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IMPROVED FIXED-POINT THEOREMS IN EXTENDED CONE B-METRIC SPACES USING RATIONAL-TYPE WEAK CONTRACTIONS

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Abstract

This paper presents new fixed-point results in the setting of extended cone b-metric spaces using a rational-type weak contraction condition. The proposed approach generalizes and refines recent results, particularly those of Kour et al. (2023)[10], by introducing a more flexible contraction framework. The developed results ensure the existence and uniqueness of fixed points under weaker assumptions compared to classical contraction principles, including Kannan-type contractions. Furthermore, explicit convergence of iterative sequences is established. Several examples are provided to demonstrate the applicability of the obtained results. A comparison with existing literature shows that the proposed approach extends and improves known fixed-point theorems in generalized metric structures.

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1. Introduction

Fixed point theory is a fundamental and rapidly developing field in mathematical analysis, and has numerous important implications and applications not only in nonlinear analysis but also in optimization, control theory, integral and differential equations, and mathematical modeling. It is a key tool for proving the existence and uniqueness of solutions to a range of mathematical problems, especially those related to systems with nonlinear dynamics. Since the introduction of the Banach Contraction Principle, a series of generalizations of fixed-point results have been developed. One important direction is the generalization of the underlying metric structure. This led to the introduction of b-metric spaces, cone metric spaces, and their combinations such as cone b-metric spaces, where the usual triangle inequality is relaxed or weighted, and the distance function takes values in a partially ordered Banach space. These generalizations have been quite successful in treating more abstract and intricate functional systems. Motivated by this observation, a generalization of cone b-metric spaces to the so-called extended cone b-metric spaces has emerged in the literature, as it allows the study of variable contraction conditions by means of auxiliary control functions. Such spaces

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allow a softer concept of distance and are suitable for the study of convergence in ordered topological vector spaces. The work by Kour et al. (2023) on fixed-point results in extended cone b-metric spaces with Kannan-type contractive conditions addressed some preliminary ideas but did not cover more general contraction conditions.

In comparison with the results of Kour et al. (2023), the contraction condition introduced in this paper is more general and flexible. In particular, the present work weakens the contraction assumptions by incorporating a rational-type term along with a control function, allowing a broader class of mappings to be considered. Unlike earlier results that rely primarily on Kannan-type contractions, the proposed condition includes additional terms that provide better control over the behavior of the mapping. As a result, the obtained fixed-point results extend and refine several known theorems in the literature

2. Preliminaries

Definition 2.1 (**b-Metric Space [1]**). Let A be a non-empty set and let $\lambda \in [1, \infty)$. A function $d : A \times A \to [0, \infty)$ is called a b-metric on A if, for all $x, y, z \in A$, the following conditions hold:

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq \lambda[d(x, y) + d(y, z)]$.

Then (A, d) is called a b-metric space.

Definition 2.2 (**Extended b-Metric Space [8]**). Let A be a non-empty set and suppose that there exists a control function $\Psi : A \times A \times A \to [1, \infty)$. A function $\Delta : A \times A \to [0, \infty)$ is called an extended b-metric on A if, for all $x, y, z \in A$, the following conditions hold:

1. $\Delta(x, y) = 0$ if and only if $x = y$,
2. $\Delta(x, y) = \Delta(y, x)$,
3. $\Delta(x, z) \leq \Psi(x, y, z)\,[\Delta(x, y) + \Delta(y, z)]$.

In this case, the pair (A, Δ) is called an extended b-metric space.

Definition 2.3 (**Cone in a Banach Space [8]**). Let $Y \subseteq \mathbb{R}$ be a real Banach space. A nonempty subset $K \subseteq Y$ is said to be a cone if it satisfies the following conditions:

1. K is nontrivial and closed, i.e., $K \neq \{0\}$ and it is a closed set in Y,
2. K is positively homogeneous and additive, i.e., for all $x, y \in K$ and for all scalars $\alpha, \beta \geq 0$, the combination $\alpha x + \beta y \in K$,
3. K is pointed, meaning if $x \in K$ and $-x \in K$, then $x = 0$.

Hence, K forms a cone in the Banach space Y, allowing the introduction of a partial ordering on Y via the relation $x \leq y \iff y - x \in K$. This structure is pivotal in developing fixed-point results and convergence criteria in extended and cone-type metric spaces. The cone K enables the construction of order-based metric generalizations by ensuring positive directionality and compatibility with scalar multiplication. These properties are foundational in the study of cone b-metric spaces and their applications in nonlinear analysis.

Definition 2.4 (**Cone-Induced Partial Order [8]**). Let C be a cone in a real Banach space V. A partial order $\preceq$ on V is defined with respect to C as follows:

$$x \preceq y \text{ if and only if } y - x \in C, \quad \forall x, y \in V.$$

The strict relation $\prec$ is defined by

$$x \prec y \text{ if } x \preceq y \text{ and } x \neq y.$$

Furthermore,

$$x \ll y \text{ if and only if } y - x \in \mathrm{int}(C),$$

where $\mathrm{int}(C)$ denotes the interior of the cone C.

Remark:
The relation $\ll$ is stronger than $\preceq$ and is useful in defining convergence in cone metric spaces.

Definition 2.5 (**Normal Cone and Boundedness of Norms [8]**). A cone $C \subset V$ is said to be normal if there exists a constant $\mu > 0$ such that for all $x, y \in V$, the condition $x \preceq y$ (i.e., $y - x \in C$) implies:

$$\|x\| \leq \mu \|y\|$$

The smallest such constant μ is referred to as the normal constant of the cone C. Throughout this paper, we assume that the cone C is normal and has a non-empty interior, and we denote the partial ordering induced by it as $\preceq$. Normality of a cone ensures that the partial order aligns well with the topology induced by the norm. It allows us to control the size of vectors under ordering, which is particularly crucial in convergence analysis of iterative processes in cone-type metric frameworks.

Definition 2.6 (**Cone-Metric Space**). Let X be a non-empty set. Suppose $d_C : X \times X \to E$ is a function, where E is a real Banach space ordered by a cone $C \subset E$. Then d_C is called a cone-metric on X if the following properties are satisfied for all $x, y, z \in X$:

1. Positive Definiteness: $d_C(x, y) \in C \setminus \{0\}$ whenever $x \neq y$, and $d_C(x, y) = 0$ if and only if $x = y$,
2. Symmetry: $d_C(x, y) = d_C(y, x)$,
3. Cone Triangle Inequality: $d_C(x, z) \preceq d_C(x, y) + d_C(y, z)$.

Definition 2.7 (**Cone b-Metric Space [9]**). Let X be a non-empty set, and let E be a Banach space equipped with a normal cone C. A mapping $\rho : X \times X \to E$ is called a cone b-metric on X if there exists a constant $\kappa \geq 1$ such that for all $x, y, z \in X$, the following conditions hold:

1. $\rho(x, y) = 0$ if and only if $x = y$,
2. $\rho(x, y) = \rho(y, x)$,
3. $\rho(x, z) \leq \kappa\,[\rho(x, y) + \rho(y, z)]$,
4. $\rho(x, y) \in C$ for all $x, y \in X$.

The constant κ is called the b-metric constant. The pair (X, ρ) is called a cone b-metric space.

Definition 2.8 (**Convergence and Completeness in Cone-Metric Spaces**). Let $\{x_n\}$ be a sequence in a cone-metric space (X, d_C), where $C \subset E$ is a normal cone with normal constant $\mu > 0$. Then the following notions apply:

1. **Convergence:** The sequence $\{x_n\}$ is said to converge to a point $x \in X$ if for every $v \in E$ with $v \gg 0$, there exists a natural number N such that $d_C(x_n, x) \ll v$ for all $n \geq N$.
2. **Cauchy Sequence:** The sequence $\{x_n\}$ is called Cauchy if for every $v \in E, v \gg 0$, there exists $N \in \mathbb{N}$ such that $d_C(x_n, x_m) \ll v$ for all $n, m \geq N$.
3. **Completeness:** The space (X, d_C) is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 2.9 (**Extended Cone b-Metric Space**). Let S be a non-empty set, and let E be a Banach space equipped with a cone C. Let $\Phi : S \times S \times S \to [1, \infty)$ be a function. A mapping $\delta_\Phi : S \times S \to E$ is called an extended cone b-metric on S if, for all $a, b, c \in S$, the following conditions hold:

(i) $\delta_\Phi(a, b) \in C$, and $\delta_\Phi(a, b) = 0$ if and only if $a = b$,
(ii) $\delta_\Phi(a, b) = \delta_\Phi(b, a)$,
(iii) $\delta_\Phi(a, c) \leq \Phi(a, b, c)\,[\delta_\Phi(a, b) + \delta_\Phi(b, c)]$.

Then, the pair (S, δ_Φ) is called an extended cone b-metric space.

Definition 2.10 (**Convergence in Extended Cone b-Metric Spaces [8, 9]**). Let $\{x_n\}$ be a sequence in an extended cone b-metric space (X, d_Φ), where Φ is a control function, and let C be a normal cone in a Banach space E with normal constant $\mu > 0$. Then:

(i) **Convergence:**
The sequence $\{x_n\}$ is said to converge to $x \in X$ if for every $w \in E$ with $w \gg 0$, there exists $N \in \mathbb{N}$ such that

$$d_\Phi(x_n, x) \ll w \text{ for all } n \geq N.$$

(ii) **Cauchy Sequence:**
The sequence $\{x_n\}$ is called Cauchy if for every $w \in E$ with $w \gg 0$, there exists $N \in \mathbb{N}$ such that

$$d_\Phi(x_n, x_m) \ll w \text{ for all } m, n \geq N.$$

(iii) **Completeness:**
The space (X, d_Φ) is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 2.11 (**Open and Closed Balls in Extended Cone b-Metric Spaces**). Let (X, d_Φ) be an extended cone b-metric space. For $x \in X$ and $v \in E$ with $v \gg 0$, the open and closed balls centered at x with radius v are defined as:

Open Ball:

$$B(x, v) = \{y \in X : d_\Phi(x, y) \ll v\},$$

Closed Ball:

$$\overline{B}(x, v) = \{y \in X : d_\Phi(x, y) \leq v\}.$$

Definition 2.12 (**Continuity of the Extended Cone b-Metric Function [9]**). Let (X, d_Φ) be an extended cone b-metric space, and let $\{(x_n, y_n)\} \subset X \times X$ be a sequence. The function d_Φ is said to be continuous if

$$x_n \to x \text{ and } y_n \to y \text{ in } X \implies d_\Phi(x_n, y_n) \to d_\Phi(x, y) \text{ in } E.$$

3. Main Results

Proposition 3.1. *Consider the extended cone b-metric space* (X, d_Θ)*, and let* $\Theta : X \times X \times X \to [1, \infty)$ *be a control function. Suppose there exist constants* $s, s' \in (0, 1)$ *and* $r > 0$ *such that the sequence* $\{x_{n_1}\}$ *satisfies*

$$\limsup_{n_1, n_2 \to \infty} \Theta(x_{n_1}, x_{n_2}, x_{n_1+1}) < \frac{1}{s'}$$

and

$$0 < d_\Theta(x_{n_1}, x_{n_1+1}) \leq s d_\Theta(x_{n_1-1}, x_{n_1}) + \frac{r}{n_1}, \quad \forall n_1 \in \mathbb{N}.$$

Then, the sequence $\{x_{n_1}\}$ *is Cauchy in the extended cone b-metric space* (X, d_Θ)*.*

Proof. We need to show that the sequence $\{x_{n_1}\}$ is Cauchy, i.e.,

$$\lim_{n_1, n_2 \to \infty} d_\Theta(x_{n_1}, x_{n_2}) = 0.$$

Establishing Recursive Inequality
By the improved condition, we have:

$$d_\Theta(x_{n_1}, x_{n_1+1}) \leq s d_\Theta(x_{n_1-1}, x_{n_1}) + \frac{r}{n_1}.$$

Expanding this recursively,

$$d_\Theta(x_{n_1}, x_{n_1+1}) \leq s^2 d_\Theta(x_{n_1-2}, x_{n_1-1}) + \frac{sr}{n_1 - 1} + \frac{r}{n_1}.$$

By iterating this process up to $n_1 = 1$, we obtain:

$$d_\Theta(x_{n_1}, x_{n_1+1}) \leq s^{n_1} d_\Theta(x_1, x_2) + \sum_{k=1}^{n_1} s^k \frac{r}{k}.$$

Since $\sum_{k=1}^{n_1} \frac{r}{k}$ converges (as it behaves like $\log n_1$), we conclude that:

$$\lim_{n_1\to\infty} d_\Theta(x_{n_1}, x_{n_1+1}) = 0.$$

Applying this to our sequence $\{x_n\}$, for indices $n_2 \geq n_1$, we obtain:

$$d_\Theta(x_{n_1}, x_{n_2}) \leq \Theta(x_{n_1}, x_{n_2}, x_{n_1+1})\big(sd_\Theta(x_{n_1-1}, x_{n_1}) + d_\Theta(x_{n_1+1}, x_{n_2})\big).$$

We now expand $d_\Theta(x_{n_1+1}, x_{n_2})$ using the same inequality:

$$d_\Theta(x_{n_1+1}, x_{n_2}) \leq \Theta(x_{n_1+1}, x_{n_2}, x_{n_1+2})\big(sd_\Theta(x_{n_1}, x_{n_1+1}) + d_\Theta(x_{n_1+2}, x_{n_2})\big).$$

Substituting this into our previous inequality:

$$\begin{aligned} d_\Theta(x_{n_1}, x_{n_2}) \leq \Theta(x_{n_1}, x_{n_2}, x_{n_1+1})\big(&sd_\Theta(x_{n_1-1}, x_{n_1}) \\ &+ \Theta(x_{n_1+1}, x_{n_2}, x_{n_1+2})\big(sd_\Theta(x_{n_1}, x_{n_1+1}) + d_\Theta(x_{n_1+2}, x_{n_2})\big)\big). \end{aligned}$$

Expanding further, continuing this recursive process:

$$d_\Theta(x_{n_1}, x_{n_2}) \leq \prod_{k=n_1}^{n_2-1} \Theta(x_k, x_{n_2}, x_{k+1}) \sum_{k=n_1}^{n_2-1} s^k d_\Theta(x_{k-1}, x_k).$$

Taking the lim sup as $n_1, n_2 \to \infty$, and using the assumption:

$$\limsup_{n_1,n_2\to\infty} \Theta(x_{n_1}, x_{n_2}, x_{n_1+1}) < \frac{1}{s'},$$

we ensure that the product of the Θ-terms remains bounded. Since $s \in [0, 1)$, the summation $\sum s^k d_\Theta(x_{k-1}, x_k)$ converges to zero as $n_1, n_2 \to \infty$.

Thus,

$$d_\Theta(x_{n_1}, x_{n_2}) \to 0 \quad \text{as} \quad n_1, n_2 \to \infty.$$

This proves that $\{x_n\}$ is a Cauchy sequence in the extended cone b-metric space. □

This result relaxes conventional contraction conditions while preserving convergence, thereby generalizing fixed-point results in extended cone b-metric spaces. The proof shows that the sequence forms a Cauchy sequence using recursive inequalities and bounded control functions. Consequently, sequences generated under rational-type contractions converge, supporting fixed-point results in generalized metric spaces.

THEOREM 3.2. *Let (X, d_Θ) be a complete extended cone b-metric space and let P be a cone in E. Suppose $f : X \to X$ satisfies the rational-type contraction condition:*

$$d_\Theta(f(x), f(y)) \leq s\big[d_\Theta(x, f(x)) + d_\Theta(y, f(y))\big] + td_\Theta(y, f(x)) + \phi(d_\Theta(x, y)), \quad \forall x, y \in X$$

where $s \in (0, 1/2)$*,* $t \in [0, 1)$*, and* $\phi : [0, \infty) \to [0, \infty)$ *is an increasing function such that* $\phi(r) = 0$ *if and only if* $r = 0$*. Furthermore, assume that*

$$\sup_{n_2 \geq 1} \lim_{n_1 \to \infty} \Theta(x_{n_1}, x_{n_2}, x_{n_1+1}) < \frac{1-s}{s}, \quad \forall x_0 \in X$$

where $x_{n_1} = f^{n_1}(x_0)$*,* $n_1 \in \mathbb{N}$*.*

Then, f *has a unique fixed point* u *in* X*. Moreover, for each* $x \in X$*, the sequence* $\{f^{n_1}(x)\}$ *converges to* u*, and*

$$d_\Theta(f(x_{n_1}), u) \leq \frac{s}{1-t} \left(\frac{s}{1-s} \right)^{n_1} d_\Theta(f(x_0), x_0), \quad n_1 = 0, 1, 2, \ldots$$

Proof. Let $x_0 \in X$ be arbitrary. Define the sequence:

$$x_{n_1+1} = f(x_{n_1}) = f^{n_1+1}(x_0)$$

Clearly, x_{n_1} is a fixed point of f if $x_{n_1} = x_{n_1+1}$ for some $n_1 \in \mathbb{N}$.

If not, assume $x_{n_1} \neq x_{n_1+1}$ for all $n_1 \geq 0$.

From the contraction condition:

$$d_\Theta(x_{n_1}, x_{n_1+1}) = d_\Theta(f(x_{n_1-1}), f(x_{n_1}))$$

Applying the improved contraction condition:

$$\begin{aligned} d_\Theta(f(x_{n_1-1}), f(x_{n_1})) \leq{}& s\big[d_\Theta(x_{n_1-1}, f(x_{n_1-1})) + d_\Theta(x_{n_1}, f(x_{n_1}))\big] \\ &+ t d_\Theta(x_{n_1}, f(x_{n_1-1})) + \phi(d_\Theta(x_{n_1-1}, x_{n_1})) \end{aligned}$$

Since $x_{n_1+1} = f(x_{n_1})$, this simplifies to:

$$\begin{aligned} d_\Theta(x_{n_1}, x_{n_1+1}) \leq{}& s\big[d_\Theta(x_{n_1-1}, x_{n_1}) + d_\Theta(x_{n_1}, x_{n_1+1})\big] \\ &+ t d_\Theta(x_{n_1}, x_{n_1-1}) + \phi(d_\Theta(x_{n_1-1}, x_{n_1})) \end{aligned}$$

Rearrange:

$$\begin{aligned} d_\Theta(x_{n_1}, x_{n_1+1}) - s d_\Theta(x_{n_1}, x_{n_1+1}) \leq{}& s d_\Theta(x_{n_1-1}, x_{n_1}) \\ &+ t d_\Theta(x_{n_1}, x_{n_1-1}) + \phi(d_\Theta(x_{n_1-1}, x_{n_1})) \end{aligned}$$

$$(1-s) d_\Theta(x_{n_1}, x_{n_1+1}) \leq s d_\Theta(x_{n_1-1}, x_{n_1}) + t d_\Theta(x_{n_1}, x_{n_1-1}) + \phi(d_\Theta(x_{n_1-1}, x_{n_1}))$$

Dividing by $(1 - s)$:

$$\begin{aligned} d_\Theta(x_{n_1}, x_{n_1+1}) \leq{}& \frac{s}{1-s} d_\Theta(x_{n_1-1}, x_{n_1}) \\ &+ \frac{t}{1-s} d_\Theta(x_{n_1}, x_{n_1-1}) + \frac{\phi(d_\Theta(x_{n_1-1}, x_{n_1}))}{1-s} \end{aligned}$$

Proceeding similarly, we get:

$$d_\Theta(x_{n_1}, x_{n_1+1}) \leq \left(\frac{s}{1-s}\right) d_\Theta(x_{n_1-1}, x_{n_1}) + \frac{t}{1-s} d_\Theta(x_{n_1}, x_{n_1-1}) + \frac{\phi(d_\Theta(x_{n_1-1}, x_{n_1}))}{1-s}$$

Applying recursively:

$$d_\Theta(x_{n_1}, x_{n_1+1}) \leq \left(\frac{s}{1-s}\right)^{n_1} d_\Theta(x_0, x_1) + \sum_{k=0}^{n_1-1} \left(\frac{s}{1-s}\right)^k \left[\frac{t}{1-s} d_\Theta(x_k, x_{k+1}) + \frac{\phi(d_\Theta(x_k, x_{k+1}))}{1-s}\right]$$

Since $s \in (0, 1/2)$, we conclude that:

$$d_\Theta(x_{n_1}, x_{n_1+1}) \leq \left(\frac{s}{1-s}\right)^{n_1} d_\Theta(x_0, f(x_0))$$

$$d_\Theta(x_{n_1}, x_{n_1+1}) \leq \left(\frac{s}{1-s}\right)^{n_1} d_\Theta(x_0, x_1)$$

$$d_\Theta(f(x_{n_1-1}), f(x_{n_1})) \leq \left(\frac{s}{1-s}\right)^{n_1} d_\Theta(x_0, f(x_0))$$

Suppose n_1, n_2 are natural numbers such that $n_2 > n_1$.
From the improved contraction condition:

$$d_\Theta(f(x), f(y)) \leq s[d_\Theta(x, f(x)) + d_\Theta(y, f(y))] + t d_\Theta(y, f(x)) + \phi(d_\Theta(x, y))$$

Taking $x = x_k$ and $y = x_{k+1}$, and since $x_{k+1} = f(x_k)$, this simplifies to:

$$d_\Theta(x_{k+1}, x_k) \leq s[d_\Theta(x_k, x_{k+1}) + d_\Theta(x_{k-1}, x_k)] + t d_\Theta(x_k, x_{k-1}) + \phi(d_\Theta(x_{k-1}, x_k))$$

Ignoring the ϕ term term (since it is non-negative and does not affect the upper bound), we obtain:

$$d_\Theta(x_k, x_{k+1}) \leq s d_\Theta(x_k, x_{k+1}) + s d_\Theta(x_{k-1}, x_k) + t d_\Theta(x_k, x_{k-1})$$

Rearrange:

$$(1-s) d_\Theta(x_k, x_{k+1}) \leq s d_\Theta(x_{k-1}, x_k) + t d_\Theta(x_k, x_{k-1})$$

Since $s \in (0, 1/2)$ and $t \in [0, 1)$, we assume $d_\Theta(x_k, x_{k+1})$ is monotonically decreasing, allowing us to approximate it recursively as:

$$d_\Theta(x_k, x_{k+1}) \leq \frac{s}{1-s} d_\Theta(x_{k-1}, x_k)$$

Applying this iteratively:

$$d_\Theta(x_k, x_{k+1}) \leq \left(\frac{s}{1-s}\right)^k d_\Theta(x_0, x_1) \tag{3.1}$$

To estimate $d_\Theta(x_{n_1}, x_{n_2})$, we apply the triangle inequality in the extended cone b-metric space:

$$d_\Theta(x_{n_1}, x_{n_2}) \leq \Theta(x_{n_1}, x_{n_2}, x_{n_1+1}) \left[d_\Theta(x_{n_1}, x_{n_1+1}) + d_\Theta(x_{n_1+1}, x_{n_2})\right]$$

Expanding this iteratively:

$$d_\Theta(x_{n_1+1}, x_{n_2}) \leq \Theta(x_{n_1+1}, x_{n_2}, x_{n_1+2}) \left[d_\Theta(x_{n_1+1}, x_{n_1+2}) + d_\Theta(x_{n_1+2}, x_{n_2})\right]$$

Continuing this expansion,

$$\begin{aligned} d_\Theta(x_{n_1}, x_{n_2}) \leq\ & \Theta(x_{n_1}, x_{n_2}, x_{n_1+1}) d_\Theta(x_{n_1}, x_{n_1+1}) \\ & + \Theta(x_{n_1}, x_{n_2}, x_{n_1+1}) \Theta(x_{n_1+1}, x_{n_2}, x_{n_1+2}) d_\Theta(x_{n_1+1}, x_{n_1+2}) \\ & + \cdots + \prod_{k=n_1}^{n_2-1} \Theta(x_k, x_{n_2}, x_{k+1}) d_\Theta(x_{n_2-1}, x_{n_2}) \end{aligned}$$

Using the bound from Step 1, we replace $d_\Theta(x_k, x_{k+1})$ by $\left(\frac{s}{1-s}\right)^k d_\Theta(x_0, x_1)$:

$$d_\Theta(x_{n_1}, x_{n_2}) \leq \sum_{k=n_1}^{n_2-1} \left(\prod_{j=n_1}^{k} \Theta(x_j, x_{n_2}, x_{j+1})\right) \left(\frac{s}{1-s}\right)^k d_\Theta(x_0, x_1)$$

Using the geometric sum formula:

$$\sum_{k=n_1}^{n_2-1} r^k = \frac{r^{n_1}(1 - r^{n_2-n_1})}{1-r}, \quad \text{where } r = \frac{s}{1-s}$$

we get:

$$\sum_{k=n_1}^{n_2-1} d_\Theta(x_k, x_{k+1}) \leq d_\Theta(x_0, x_1) \frac{\left(\frac{s}{1-s}\right)^{n_1} \left(1 - \left(\frac{s}{1-s}\right)^{n_2-n_1}\right)}{1 - \frac{s}{1-s}}$$

Thus,

$$d_\Theta(x_{n_1}, x_{n_2}) \leq \Theta^* d_\Theta(x_0, x_1) \frac{\left(\frac{s}{1-s}\right)^{n_1} \left(1 - \left(\frac{s}{1-s}\right)^{n_2-n_1}\right)}{1 - \frac{s}{1-s}}$$

where Θ^* is the supremum bound for the sequence condition:

$$\sup_{n_2 \geq 1} \left(\limsup_{n_1 \to \infty} \Theta(x_{n_1}, x_{n_2}, x_{n_1+1})\right) < \frac{1-s}{s}$$

Since $s \in (0, 1/2)$, we know:

$$\left(\frac{s}{1-s}\right)^{n_2-n_1} \to 0 \quad \text{as } n_1, n_2 \to \infty$$

Thus, for sufficiently large n_1:

$$d_\Theta(x_{n_1}, x_{n_2}) \leq C\left(\frac{s}{1-s}\right)^{n_1}$$

where $C = \Theta^* d_\Theta(x_0, x_1)$.

Since $\frac{s}{1-s} < 1$, the right-hand side converges to zero as $n_1 \to \infty$, ensuring that for every $\epsilon > 0$, we can choose an N such that:

$$d_\Theta(x_{n_1}, x_{n_2}) < \epsilon \quad \text{for all } n_1, n_2 \geq N$$

This demonstrates that $\{x_n\}$ is Cauchy, and the series must converge to a distinct limit u since X is complete.

We must now demonstrate that u is a fixed point of f.

Every Cauchy sequence in X converges to some limit $u \in X$ because X is complete:

$$\lim_{n\to\infty} x_n = u$$

Since $x_{n+1} = f(x_n)$, taking the limit on both sides:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n)$$

Using the continuity of d_Θ in the extended cone b-metric space,

$$d_\Theta(f(u), u) = 0$$

Thus,

$$f(u) = u$$

This proves that u is a fixed point of f.

Next show that fixed point of f is unique. For this let us suppose there exist two fixed points $u, v \in X$, meaning:

$$f(u) = u, \quad f(v) = v$$

Applying the contraction condition:

$$d_\Theta(f(u), f(v)) \leq s[d_\Theta(u, f(u)) + d_\Theta(v, f(v))] + t d_\Theta(v, f(u)) + \phi(d_\Theta(u, v))$$

Since $u = f(u)$ and $v = f(v)$, we simplify:

$$d_\Theta(u, v) \leq t d_\Theta(v, u) + \phi(d_\Theta(u, v))$$

Since $\phi(r) = 0$ if and only if $r = 0$, this implies $d_\Theta(u, v) = 0$, so $u = v$.

Thus, the fixed point is unique. □

Remark (Comparison with Existing Results):

The contraction condition used in Theorem 3.2 generalizes the Kannan-type contraction considered in Kour et al. (2023). In particular:

1. The classical Kannan-type condition depends only on distances of the form $d(x, Tx)$ and $d(y, Ty)$, whereas the present condition incorporates additional rational-type terms.
2. The inclusion of a control function allows greater flexibility in handling nonlinear behaviours.
3. The contraction parameters are less restrictive, thereby enlarging the class of admissible mappings.

Hence, the present result extends the applicability of existing fixed-point theorems in extended cone b-metric spaces.

Corollary 3.3. *Weak Contractions in Extended Cone b-Metric Spaces have a Fixed Point.*

If $f : X \to X$ satisfies a weaker form of the rational-type contraction:

$$d_\Theta(f(x), f(y)) \leq s d_\Theta(x, y) + \phi(d_\Theta(x, y)), \quad \forall x, y \in X$$

where $s \in (0, 1/2)$ and $\phi : [0, \infty) \to [0, \infty)$ is an increasing function with $\phi(r) = 0$ if and only if $r = 0$, then f has a unique fixed point in X, and the sequence $\{f^n(x)\}$ converges to the fixed point u.

Proof. This follows from Theorem 3.2 by setting $t = 0$ and considering only the function $\phi(d_\Theta(x, y))$, which still ensures contraction. □

Example 3.4. *Fixed Point in an Extended Cone b-Metric Space*

Step 1: Define the Space and Cone b-Metric

Let $X = [0, \infty)$ with the extended cone b-metric:

$$d_\Theta(x, y) = |x - y| + \frac{|x - y|}{1 + |x - y|}, \quad \forall x, y \in X$$

This satisfies the triangle inequality for an extended cone b-metric space.

Let the function $f : X \to X$ be:

$$f(x) = \frac{x}{3}.$$

We verify that f satisfies the rational-type contraction:

$$d_\Theta(f(x), f(y)) \leq s[d_\Theta(x, f(x)) + d_\Theta(y, f(y))] + t d_\Theta(y, f(x)) + \phi(d_\Theta(x, y)).$$

For $s = 0.4$, $t = 0.3$, and $\phi(r) = \frac{r}{2(1+r)}$, let's check with $x = 6$ and $y = 3$:

$$f(6) = 2, \quad f(3) = 1.$$

Computing distances:

$$d_\Theta(6,3) = |6-3| + \frac{|6-3|}{1+|6-3|} = 3 + \frac{3}{4} = \frac{15}{4}.$$

$$d_\Theta(f(6), f(3)) = d_\Theta(2,1) = |2-1| + \frac{|2-1|}{1+|2-1|} = 1 + \frac{1}{2} = \frac{3}{2}.$$

Verifying the contractive condition:

$$\frac{3}{2} \leq s\left(\frac{15}{4} + \frac{3}{2}\right) + \frac{\phi(d_\Theta(6,3))}{1 + d_\Theta(6,3)}.$$

Approximating:

$$\begin{aligned}
&= 0.4 \times \left(\frac{30}{8} + \frac{12}{8}\right) + \frac{15/8}{1+15/4}.\\
&= 0.4 \times \frac{42}{8} + \frac{15}{23}\\
&= \frac{16.8}{8} + \frac{15}{23}\\
&= 2.1 + 0.65 = 2.75.
\end{aligned}$$

Since LHS $\leq$ RHS, the function satisfies the contraction, ensuring the existence of a unique fixed point at $u = 0$.

Theorem 3.5. *Let $(X^{\circledast}, d_\Theta)$ be a complete extended cone b-metric space, and let $N \subset X$ be a closed subset. Suppose that $f : N \to N$ is a mapping satisfying the rational-type contraction condition:*

$$d_\Theta(f(x), f(y)) \leq s\big[d_\Theta(x, f(x)) + d_\Theta(y, f(y))\big] + t\, d_\Theta(y, f(x)) + \frac{\alpha\, d_\Theta(x,y)}{1 + d_\Theta(x,y)}, \quad \forall x, y \in N,$$

where $s \in (0, \frac{1}{2})$, $t \in [0,1)$, and $\alpha > 0$.
Additionally, assume that there exist constants $\gamma \in (0,1)$ and $\delta > 0$ such that for any $x \in N$, there exists $x^* \in N$ satisfying:

$$d_\Theta(x^*, f(x^*)) \leq \gamma\, d_\Theta(x, f(x)),$$

$$d_\Theta(x^*, x) \leq \delta\, d_\Theta(x, f(x)).$$

Furthermore, for an arbitrary $x_0 \in N$, suppose that the sequence $\{x_n\}$ defined by

$$x_n = f^n(x_0)$$

satisfies:

$$\sup_{n_2 \geq 1} \lim_{n_1 \to \infty} \Theta(x_{n_1}, x_{n_1+1}, x_{n_2}) < \frac{1}{\gamma}.$$

Then f has a unique fixed point $u \in N$, and for each $x \in N$, the sequence $\{f^n(x)\}$ converges to u.
Moreover, the rate of convergence is given by:

$$d_\Theta(f(x_n), u) \leq \frac{s}{1-t}\left(\frac{s}{1-s}\right)^n d_\Theta(f(x_0), x_0), \quad n = 0, 1, 2, \ldots$$

Proof. Define the Iterative Sequence
Let $x_0 \in N$ be an arbitrary initial point, and define the sequence $\{x_n\}$ as follows:

$$x_{n+1} = f(x_n), \quad \forall n \geq 0.$$

We aim to prove that $\{x_n\}$ is Cauchy and converges to a unique fixed point u in N.
We analyze the given sequence:

$$d_\Theta(x_{n_1+1}, x_{n_1}) = d_\Theta(f(x_{n_1}), x_{n_1}).$$

By applying the contraction condition with $x = x_{n_1}$ and $y = x_{n_1+1}$, we get:

$$\begin{aligned} d_\Theta(f(x_{n_1+1}), f(x_{n_1})) \leq\ & s\big[d_\Theta(x_{n_1}, f(x_{n_1})) + d_\Theta(x_{n_1+1}, f(x_{n_1+1}))\big] \\ & + t\, d_\Theta(x_{n_1+1}, f(x_{n_1})) + \frac{\alpha d_\Theta(x_{n_1}, x_{n_1+1})}{1 + d_\Theta(x_{n_1}, x_{n_1+1})}. \end{aligned}$$

Using the assumption that:

$$d_\Theta(f(x_{n_1+1}), x_{n_1+1}) \leq \gamma d_\Theta(f(x_{n_1}), x_{n_1}),$$

$$d_\Theta(x_{n_1+1}, x_{n_1}) \leq \delta d_\Theta(f(x_{n_1}), x_{n_1}),$$

Substituting these into the contraction inequality:

$$\begin{aligned} d_\Theta(f(x_{n_1+1}), f(x_{n_1})) \leq\ & s\big[d_\Theta(x_{n_1}, f(x_{n_1})) + \gamma d_\Theta(f(x_{n_1}), x_{n_1})\big] \\ & + t\, d_\Theta(x_{n_1+1}, f(x_{n_1})) + \frac{\alpha\delta d_\Theta(f(x_{n_1}), x_{n_1})}{1 + \delta d_\Theta(f(x_{n_1}), x_{n_1})}. \end{aligned}$$

Factoring out $d_\Theta(f(x_{n_1}), x_{n_1})$:

$$d_\Theta(f(x_{n_1+1}), f(x_{n_1})) \leq (s(1+\gamma) + t\delta) d_\Theta(f(x_{n_1}), x_{n_1}) + \frac{\alpha\delta d_\Theta(f(x_{n_1}), x_{n_1})}{1 + \delta d_\Theta(f(x_{n_1}), x_{n_1})}.$$

Since $s \in (0, 1/2)$ and $t \in [0, 1)$, we ensure that:

$$s(1+\gamma) + t\delta < 1.$$

Thus, we get:

$$d_\Theta(f(x_{n_1+1}), f(x_{n_1})) \leq \rho d_\Theta(f(x_{n_1}), x_{n_1}),$$

where

$$\rho = s(1+\gamma) + t\delta + \frac{\alpha\delta}{1 + \delta d_\Theta(f(x_{n_1}), x_{n_1})}.$$

Since $\rho < 1$, this ensures the sequence is contractive and leads to convergence to the unique fixed point.
Applying this iteratively,

$$d_\Theta(f(x_{n_1+2}), f(x_{n_1+1})) \le \rho d_\Theta(f(x_{n_1+1}), x_{n_1+1}),$$

$$d_\Theta(f(x_{n_1+3}), f(x_{n_1+2})) \le \rho^2 d_\Theta(f(x_{n_1}), x_{n_1}).$$

Thus, by induction,

$$d_\Theta(f(x_{n_1+k+1}), f(x_{n_1+k})) \le \rho^k d_\Theta(f(x_0), x_0).$$

Since $\rho < 1$, this proves that the sequence converges exponentially fast to the unique fixed point u.
The given inequality follows from the extended cone b-metric triangular inequality:

$$d_\Theta(x_{n_1}, x_{n_2}) \le \Theta(x_{n_1}, x_{n_2}, x_{n_1+1})\big[d_\Theta(x_{n_1}, x_{n_1+1}) + d_\Theta(x_{n_1+1}, x_{n_2})\big].$$

Expanding further:

$$\begin{aligned} d_\Theta(x_{n_1+1}, x_{n_2}) \le \Theta(x_{n_1}, x_{n_2}, x_{n_1+1})&\Theta(x_{n_1+1}, x_{n_2}, x_{n_1+2}) \\ &\times \big[d_\Theta(x_{n_1+1}, x_{n_1+2}) + d_\Theta(x_{n_1+2}, x_{n_2})\big]. \end{aligned}$$

Thus, by iterating this process, we obtain:

$$d_\Theta(x_{n_1}, x_{n_2}) \le \prod_{k=n_1}^{n_2-1} \Theta(x_k, x_{n_2}, x_{k+1}) \sum_{k=n_1}^{n_2-1} d_\Theta(x_k, x_{k+1}).$$

Since the mapping f satisfies the rational-type contraction condition, we apply it to the term $d_\Theta(x_k, x_{k+1})$:

$$\begin{aligned} d_\Theta(x_k, x_{k+1}) &= d_\Theta(f(x_{k-1}), f(x_k)) \\ &\le s\big[d_\Theta(x_{k-1}, f(x_{k-1})) + d_\Theta(x_k, f(x_k))\big] \\ &\quad + t\, d_\Theta(x_k, f(x_{k-1})) + \frac{\alpha d_\Theta(x_{k-1}, x_k)}{1 + d_\Theta(x_{k-1}, x_k)}. \end{aligned}$$

By using the recurrence relation:

$$d_\Theta(x_k, f(x_k)) \le \gamma d_\Theta(x_{k-1}, f(x_{k-1})),$$

we substitute:

$$d_\Theta(x_k, x_{k+1}) \le s(1+\gamma) d_\Theta(x_{k-1}, x_k) + t\, d_\Theta(x_k, x_{k-1}) + \frac{\alpha d_\Theta(x_{k-1}, x_k)}{1 + d_\Theta(x_{k-1}, x_k)}.$$

Rewriting the sum:

$$\sum_{k=n_1}^{n_2-1} d_\Theta(x_k, x_{k+1}) \le \sum_{k=n_1}^{n_2-1} \Big[s(1+\gamma) d_\Theta(x_{k-1}, x_k) + t\, d_\Theta(x_k, x_{k-1}) + \frac{\alpha d_\Theta(x_{k-1}, x_k)}{1 + d_\Theta(x_{k-1}, x_k)}\Big].$$

Factoring out terms:

$$\sum_{k=n_1}^{n_2-1} d_\Theta(x_k, x_{k+1}) \leq \sum_{k=n_1}^{n_2-1} \rho d_\Theta(x_{k-1}, x_k),$$

where

$$\rho = s(1+\gamma) + t + \frac{\alpha}{1 + d_\Theta(x_{k-1}, x_k)}.$$

Since $\rho < 1$, this results in geometric convergence, meaning:

$$d_\Theta(x_{n_1}, x_{n_2}) \to 0 \quad \text{as} \quad n_2 \to \infty.$$

Thus, we obtain:

$$d_\Theta(x_{n_1}, x_{n_2}) \leq \prod_{k=n_1}^{n_2-1} \Theta(x_k, x_{n_2}, x_{k+1})\, \rho^k d_\Theta(x_0, x_1).$$

Since ρ^k tends to zero exponentially fast, this shows that:

$$\lim_{n_2\to\infty} d_\Theta(x_{n_1}, x_{n_2}) = 0.$$

Thus, the sequence $\{x_n\}$ is Cauchy and converges to a unique fixed point.

We shall show that u is a fixed point of f. Since $\{x_n\}$ is Cauchy in the complete space X, there exists $u \in N$ such that:

$$\lim_{n\to\infty} x_n = u.$$

Taking limits on both sides of $x_{n+1} = f(x_n)$:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n).$$

Since $x_n \to u$ and f is continuous, we obtain:

$$f(u) = u.$$

Thus, u is a fixed point.

Next we shall show that f has unique fixed point. Suppose there exists another fixed point $v \neq u$, such that $f(v) = v$. Then:

$$d_\Theta(u, v) = d_\Theta(f(u), f(v)).$$

Applying the contraction condition:

$$d_\Theta(u, v) \leq s\big[d_\Theta(u, f(u)) + d_\Theta(v, f(v))\big] + t\, d_\Theta(v, f(u)) + \frac{\alpha d_\Theta(u, v)}{1 + d_\Theta(u, v)}.$$

Since $f(u) = u$ and $f(v) = v$, we simplify:

$$d_\Theta(u, v) \leq t d_\Theta(v, u) + \frac{\alpha d_\Theta(u, v)}{1 + d_\Theta(u, v)}.$$

Rearranging:

$$(1-t)d_\Theta(u,v) \leq \frac{\alpha d_\Theta(u,v)}{1+d_\Theta(u,v)}.$$

Since $\frac{\alpha d_\Theta(u,v)}{1+d_\Theta(u,v)} \to 0$ as $d_\Theta(u,v) \to 0$, we conclude:

$$d_\Theta(u,v) = 0.$$

Thus, $u = v$, proving uniqueness. □

Remark:
Theorem 3.5 extends the fixed-point results to closed subsets of extended cone b-metric spaces under a generalized rational-type contraction. The inclusion of additional parameters provides greater flexibility and ensures convergence with an explicit rate.

Corollary 3.6. *Fixed Point Theorem for Rational-Type Weak Contractions.*
If $f : N \to N$ satisfies:

$$d_\Theta(f(x), f(y)) \leq \frac{\alpha d_\Theta(x,y)}{1+d_\Theta(x,y)}$$

where $\alpha > 0$, then f has a unique fixed point.

Proof. This follows by setting $s = 0, t = 0$ in our theorem. □

Example 3.7. *Let $X = [0,\infty)$ and define the extended cone b-metric $d_\Theta : X \times X \to \mathbb{R}$ as:*

$$d_\Theta(x,y) = |x-y| + \frac{|x-y|}{1+|x-y|}$$

for all $x, y \in X$.
This satisfies the extended cone b-metric properties:

1. $d_\Theta(x,y) = 0$ *if and only if* $x = y$.
2. $d_\Theta(x,y) = d_\Theta(y,x)$.
3. *There exists a function* $\Theta : X \times X \times X \to [1,\infty)$ *defined by*

$$\Theta(x,y,z) = 1 + |x-y| + |y-z|$$

such that

$$d_\Theta(x,z) \leq \Theta(x,y,z)[d_\Theta(x,y) + d_\Theta(y,z)].$$

Consider the mapping $f : X \to X$ defined by:

$$f(x) = \frac{x}{2}.$$

This function satisfies the rational-type contraction condition:

$$d_\Theta(f(x), f(y)) \leq s\big[d_\Theta(x, f(x)) + d_\Theta(y, f(y))\big] + t\, d_\Theta(y, f(x)) + \frac{\alpha\, d_\Theta(x, y)}{1 + d_\Theta(x, y)}.$$

Let $s = 0.4$*,* $t = 0.3$*, and* $\alpha = 0.5$*. Consider* $x = 4$*,* $y = 2$*.*
Consider

$$f(4) = 2, \quad f(2) = 1.$$

We compute:

$$d_\Theta(4, 2) = |4 - 2| + \frac{|4 - 2|}{1 + |4 - 2|} = 2 + \frac{2}{3} = \frac{8}{3}.$$

$$d_\Theta(f(4), f(2)) = d_\Theta(2, 1) = |2 - 1| + \frac{|2 - 1|}{1 + |2 - 1|} = 1 + \frac{1}{2} = \frac{3}{2}.$$

Now, verifying the contractive condition:
LHS:

$$d_\Theta(f(4), f(2)) = \frac{3}{2}.$$

RHS:

$$\begin{aligned}
&s\big[d_\Theta(4, 2) + d_\Theta(2, 1)\big] + t\, d_\Theta(2, 2) + \frac{\alpha\, d_\Theta(4, 2)}{1 + d_\Theta(4, 2)} \\
&= 0.4\left(\frac{8}{3} + \frac{3}{2}\right) + 0 + \frac{0.5 \cdot \frac{8}{3}}{1 + \frac{8}{3}} \\
&= 0.4\left(\frac{16}{6} + \frac{9}{6}\right) + \frac{\frac{4}{3}}{\frac{11}{3}} \\
&= 0.4 \cdot \frac{25}{6} + \frac{4}{11} \\
&= \frac{10}{6} + \frac{4}{11} \\
&= \frac{5}{3} + \frac{4}{11}.
\end{aligned}$$

$$\begin{aligned}
&\frac{5}{3} \approx 1.67, \quad \frac{4}{11} \approx 0.36, \\
&\approx 1.67 + 0.36 = 2.03.
\end{aligned}$$

Since, LHS $\leq$ *RHS, the function satisfies the rational-type contraction condition. Thus, the function has a unique fixed point at u=0, because:*
To find the fixed point u:

$$f(u) = \frac{u}{2} = u \implies u = 0.$$

This confirms that our improved theorem applies.

4. Conclusion

In this work, we introduced a new rational-type weak contraction mapping in the context of extended cone b-metric spaces, which also encompasses classical cone b-metric spaces. This extension enables the development of more general fixed-point theorems that guarantee the existence and uniqueness of fixed points under weaker conditions. The main novelty lies in relaxing the contraction assumptions and incorporating a rational-type control function, which broadens the class of mappings considered. Our results provide a generalization and refinement of the recent results of Kour et al. (2023) [10] by weakening restrictive assumptions. Furthermore, the obtained results improve those of Huaping et al. (2021) and Alqahtani et al. (2019) [20]in terms of convergence behavior and contraction conditions. The sequence-dependent convergence analysis and explicit rate bounds contribute to the stability and robustness of iterative methods. This structure is flexible and can be applied to nonlinear and weakly contractive mappings. The results are also applicable to nonlinear differential equations, integral equations, dynamical systems, and optimization.

5. Future Work

The scope for future extensions is substantial. One natural progression is the application of the current fixed-point results to broader classes of generalized spaces, such as quasi-metric, fuzzy metric, and probabilistic metric spaces. These spaces play crucial roles in areas like decision-making, machine learning, and uncertainty modeling. Moreover, a promising research direction involves the application of the developed results to solve real-world problems in optimization theory, stability analysis, and control theory. For instance, the extended cone b-metric framework may be used to investigate equilibrium conditions in dynamic systems or solve integral equations in engineering. Finally, hybrid contraction principles combining probabilistic and rational-type conditions may further enhance the robustness of the framework. These refinements can yield stronger theoretical guarantees and support more complex systems under uncertainty, paving the way for interdisciplinary applications in science and engineering.

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