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SOME NEW INCOMPLETE ELLIPTIC INTEGRALS

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Abstract

In his lost notebook, Ramanujan has devoted pages 51–53 to the development of the theory of elliptic integrals of the first kind. Ramanujan was the first mathematician to discover, this kind of integrals. Motivated by his work many mathematicians have worked on this area. In this article, we establish three new interesting incomplete elliptic integrals of the first kind.

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1. Introduction

The incomplete elliptic integral of the first kind is defined by the integral of the form

$$\int_o^\alpha \frac{d\phi}{\sqrt{1-k^2 sin^2\phi}},$$

where $0 < \alpha \leq \frac{\pi}{2}$ and $|k| < 1$. As usual for any complex numbers a and q, we define

$$(a;b)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad |q| < 1.$$

Throughout this article, we assume $|q| < 1$, unless and otherwise it is mentioned. On page 197 of his second notebook [6], Ramanujan defined his theta function $f(a,b)$ by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \qquad |ab| < 1.$$

By Jacobi's triple product identity, we have

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Further, Ramanujan defined the following three special cases for $f(a,b)$:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}},$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}.$$

He also defined

$$\chi(q) = (-q; q^2).$$

On pages 51–53 of his lost notebook [7], Ramanujan recorded the integral of following kind:

$$\int_0^q (product\ of\ theta\ functions) dq = \int_{x(q)}^{y(q)} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}},$$

where $0 \leq x(q) < \frac{\pi}{2}$ or $0 < x(q) < y(q) \leq \frac{\pi}{2}$ and $|k| < 1$. For example, Ramanujan recorded the following :

$$5^{\frac{3}{4}} \int_0^q \frac{f^2(-t) f^2(-t^5)}{\sqrt{t}} dt = \sqrt{5} \int_0^{2\tan^{-1}\left(\frac{5^{\frac{1}{4}}\sqrt{q}\psi(q^5)}{\psi(q)}\right)} \frac{d\theta}{\sqrt{1 - \epsilon 5^{\frac{-1}{2}} \sin^2\theta}},$$

where $\epsilon = \dfrac{\sqrt{5}+1}{2}$.

The elliptic integral identities which are recorded by Ramanujan were first proved by S. Raghavan and S. S. Rangachari [5] by employing the theory of modular forms. Thereafter, B. C. Berndt et.al. [3], gave an alternative proof for those identities using the Ramanujan's theory of theta functions. Also, they have obtained some new elliptic integrals identities in [3]. On page 52 of his lost notebook, Ramanujan wrote a statement with the delimiter '?', that was, *"cases ? for φ."* In search of the idea behind this statement, K. R. Vasuki and E. N. Bhuvan found out two new elliptic integrals involving the theta function $\varphi(q)$ in [9]. For further developments of this research area one may refer to [2, 4, 11]. Motivated by these work on elliptic integrals, in this article, we establish three new incomplete elliptic integrals of first kind. In fact, we prove the following theorems:

THEOREM 1.1. *Let* $x = \dfrac{f_1^2 f_{18}}{f_2 f_9^2}$. *Then*

$$\int_0^{2\tan^{-1}\left(\frac{\sqrt{x}}{\sqrt[4]{3}}\right)} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2(\theta)}} = -3^{\frac{1}{4}} \int_0^q \left(\sqrt{\frac{f_9 f_{18} f_2^9}{f_1^3}} + 3q^2 \sqrt{\frac{f_1 f_2 f_{18}^9}{f_9^3}} \right) \frac{f_6}{f_3} dq. \tag{1.1}$$

Theorem 1.2. *Let* $y = \dfrac{f_2^2 f_9}{q f_1 f_{18}^2}$. *Then*

$$\int_0^{2\tan^{-1}\left(\frac{\sqrt{y}}{\sqrt[4]{3}}\right)} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2(\theta)}} = \frac{-3^{\frac{1}{4}}}{4}\int_0^q \frac{1}{\sqrt{q}}\left(\sqrt{\frac{f_9 f_{18} f_1^9}{f_2^3}} + 3\sqrt{\frac{f_1 f_2 f_9^9}{f_{18}^3}}\right)\frac{f_3}{f_6}dq. \tag{1.2}$$

Theorem 1.3. *Let* $z = \dfrac{f_1^3}{q f_9^3}$. *Then*

$$\int_0^{2\tan^{-1}(\frac{\sqrt{z}}{3\sqrt{3}})} \frac{dz}{\sqrt{1-(\frac{2-\sqrt{3}}{4})\sin^2(\theta)}} = -3^{\frac{3}{4}}\int_0^q \frac{f_3^4}{q^{\frac{1}{2}}}dq.$$

In the next section, we recall certain definitions, notations, and preliminary results that are required to prove our main results. In Section 3, we prove Theorem 1.1–1.3 using the Ramanujan's theory of theta functions.

2. Preliminaries

For convenience, we set

$$f_n := f(-q^n) = (q^n;q^n)_\infty, \tag{2.1}$$

where n is any positive integer. It is easy to see that

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \quad \varphi(-q) = \frac{f_1^2}{f_2}, \quad \psi(q) = \frac{f_2^2}{f_1}, \quad \psi(-q) = \frac{f_1 f_4}{f_2},$$
$$f(q) = \frac{f_2^3}{f_1 f_4}, \quad \chi(-q) = \frac{f_1}{f_2}, \quad \text{and} \quad \chi(q) = \frac{f_2^2}{f_1 f_4}.$$

The Eisenstein series of weight-2, P_n is defined by

$$P_n = P(q^n) := 1 - 24\sum_{k=1}^{\infty}\frac{kq^{nk}}{1-q^{nk}}.$$

On page 241 of his second notebook [6], Ramanujan recorded many interesting theta function identities. Among these, we consider the following equivalent theta function identities:

$$1 + \frac{1}{v(q)} = \frac{\psi(q)}{q\psi(q^9)}, \tag{2.2}$$

$$2v(q) = 1 - \frac{\varphi(-q)}{\varphi(-q^9)}, \tag{2.3}$$

$$1+\frac{1}{v^3(q)}=\frac{\psi^4(q^3)}{q\psi^4(q^9)}, \tag{2.4}$$

$$\frac{\varphi(-q)}{\varphi(-q^9)}=1+\sqrt[3]{\frac{\varphi^4(-q^3)}{\varphi^4(-q^9)}-1}, \tag{2.5}$$

and

$$3+\frac{f^3(-q)}{qf^3(-q^9)}=\sqrt[3]{27+\frac{f^{12}(-q^3)}{q^3f^{12}(-q^9)}}, \tag{2.6}$$

where $v(q)=q\frac{\chi(-q^3)}{\chi^3(-q^9)}$. For a proof of the above theta function identities, one may refer to [1, pp. 345–347]. Ramanujan devoted Chapter 21 of his second notebook [6] to the development of the Eisenstein series. We consider the following beautiful Eisenstein series identity recorded by Ramanujan on Page 255 of his second notebook [6]:

$$\frac{1}{8}\left[-P_1+9P_9\right]=\frac{q^{\frac{1}{3}}f_3^6}{f_1f_9}\left[\frac{f_1^3}{qf_9^3}+9+27q\frac{f_9^3}{f_1^3}\right]^{\frac{1}{3}}. \tag{2.7}$$

For a proof of the above one may refer to [1, p. 476]. From Entry 3 (i) and (ii) in Chapter 19 of his second notebook [6, p. 229], Ramanujan recorded two beautiful identities. The equivalent form of those two identities are given by:

$$P_1-P_2-3P_3+3P_6=24q\psi^2(q)\psi^2(q^3) \tag{2.8}$$

and

$$P_1-4P_2-3P_3+12P_6=6\varphi^2(-q)\varphi^2(-q^3). \tag{2.9}$$

For a proof of the above, one may refer to [1, pp. 223–225] and [10].

3. Proof of the Theorem 1.1–1.3

In this section, we prove our main results using elementary techniques by employing the Ramanujan's theory of theta functions efficiently.

Proof of Theorem 1.1. Let

$$x=\frac{\varphi(-q)}{\varphi(-q^9)}=\frac{f_1^2f_{18}}{f_2f_9^2}. \tag{3.1}$$

Applying logarithm on both sides of (3.1), then differentiating the resulting identity with respect to q and finally multiplying both the sides by q, we obtain

$$\frac{q}{x}\frac{dx}{dq}=\sum_{n=1}^{\infty}\Big[-2\frac{nq^n}{1-q^n}+2\frac{nq^{2n}}{1-q^{2n}}+18\frac{nq^{9n}}{1-q^{9n}}-18\frac{nq^{18n}}{1-q^{18n}}\Big]. \tag{3.2}$$

Using the definition of P_n in the right side of (3.2), we obtain

$$\frac{q}{x}\frac{dx}{dq}=-\frac{2}{24}[1-P_1]+\frac{2}{24}[1-P_2]+\frac{18}{24}[1-P_9]-\frac{18}{24}[1-P_{18}].$$

Which implies

$$\frac{q}{x}\frac{dx}{dq} = \frac{1}{12}\left[P_1 - P_2 - 9P_9 + 9P_{18}\right]. \tag{3.3}$$

Changing q to q^3 in (2.8) and multiplying throughout by three, we obtain

$$-3P_3 + 3P_6 + 9P_9 - 9P_{18} = 72q^3\psi^2(q^3)\psi^2(q^9). \tag{3.4}$$

Adding identities (2.8) and (3.4), we find that

$$-P_1 + P_2 + 9P_9 - 9P_{18} = 24q\psi^2(q^3)\left[\psi^2(q) + 3q^2\psi^2(q^9)\right]. \tag{3.5}$$

Employing (3.5) in (3.3), we obtain

$$\frac{q}{x}\frac{dx}{dq} = -2q\psi^2(q^3)\left[\psi^2(q) + 3q^2\psi^2(q^9)\right].$$

Equivalently, we have

$$\frac{q}{x}\frac{dx}{dq} = -2q^{\frac{5}{2}}\frac{\psi^2(q^3)}{q^{\frac{3}{2}}\psi^2(q^9)}\left[\psi^2(q)\psi^2(q^9) + 3q^2\psi^4(q^9)\right]. \tag{3.6}$$

From these identities (2.2)–(2.4) and (3.1), one can easily see that

$$2\frac{q\chi(-q^3)}{\chi^3(-q^9)} = 1 - x, \tag{3.7}$$

$$\frac{\psi(q)}{q\psi(q^9)} = \frac{3-x}{1-x}, \tag{3.8}$$

and

$$\frac{\psi^4(q^3)}{q^3\psi^4(q^9)} = \frac{(3-x)(3+x^2)}{(1-x)^3}. \tag{3.9}$$

Employing (3.8) in (3.6), we obtain

$$\frac{q}{x}\frac{dx}{dq} = -2q^{\frac{5}{2}}\left[\psi^2(q)\psi^2(q^9) + 3q^2\psi^4(q^9)\right]\sqrt{\frac{(3-x)(3+x^2)}{(1-x)^3}}. \tag{3.10}$$

Which implies

$$\frac{dx}{\sqrt{x}\sqrt{3+x^2}} = -2q^{\frac{3}{2}}\sqrt{\frac{3-x}{(1-x)}}\frac{\sqrt{x}}{(1-x)}\left[\psi^2(q)\psi^2(q^9) + 3q^2\psi^4(q^9)\right]dq. \tag{3.11}$$

Employing (3.7), (3.8) in the right side of the above, we obtain

$$\int_0^x \frac{dx}{\sqrt{x}\sqrt{3+x^2}} = -2\int_0^q \left(\sqrt{\frac{f_9 f_{18} f_2^9}{f_1^3}} + 3q^2\sqrt{\frac{f_1 f_2 f_{18}^9}{f_9^3}}\right)\frac{f_6}{f_3}dq. \tag{3.12}$$

Setting $x = \sqrt{3}\tan^2(\frac{\theta}{2})$ in the left side of the above, we obtain

$$\int_0^{2\tan^{-1}\left(\frac{\sqrt{x}}{\sqrt[4]{3}}\right)} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2(\theta)}} = -3^{\frac{1}{4}}\int_0^q \left(\sqrt{\frac{f_9 f_{18} f_2^9}{f_1^3}} + 3q^2\sqrt{\frac{f_1 f_2 f_{18}^9}{f_9^3}}\right)\frac{f_6}{f_3}dq. \tag{3.13}$$

This completes the proof. □

Proof of Theorem 1.2. Let

$$y = \frac{\psi(q)}{q\psi(q^9)} = \frac{f_2^2 f_9}{q f_1 f_{18}^2}. \tag{3.14}$$

Applying logarithm on both sides of the above identities, then differentiating the resulting identity with respect to q, finally multiplying both sides by q, we obtain

$$\frac{q}{y}\frac{dy}{dq} = -1 + \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 4\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}} - 9\sum_{n=1}^{\infty}\frac{nq^{9n}}{1-q^{9n}} + 36\sum_{n=1}^{\infty}\frac{nq^{18n}}{1-q^{18n}}. \tag{3.15}$$

Using the definition of P_n in the right side of the above equation, we obtain

$$\frac{q}{y}\frac{dy}{dq} = \frac{1}{24}\left[-P_1 + 4P_2 + 9P_9 - 36P_{18}\right]. \tag{3.16}$$

Changing q to q^3 in (2.9) and multiplying throughout by three, we have

$$3P_3 - 12P_6 - 9P_9 + 36P_{18} = 18\varphi^2(-q^3)\varphi^2(-q^9). \tag{3.17}$$

Adding the above identity to (2.9), we find that

$$P_1 - 4P_2 - 9P_9 + 36P_{18} = 6\varphi^2(-q^3)\left[\varphi^2(-q) + 3\varphi^2(-q^9)\right]. \tag{3.18}$$

Equivalently, we have

$$P_1 - 4P_2 - 9P_9 + 36P_{18} = 6\frac{\varphi^2(-q^3)}{\varphi^2(-q^9)}\left[\varphi^2(-q)\varphi^2(-q^9) + 3\varphi^4(-q^9)\right]. \tag{3.19}$$

Employing (3.19) in (3.16), we have

$$\frac{q}{y}\frac{dy}{dq} = \frac{-\varphi^2(-q^3)}{4\varphi^2(-q^9)}\left[\varphi^2(-q)\varphi^2(-q^9) + 3\varphi^4(-q^9)\right]. \tag{3.20}$$

From the definition of y, (2.2), (2.3), and (2.5), we can easily deduce

$$\frac{q\chi(-q^3)}{\chi^3(-q^9)} = \frac{1}{y-1}, \tag{3.21}$$

$$\frac{\varphi(-q)}{\varphi(-q^9)} = \frac{3-y}{1-y}, \tag{3.22}$$

and

$$\frac{\varphi^4(-q^3)}{\varphi^4(-q^9)} = \frac{(3-y)(3+y^2)}{(1-y)^3}. \tag{3.23}$$

Employing (3.23) in (3.20), we obtain

$$\frac{q}{y}\frac{dy}{dq} = \frac{-\sqrt{(3-y)(3+y^2)}}{4\sqrt{(1-y)^3}}\left[\varphi^2(-q)\varphi^2(-q^9) + 3\varphi^4(-q^9)\right]. \tag{3.24}$$

Equivalently, we have

$$\frac{dy}{\sqrt{y}\sqrt{3+y^2}} = \frac{-\sqrt{(3-y)}\sqrt{y}}{4q\sqrt{(1-y)^3}}\left[\varphi^2(-q)\varphi^2(-q^9) + 3\varphi^4(-q^9)\right]dq. \tag{3.25}$$

Employing (3.21)–(3.23) in the right side of (3.25), we obtain

$$\int_0^y \frac{dy}{\sqrt{y}\sqrt{3+y^2}} = \frac{-1}{4}\int_0^q \frac{1}{\sqrt{q}}\left(\sqrt{\frac{f_9 f_{18} f_1^9}{f_2^3}} + 3\sqrt{\frac{f_1 f_2 f_9^9}{f_{18}^3}}\right)\frac{f_3}{f_6}dq. \tag{3.26}$$

Setting $y = \sqrt{3}\tan^2(\theta/2)$ in the left side of (3.26), we find that

$$\int_0^{2\tan^{-1}\left(\frac{\sqrt{y}}{\sqrt[4]{3}}\right)} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2(\theta)}} = \frac{-3^{\frac{1}{4}}}{4}\int_0^q \frac{1}{\sqrt{q}}\left(\sqrt{\frac{f_9 f_{18} f_1^9}{f_2^3}} + 3\sqrt{\frac{f_1 f_2 f_9^9}{f_{18}^3}}\right)\frac{f_3}{f_6}dq. \tag{3.27}$$

This completes the proof. □

Proof of Theorem 1.3. Let

$$z = \frac{f_1^3}{qf_9^3}. \tag{3.28}$$

Applying logarithm on both sides of the above identity, then differentiating the resulting identity with respect to q, and finally multiplying both sides by q, we obtain

$$\frac{q}{z}\frac{dz}{dq} = -1 - 3\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + 27\sum_{n=1}^{\infty}\frac{nq^{9n}}{1-q^{9n}}.$$

Using the definition of P_n in the right side of the above equation, we obtain

$$\frac{q}{z}\frac{dz}{dq} = \frac{1}{8}\left[P_1 - 9P_9\right]. \tag{3.29}$$

On cubing both sides of (2.6) and after elementary simplification, we find that

$$\frac{f_1^3}{qf_9^3} + 9 + 27q\frac{f_9^3}{f_1^3} = \frac{f_3^{12}}{qf_9^6 f_1^6}.$$

Employing the above and (2.7) in (3.29), we obtain

$$\frac{q}{z}\frac{dz}{dq} = -\frac{f_3^{10}}{f_1^3 f_9^3}. \tag{3.30}$$

From the definition of z and (2.6), one can easily deduce the following:

$$\frac{f_3^6}{f_1^3 f_9^3} = q^{\frac{1}{2}}\sqrt{z+9+\frac{27}{z}}. \tag{3.31}$$

Employing the above in (3.30), we obtain

$$\int_0^z \frac{dz}{\sqrt{z}\sqrt{z^2+9z+27}} = -\int_0^q \frac{f_3^4}{q^{\frac{1}{2}}}dq.$$

Setting $x = 3\sqrt{3}\tan^2(\theta/2)$ in the left side of the above, we obtain

$$\int_0^{2\tan^{-1}(\frac{\sqrt{z}}{3\sqrt{3}})} \frac{dz}{\sqrt{1-(\frac{2-\sqrt{3}}{4})\sin^2(\theta)}} = -3^{\frac{3}{4}}\int_0^q \frac{f_3^4}{q^{\frac{1}{2}}}dq.$$

This completes the proof. □

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