]}(r_1r_2...r_n)+\exp^{[t]}L(r_1r_2...r_n)) \quad (3.15)$$

So, combining (3.14) and (3.15), we get

$$\frac{\log^{[p]}|\nu_{fog}(r_1,r_2,...,r_n)|}{\log^{[m]}|\nu_g(r_1,r_2,...,r_n)|} \leq \frac{\lambda^{(p,q,t)L}(fog)+\epsilon}{\lambda^{(m,q,t)L}(g)-\epsilon}$$

Since $\epsilon(>0)$ is arbitrary, it follows that

$$\liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[p]}|\nu_{fog}(r_1,r_2,...,r_n)|}{\log^{[m]}|\nu_g(r_1,r_2,...,r_n)|} \leq \frac{\lambda^{(p,q,t)L}(fog)}{\lambda^{(m,q,t)L}(g)} \quad (3.16)$$

Also, for a sequence of values of $(r_1,r_2,...,r_n)\to\infty$

$$\log^{[m]}|\nu_{fog}(r_1,r_2,...,r_n)| \leq (\lambda^{(m,q,t)L}(g)+\epsilon)(\log^{[q]}(r_1r_2...r_n)+exp^{[t]}L(r_1r_2...r_n)) \quad (3.17)$$

Now, from (3.11) and (3.17) we obtain for a sequence of values of $(r_1,r_2,...,r_n)\to\infty$,

$$\frac{\log^{[p]}|\nu_{fog}(r_1,r_2,...,r_n)|}{\log^{[m]}|\nu_g(r_1,r_2,...,r_n)|} \geq \frac{\lambda^{(p,q,t)L}(fog)-\epsilon}{\lambda^{(m,q,t)L}(g)+\epsilon} \quad (3.18)$$

Choosing $\epsilon\to 0$ we get that

$$\limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[p]}|\nu_{fog}(r_1,r_2,...,r_n)|}{\log^{[m]}|\nu_g(r_1,r_2,...,r_n)|} \leq \frac{\lambda^{(p,q,t)L}(fog)}{\lambda^{(m,q,t)L}(g)} \quad (3.19)$$

Also, for all large values of $r_1,r_2,...,r_n$

$$\log^{[p]}|\nu_{fog}(r_1,r_2,...,r_n)| \leq (\rho^{(p,q,t)L}(fog)+\epsilon)(\log^{[q]}(r_1r_2...r_n)+exp^{[t]}L(r_1r_2...r_n)) \quad (3.20)$$

So, from (3.15) and (3.20) it follows for a sequence of values of $(r_1, r_2, ..., r_n) \to \infty$,

$$\frac{\log^{[p]}|\nu_{fog}(r_1, r_2, ..., r_n)|}{\log^{[m]}|\nu_g(r_1, r_2, ..., r_n)|} \leq \frac{\rho^{(p,q,t)L}(fog) + \epsilon}{\lambda^{(m,q,t)L}(g) - \epsilon}$$

As $\epsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[p]}|\nu_{fog}(r_1, r_2, ..., r_n)|}{\log^{[m]}|\nu_g(r_1, r_2, ..., r_n)|} \leq \frac{\rho^{(p,q,t)L}(fog)}{\lambda^{(m,q,t)L}(g)} \tag{3.21}$$

Hence, the theorem follows from (3.13), (3.16), (3.19) and (3.21). □

Theorem 3.3. *If f and g are entire functions of n-complex variables such that $0 < \lambda^{(p,q,t)L}(fog) \leq \rho^{(p,q,t)L}(fog) < \infty$ and $0 < \lambda^{(m,q,t)L}(g) \leq \rho^{(m,q,t)L}(g) < \infty$, then*

$$\liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[p]}|\nu_{fog}(r_1, r_2, ..., r_n)|}{\log^{[m]}|\nu_g(r_1, r_2, ..., r_n)|} \leq \frac{\rho^{(p,q,t)L}(fog)}{\rho^{(m,q,t)L}(g)}$$
$$\leq \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[p]}|\nu_{fog}(r_1, r_2, ..., r_n)|}{\log^{[m]}|\nu_g(r_1, r_2, ..., r_n)|}$$

where p, q and m are positive integers with $p \geq q \geq m$.

The proof follows from Theorem 3.1 and Theorem 3.2 together with standard lim sup and lim inf arguments similar to those used in [8]. Hence, the details are omitted.

The following theorem is also a natural consequence of Theorem 3.2 and Theorem 3.3.

Theorem 3.4. *If f and g are entire functions of n-complex variables such that $0 < \lambda^{(p,q,t)L}(fog) \leq \rho^{(p,q,t)L}(fog) < \infty$ and $0 < \lambda^{(m,q,t)L}(g) \leq \rho^{(m,q,t)L}(g) < \infty$, then*

$$\liminf_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[p]}|\nu_{fog}(r_1, r_2, ..., r_n)|}{\log^{[m]}|\nu_g(r_1, r_2, ..., r_n)|} \leq min\left\{\frac{\lambda^{(p,q,t)L}(fog)}{\lambda^{(m,q,t)L}(g)}, \frac{\rho^{(p,q,t)L}(fog)}{\rho^{(m,q,t)L}(g)}\right\}$$
$$\leq max\left\{\frac{\lambda^{(p,q,t)L}(fog)}{\lambda^{(m,q,t)L}(g)}, \frac{\rho^{(p,q,t)L}(fog)}{\rho^{(m,q,t)L}(g)}\right\} \leq \limsup_{r_1,r_2,...,r_n\to\infty} \frac{\log^{[p]}|\nu_{fog}(r_1, r_2, ..., r_n)|}{\log^{[m]}|\nu_g(r_1, r_2, ..., r_n)|}$$

where p, q and m are positive integers with $p \geq q \geq m$.

Remark 3.5. *If we take $0 < \lambda^{(m,q,t)L}(f) \leq \rho^{(m,q,t)L}(f) < \infty$ instead of $0 < \lambda^{(m,q,t)L}(g) \leq \rho^{(m,q,t)L}(g) < \infty$ and the other conditions remain the same, then also Theorem 3.2, Theorem 3.3 and Theorem 3.4 hold with g replaced by f in the denominator.*

Acknowledgement

The present work is closely related to earlier studies on the growth of entire functions based on the central index, particularly the work of Pramanik, Biswas

and Roy [8]. The proofs in this paper rely on known central index techniques and classical lemmas due to Krishna [1], and several arguments follow similar lines with suitable modifications. The contribution of this paper is to extend these methods to the $(p, q, t)L$-growth framework for entire functions of several complex variables and to obtain corresponding growth estimates for composite entire functions within this setting.

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