GANITA, Vol. 76(1), 2026, 75–86

THE STUDY OF PARA SASAKIAN MANIFOLDS ADMITTING $*$-CONFORMAL η-RICCI SOLITONS

ABHISHEK SINGH and SANGEETA GAUTAM ✉

Abstract

The objective of this paper is to investigate the para-Sasakian manifolds admitting $*$-Conformal η-Ricci solitons. We characterize a para-Sasakian manifold admitting $*$-Conformal η-Ricci solitons. Further, we study Ricci-recurrent para-Sasakian manifolds admitting $*$-Conformal η-Ricci solitons. We discuss η-Parallel ϕ-tensor in the manifold admitting $*$-Conformal η-Ricci solitons. We have also studied Conformal curvature tensor on para-Sasakian manifolds admitting $*$-Conformal η-Ricci solitons. Finally we have given an example of a 5-dimensional para-Sasakian manifold which verify our results.

2010 *Mathematics subject classification:* 53D15, 53C25, 53C50.

Keywords and phrases: Para Sasakian manifold, $*$-Conformal η-Ricci solitons, η-Einstein manifold, generalized η-Einstein manifold, Conformal curvature tensor.

1. Introduction

T. Takahashi [31] introduced almost contact manifolds equipped with an associated pseudo-Riemannian metric. In particular, he studied Sasakian manifolds equipped with an associated pseudo-Riemannian metric. T. Adati and K. Matsumoto [1] defined para-Sasakian and special para-Sasakian manifolds. These manifolds are special cases of an almost paracontact manifold introduced by I. Sato and K. Matsumoto [27]. In the same paper, the authors studied conformally symmetric para-Sasakian manifolds and they proved that an n-dimensional conformally symmetric para-Sasakian manifold is conformally flat and special para-Sasakian. U. C. De and N. Guha [13] showed that an n-dimensional Weyl-semisymmetric para-Sasakian manifold is conformally flat. Sharma [28] investigated the Ricci solitons in contact geometry in 2008.
Later on, Ricci solitons in contact metric manifolds have been studied by several authors like Bagewadi et al.[2, 3], Bejan and Crasmareanu [4], Blaga [6], Chandra et al. [10], Chen and Deshmukh [11], Chenxu and Meng [21], Nagaraja and Premalatha [23]. η-Ricci soliton have been studied by various authors [7, 8, 25, 34, 36] and many others. Recently $*$-Conformal η-Ricci solitons in ϵ-Kenmotsu and ϵ-para Sasakian manifolds have been studied by R. Prasad and U. C. De [14, 20]. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is generalization of an Einstein metric such

that it satisfies the following condition [19]

$$£_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is the Ricci tensor, $£_V$ is the Lie derivative operator along the vector field V on (M, g) and λ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero or positive.

As a generalization of the Ricci soliton, the notion of an η-Ricci soliton was introduced by Chao and Kimura [12] and given by the equation

$$£_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where λ and μ are constants, g is a Riemannian metric and S is the Ricci tensor associated with g. The conformal Ricci flow on M is defined by A. E. Fischer [17]

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg \qquad r = -1, \tag{1.3}$$

where p is a scalar non-dynamical field (time dependent scalar field), r is scalar curvature of manifold and n is n-dimensional manifold. In the Riemannian setting, the notion of a conformal Ricci soliton was introduced by Basu and Bhattacharyya [5] on a Kenmotsu manifold of dimension n as

$$£_V g + 2S = -[2\lambda - (p + \frac{2}{n})]g, \tag{1.4}$$

where λ is a constant and $£_V$ is the Lie derivative along the vector field V. The notion of conformal η-Ricci soliton introduced by M. D. Siddiqui [29] is defined as

$$£_V g + 2S + [2\lambda - (p + \frac{2}{n})]g + 2\mu\eta \otimes \eta = 0, \tag{1.5}$$

Tachibana [32] was introduced the notion of $*$-Ricci tensor on almost Hermitian manifolds in 1959. Again Hamada [18] studied real hypersurfaces of non-flat complex space forms. A Riemannian metric g on a smooth manifold M is called a $*$-Ricci soliton if their exits [22]

$$(£_V g)(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) = 0, \tag{1.6}$$

where V is a smooth vector field and λ is a real number.

$$S^*(X, Y) = g(Q^* X, Y) = Trace[\phi o R(X, \phi Y)] \tag{1.7}$$

where ϕ is a tensor field of $(0, 2)$ type and Q^* is $(1, 1)$ type $*$-Ricci operator, $\forall X, Y \in M$. In an n-dimesional Riemannian manifold (M, g), $*$-Conformal η-Ricci soliton is defined as

$$£_\xi g + 2S^* + (2\lambda - (p + \frac{2}{n}))g + 2\mu\eta \otimes \eta = 0, \tag{1.8}$$

where $£_\xi$ is the Lie derivative along the vector field ξ, S^* is the $*$-Ricci tensor and λ, μ are constant.

The present work is organized as follows: In section-1 introduction is discussed. Section-2 is equipped with some prerequisites about para-Sasakian manifolds. Section-3, deals with $*$-Conformal η-Ricci solitons on para-Sasakian manifolds with some interesting results. Section-4 concerned with the study of Ricci recurrent para-Sasakian manifold admitting $*$-Conformal η-Ricci solitons. In section-5, we study η-Parallel ϕ-tensor in para-Sasakian manifolds admitting $*$-Conformal η-Ricci solitons. Section-6 is devoted to the study of Conformal curvature tensor on para-Sasakian manifolds with respect to the $*$-Conformal η-Ricci solitons. In the last section, we have given an example of 5-dimensional para-Sasakian manifolds admitting $*$-Conformal η-Ricci solitons in the support of our results.

2. Preliminaries

A n-dimensional smooth manifold M together with a $(1,1)$-tensor field ϕ, a vector field ξ, η is 1-form and pseudo-Riemannian metric g is called an almost paracontact metric manifold if

$$\phi^2 X = X - \eta(X)\xi \qquad \eta(\xi) = 1, \qquad \eta(\phi X) = 0 \qquad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = g(\phi X, Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$g(\xi, \xi) = \eta(\xi) = 1, \tag{2.5}$$

$$(\nabla_X \phi)Y = -g(\phi X, \phi Y)\xi - \eta(Y)\phi^2 X, \tag{2.6}$$

where ∇ denotes the Levi-Civita connection with respect to g, then manifold (M, ϕ, ξ, η, g) is called a para-Sasakian Manifold [33], it can be shown that

$$(\nabla_X \xi) = \phi(X), \tag{2.7}$$

$$(\nabla_X \eta)Y = g(Y, \phi X), \tag{2.8}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.9}$$

$$R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.10}$$

$$\eta(R(X, Y)Z) = (g(X, Z)\eta(Y) - g(Y, Z)\eta(X)) \tag{2.11}$$

$$S(X, \xi) = -(n-1)\eta(X) \tag{2.12}$$

$$Q\xi = -(n-1)\xi. \tag{2.13}$$

for any vector fields X, Y, Z on M. Where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator.

Definition 2.1. A para-Sasakian manifold M is said to be a generalized η-Einstein manifold if the following condition [35]

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + cg(\phi X, Y). \tag{2.14}$$

holds on M, where a, b and c are smooth functions on M. If $c = 0, b = c = 0$, and $a = c = 0$, then the manifold is called an η-Einstein manifold, an Einstein and a special type of η-Einstein manifold respectively.

Lemma 2.2. *In n-dimensional para-Sasakian manifold the $*$-Ricci tensor is given by [16]*

$$S^*(Y,Z) = S(Y,Z) - (n-2)g(Y,Z) - \psi g(Y,\phi Z) + (2n-3)\eta(Y)\eta(Z). \tag{2.15}$$

where S and S^* are the Ricci tensor and $*$-Ricci tensor of type of (0,2) respectively and ψ =trace ϕ.

3. $*$-Conformal η-Ricci solitons on para-Sasakian manifolds

Let us consider an n-dimensional para-Sasakian manifold equipped with $*$-Conformal η-Ricci solitons (g, ξ, λ, μ), then equation (1.8) can be written as

$$(\pounds_\xi g)(Y,Z) + 2S^*(Y,Z) + (2\lambda - (p + \frac{2}{n}))g(Y,Z) + 2\mu\eta(Y)\eta(Z) \quad = \quad 0, \tag{3.1}$$

$\forall\ X, Y \in M$.
Now

$$(\pounds_\xi g)(Y,Z) \quad = \quad g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) = 2g(\phi Y, Z). \tag{3.2}$$

In view of (3.1) and (3.2), we have

$$S^*(Y,Z) = -[\lambda - \frac{1}{2}(p + \frac{2}{n})]g(Y,Z) - \mu\eta(Y)\eta(Z) - g(\phi Y, Z). \tag{3.3}$$

Using (3.3) in (2.15), we have

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z) + cg(\phi Y, Z), \tag{3.4}$$

where $a = (n-2) - \lambda + \frac{1}{2}(p + \frac{2}{n})$, $b = -(2n - 3 + \mu)$ and $c = (\psi - 1)$.
Replacing Z by ξ in (3.4), we get

$$S(Y,\xi) = (a+b)\eta(Y). \tag{3.5}$$

$$Q\xi = (a+b)\xi. \tag{3.6}$$

From (2.12) and (3.5), we get

$$\lambda + \mu = \frac{1}{2}(p + \frac{2}{n}). \tag{3.7}$$

Thus, we leads to the following theorem:

THEOREM 3.1. *If the para-Sasakian manifold admits a $*$-Conformal η-Ricci solitons, then M is a generalized η-Einstein manifold of the form (3.4) and the scalars λ and μ are related by $\lambda + \mu = \frac{1}{2}(p + \frac{2}{n})$.*

Let (g, V, λ, μ) be a $*$-Conformal η-Ricci solitons on a para-Sasakian manifold such that V is pointwise collinear with ξ i.e. V=$h\xi$, where h is a function. Then (1.1) implies that

$$\pounds_{h\xi} g + 2S^* + (2\lambda - (p + \frac{2}{n}))g + 2\mu\eta \otimes \eta \quad = \quad 0. \tag{3.8}$$

Using the property of the Lie derivative and Levi-Civita cinnection in (3.8), we have

$$hg(\nabla_X \xi, Y) + (Xh)\eta(Y) + hg(X, \nabla_Y \xi) + (Yh)\eta(X)$$
$$+ 2S^*(X, Y) + (2\lambda - (p + \frac{2}{n}))g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

In view of (2.7) and (2.15), we have

$$\begin{aligned} &2hg(\phi X, Y) + (Xh)\eta(Y) + (Yh)\eta(X) + 2S(X, Y) + (2\lambda - 2(n-2) \\ &- (p + \frac{2}{n}))g(X, Y) - 2\psi g(X, \phi Y) + (4n + 2\mu - 6)\eta(X)\eta(Y) = 0. \end{aligned} \tag{3.9}$$

Replacing Y by ξ and using (2.1) and (2.12) in (3.9), we have

$$(Xh) + [(\xi h) + 2\lambda - (p + \frac{2}{n}) + 2\mu]\eta(X) = 0. \tag{3.10}$$

Again replacing X by ξ in (3.10) and using (2.1), we have

$$(\xi h) = -[\lambda - \frac{1}{2}(p + \frac{2}{n}) + 2\mu]. \tag{3.11}$$

Using (3.11) in (3.10), we have

$$(Xh) = -[\lambda + \mu - \frac{1}{2}(p + \frac{2}{n})]\eta(X). \tag{3.12}$$

It is clear that h is constant if $\lambda + \mu = \frac{1}{2}(p + \frac{2}{n})$. Therefore from (3.9), we have

$$\begin{aligned} S(X, Y) \quad = \quad &-[\lambda - (n-2) - \frac{1}{2}(p + \frac{2}{n})]g(X, Y) + (\psi - h)g(\phi X, Y) \\ &+(2n + \mu - 3)\eta(X)\eta(Y). \end{aligned}$$

Hence we leads to the following theorem:

THEOREM 3.2. *If (g, V, λ, μ) is a $*$-Conformal η-Ricci solitons on para-Sasakian manifold such that V is a pointwise collinear with ξ, then V is constant multiple of ξ and the manifold is a generalized η-Einstein manifold, provided $\lambda + \mu = \frac{1}{2}(p + \frac{2}{n})$.*

4. Ricci recurrent para-Sasakian manifolds admitting $*$-Conformal η-Ricci solitons

In this section, we study Ricci recurrent para-Sasakian manifolds admitting $*$-Conformal η-Ricci solitons

Definition 4.1. An n-dimensional para-Sasakian manifold is said to be Ricci recurrent if their exists a non zero 1-form A [24].

$$(\nabla_X S)(Z,W) = A(X)S(Z,W), \tag{4.1}$$

$\forall\ X, Z, W \in M$.

Let M is a Ricci recurrent para-Sasakian manifolds admitting $*$-Conformal η-Ricci solitons. Therefore, the Ricci tensor of the manifold satisfies

$$(\nabla_X S)(Z,W) = A(X)S(Z,W). \tag{4.2}$$

$$\nabla_X S(Z,W) - S(\nabla_X Z, W) - S(Z, \nabla_X W) = A(X)S(Z,W). \tag{4.3}$$

Taking covariant derivative of (3.4) with respect to X and using (2.6) and (2.8), we yields

$$\begin{aligned}\nabla_X S(Z,W) \;=\;& a[g(\nabla_X Z, W) + g(Z, \nabla_X W)] + b[-g(\phi X, \phi Z)\eta(W)\\ &-\eta(Z)g(\phi^2 X, W) + g(\phi\nabla_X Z, W) + g(\phi Z, \nabla_X W)]\\ &+c[g(Z,\phi X)\eta(W) + \eta(\nabla_X Z)\eta(W) + \eta(Z)g(W,\phi X)\\ &+\eta(Z)\eta(\nabla_X W)].\end{aligned} \tag{4.4}$$

Using (3.4) and (4.4) in (4.3), we have

$$\begin{aligned}&-b[g(\phi X,\phi Z)\eta(W) + \eta(Z)g(\phi^2 X, W)] + c[g(Z,\phi X)\eta(W)\\ &+\eta(Z)g(\phi X, W)] = A(X)S(Z,W).\end{aligned}$$

Taking $W{=}\xi$ and using (2.1) and (2.12) in the above equation, we have

$$-bg(X,Z) + b\eta(X)\eta(Z) + cg(Z,\phi X) = -(n-1)A(X)\eta(Z). \tag{4.5}$$

Suppose the associated 1-form A is equal to the associated 1-form η, then from (4.5), we have

$$-bg(X,Z) + (n-1+b)\eta(X)\eta(Z) + cg(Z,\phi X) = 0. \tag{4.6}$$

Putting $Z = \phi Z$ in (4.6), we have

$$-bg(X,\phi Z) + cg(\phi X, \phi Z) = 0. \tag{4.7}$$

Again putting $X = \phi X$ and in (4.7) and using (2.1), we have

$$-bg(\phi X,\phi Z) + cg(X, \phi Z) = 0. \tag{4.8}$$

By adding the equations (4.7) and (4.8), we find $c = b$ it gives $\mu = -2n + 4 - \psi$. Hence from (3.7) we get $\lambda = 2n - 4 + \psi + \frac{1}{2}(p + \frac{2}{n})$. Thus, we can leads to the following theorem:

THEOREM 4.2. *If (g, V, λ, μ) is a $*$-Conformal η-Ricci solitons in n-dimensional Ricci recurrent para-Sasakian manifold, then $\lambda = 2n-4+\psi+\frac{1}{2}(p+\frac{2}{n})$ and $\mu = -2n+4-\psi$.*

Put the value of λ and μ in (3.4), we get

$$S(X, Z) = (-n + 2 - \psi)g(X, Z) + (\psi - 1)g(\phi X, Z) - (1 - \psi)\eta(X)\eta(Z).$$

Thus, we have the following proposition:

PROPOSITION 4.3. *An n-dimensional Ricci recurrent para-Sasakian manifold admitting a $*$-Conformal η-Ricci solitons (g, V, λ, μ) is a generalized η-Einstein manifold.*

5. η-Parallel ϕ-tensor in para-Sasakian manifolds admitting $*$-Conformal η-Ricci solitons

In this section we study η-Parallel ϕ-tensor in para-Sasakian manifolds admitting $*$-Conformal η-Ricci solitons, If the (1,1) tensor ϕ is η-Parallel, then we have [9]

$$g((\nabla_X\phi)Y, Z) = 0. \tag{5.1}$$

$\forall\ X, Y, Z \in M$.
Using (2.6) and (5.1), we get

$$g(X, Y)\eta(Z) + \eta(Y)g(X, Z) - 2\eta(X)\eta(Y)\eta(Z) = 0. \tag{5.2}$$

Taking Z=ξ in (5.2), we have

$$g(X, Y) = \eta(X)\eta(Y).$$

Replacing X=ϕX in the above equation and using (3.5) yields

$$S(X, Y) = (1 - n - \lambda - \mu + \frac{1}{2}(p + \frac{2}{n}))\eta(X)\eta(Y). \tag{5.3}$$

Using (3.7) in (5.3), we have

$$S(X, Y) = (1 - n)\eta(X)\eta(Y). \tag{5.4}$$

Hence, we leads to the following theorem:

THEOREM 5.1. *If (g, V, λ, μ) is a $*$-Conformal η-Ricci soliton in an n-dimensional para-Sasakian manifold and if the tensor ϕ is η-parallel, then the para-Sasakian manifold is a special type of η-Einstein manifold.*

6. Conformal curvature tensor on para-Sasakian manifolds admitting $*$-Conformal η-Ricci solitons

The Conformal curvature tensor C is in n-dimensional para-Sasakian manifolds is defined as [16]

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y \\ &+g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X \\ &-g(X,Z)Y]. \end{aligned} \tag{6.1}$$

where R, S and r the curvature tensor, Ricci tensor and scalar curvature of M and Q is the Ricci operator defined as $S(X,Y) = g(QX,Y)$.
Let us take an n-dimensional para-Sasakian manifolds admitting $*$-Conformal η-Ricci soliton, which is conformally flat i.e.,$C(X,Y)Z = 0$. Then from (6.1) it follows that

$$\begin{aligned} R(X,Y)Z &= \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &-g(X,Z)QY] - \frac{r}{(n-1)(n-2)}[g(Y,Z)X \\ &-g(X,Z)Y]. \end{aligned} \tag{6.2}$$

Taking the inner product with ξ in (6.2), we have

$$\begin{aligned} g(X,Z)\eta(Y) - g(Y,Z)\eta(X) &= \frac{1}{(n-2)}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y) + g(Y,Z)\eta(QX) \\ &-g(X,Z)\eta(QY)] - \frac{r}{(n-1)(n-2)}[g(Y,Z)\eta(X) \\ &-g(X,Z)\eta(Y)]. \end{aligned}$$

Replacing X by ξ and using (2.1), (2.4), (3.5) and (3.6) in the above equation, we have

$$\begin{aligned} S(X,Y)Z &= [\frac{r}{(n-1)} - (a+b) - (n-2)]g(Y,Z) \\ &-[\frac{r}{(n-1)} - 2(a+b) - (n-2)]\eta(Y)\eta(Z). \end{aligned} \tag{6.3}$$

Using the value of $a + b = -(n-1)$, $r = -1$ and (3.7) in (6.3), we have

$$\begin{aligned} S(X,Y)Z &= [1 - \frac{1}{(n-1)}]g(Y,Z) \\ &-[n - \frac{1}{(n-1)}]\eta(Y)\eta(Z). \end{aligned}$$

Hence, we leads to the following theorem:

Theorem 6.1. *A conformally flat para-Sasakian manifold admitting $*$-Conformal η-Ricci soliton is an η-Einstein manifold.*

Let us take an n-dimensional para-Sasakian manifold admitting a $*$-Conformal η-Ricci soliton, which is ϕ-conformally semisymmetric [15, 26], i.e., $C.\phi = 0$. It gives

$$(C(X,Y)\phi)Z = C(X,Y)\phi Z - \phi C(X,Y)Z = 0. \tag{6.4}$$

Using (6.1) in (6.4), we have

$$\begin{aligned} &R(X,Y)\phi Z - \phi R(X,Y)Z + \frac{1}{(n-2)}[S(Y,Z)\phi X - S(X,Z)\phi Y - S(Y,\phi Z)X \\ &+ S(X,\phi Z)Y + g(Y,Z)\phi QX - g(X,Z)\phi QY - g(Y,\phi Z)QX + g(X,\phi Z)QY] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y] = 0. \end{aligned} \tag{6.5}$$

Replacing Y by ξ in (6.5) and using (2.1), (2.4), (3.5) and (3.6), we have

$$\begin{aligned} &[1 - \frac{r}{(n-1)(n-2)}](g(X,\phi Z)\xi + \eta(Z)\phi X) + \frac{1}{n-2}[S(X,\phi Z)\xi \\ &+ (a+b)(g(X,\phi Z)\xi + \eta(Z)\phi X) + \eta(Z)\phi QX] = 0. \end{aligned}$$

Taking the inner product with ξ, we have

$$S(X,\phi Z) = -[n - 2 - \frac{r}{(n-1)} + (a+b)]g(X,\phi Z). \tag{6.6}$$

Replacing Z by ϕZ in (6.6) and using (2.1) and (3.5), we have

$$\begin{aligned} S(X,Z) \quad = \quad & -[n - 2 - \frac{r}{(n-1)} + (a+b)]g(X,Z) + [n - 2 - \frac{r}{(n-1)} \\ & +2(a+b)]\eta(X)\eta(Z). \end{aligned} \tag{6.7}$$

Using the value of $a + b = -(n-1)$, $r = -1$ and (3.7) in (6.7), we have

$$S(X,Z) = [1 - \frac{1}{(n-1)}]g(X,Z) - [n - \frac{1}{(n-1)}]\eta(X)\eta(Z).$$

Hence, we leads to the following theorem:

Theorem 6.2. *A ϕ-conformally semi symmetric in para-Sasakian manifold admitting $*$-Conformal η-Ricci soliton is an η-Einstein manifold.*

7. Example

Example 7.1. *Let $M = [(y,z,u,v,w) \in R^5 : z > 0]$ be a 5-dimensional manifold, where (y,z,u,v,w) are the standard coordinates in R^5. The vector fields [30]*

$$e_1 = e^{-w}(\frac{\partial}{\partial y}), \quad e_2 = e^{-w}\frac{\partial}{\partial z}, \quad e_3 = e^{-w}\frac{\partial}{\partial u}, \quad e_4 = e^{-w}\frac{\partial}{\partial v}, \quad e_5 = \frac{\partial}{\partial w} = \xi,$$

are linearly independent at every point of M.

The indefinite Riemannian metric g is defined by

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = g(e_4,e_4) = g(e_5,e_5) = 1 \tag{7.1}$$

Suppose η be the 1-form defined as $\eta(Z)$=$g(Z,e_5)$=$g(Z,\xi)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = e_1, \quad \phi(e_2) = e_2, \quad \phi(e_3) = e_2, \quad \phi(e_4) = e_4, \quad \phi(e_5) = 0. \tag{7.2}$$

By linearity property of ϕ and g, we have

$$\eta(e_5) = 1, \quad \phi^2 Z = Z - \eta(X)\xi, \quad g(\phi Z, \phi U) = g(Z,U) - \eta(Z)\eta(U)$$

$\forall\ Z, U \in \chi(M)$; $\chi(M)$ is a set of all smooth vector fields on M.
Thus for $e_5 = \xi$, the structure (ϕ,ξ,η,g) defined an indefinite almost contact metric metric structure on M, we have

$$\begin{array}{ccccc} [e_1,e_2] = 0, & [e_1,e_3] = 0, & [e_2,e_3] = 0 & [e_2,e_4] = 0, & [e_1,e_4] = 0, \\ [e_3,e_4] = 0 & [e_1,e_5] = e_1, & [e_2,e_5] = e_2, & [e_3,e_5] = e_3 & [e_4,e_5] = e_4. \end{array}$$

Koszul's formula is given as

$$\begin{aligned} 2g(\nabla_Y Z, U) &= Yg(Z,U) + Zg(U,Y) - Ug(Y,Z) \\ &\quad -g(Y,[Z,U]) + g(Z,[U,Y]) + g(U,[Y,Z]), \end{aligned} \tag{7.3}$$

for arbitrary vector fields $X, Y, Z \in \chi(M)$.
By virtue of (7.3), we have

$$\begin{array}{lllll} \nabla_{e_1}e_1 = -e_5, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = 0, & \nabla_{e_1}e_5 = e_1 \\ \nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = -e_5, & \nabla_{e_2}e_3 = 0, & \nabla_{e_2}e_4 = 0, & \nabla_{e_2}e_5 = e_2 \\ \nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = -e_5, & \nabla_{e_3}e_4 = 0, & \nabla_{e_3}e_5 = e_3 \\ \nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = 0, & \nabla_{e_4}e_4 = -e_5, & \nabla_{e_4}e_5 = e_4 \\ \nabla_{e_5}e_1 = 0, & \nabla_{e_5}e_2 = 0, & \nabla_{e_5}e_3 = 0, & \nabla_{e_5}e_4 = 0, & \nabla_{e_5}e_5 = 0. \end{array} \tag{7.4}$$

It follows that $\nabla_Z \xi = \phi Z$. Hence the manifold is a para-Sasakian manifold. It is known that

$$R(Y,Z)U = \nabla_Y \nabla_Z U - \nabla_Z \nabla_Y U - \nabla_{[Y,Z]}U. \tag{7.5}$$

In view of above results, we obtain the non-vanishing components of curvature tensor as follows:

$$\begin{array}{llll} R(e_1,e_2)e_2 = -e_1, & R(e_1,e_3)e_3 = -e_1, & R(e_1,e_4)e_4 = -e_1 & R(e_1,e_5)e_5 = -e_1, \\ R(e_1,e_2)e_1 = e_2, & R(e_3,e_2)e_3 = e_2, & R(e_4,e_2)e_4 = e_2, & R(e_2,e_5)e_5 = -e_2 \\ R(e_1,e_3)e_1 = e_3, & R(e_4,e_3)e_4 = e_3, & R(e_2,e_3)e_2 = e_3, & R(e_3,e_5)e_5 = -e_3 \\ R(e_1,e_5)e_1 = e_5, & R(e_2,e_5)e_2 = e_5, & R(e_3,e_5)e_3 = e_5, & R(e_4,e_5)e_4 = e_5 \\ R(e_1,e_4)e_1 = e_4, & R(e_2,e_4)e_2 = e_4, & R(e_3,e_4)e_3 = e_4 & R(e_5,e_4)e_5 = e_4, \end{array} \tag{7.6}$$

The component of Ricci tensor, it follows that :

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = S(e_4, e_4) = S(e_5, e_5) = -4 \tag{7.7}$$

From (3.4), we have $S(e_5, e_5) = -4 - \lambda - \mu + \frac{1}{2}(p + \frac{2}{5})$. On equating both values of $S(e_5, e_5)$, we have

$$\lambda + \mu = \frac{1}{2}(p + \frac{2}{5}). \tag{7.8}$$

Hence, λ and μ satisfies (3.7) and so g defines a 5-dimensional para-Sasakian manifolds admitting $*$-Conformal η-Ricci solitons.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] T. Adati and K. Matsumoto, *On conformally recurrent and conformally symmetric P-Sasakian manifolds*, TRU Math.,**13**(1997), 25-32.

[2] S. R. Ashoka, C. S. Bagewadi and G. Ingalahalli, *A geometry on Ricci solitons in $(LCS)_n$-manifolds*, Diff. Geom. Dynamical Systems, **16** (2014), 50-62.

[3] C. S. Bagewadi, G. Ingalahalli and S. R. Ashoka, *A study on Ricci solitons in Kenmotsu manifolds, ISRN Geometry*,(2013), Article ID 422593, 6 pages.

[4] C. L. Bejan and M. Crasmareanu, *Ricci solitons in manifolds with Quasi-contact curvature, Publ. Math. Debrecen*, 78(1) (2011), 235-243.

[5] A. Bhattacharyya and N. Basu, *Conformal Ricci soliton in Kenmotsu manifold*, Global Journal Advanced Research on classical modern geometries, 4 (1) (2015), 15-21.

[6] A. M. Blaga, *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl., **20** (2015), 1-13.

[7] A. M. Blaga, *η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat 30, **2** (2016), 489-496.

[8] A. M. Blaga, S. Y. Perktas, B. E. Acet and F. E. Erdogan *η-Ricci solitons in ϵ-almost paracontact metric manifolds*, Glasnik Matematicki 53, **1** (2018), 205-212.

[9] D. E. Blair, *Riemannian geometryof contact and symplectic manifolds*, second edition vol 203 of progress of mathematics. Birkhauser Boston, Inc., Boston M . A (2010).

[10] S. Chandra, S. K. Hui and A. A. Shaikh, *Second order parallel tensors and Ricci solitons on $(LCS)_n$-manifolds*, Commun. Korean Math. Soc., **30** (2015), 123-130 .

[11] B. Y. Chen and S. Deshmukh, *Geometry of compact shrinking Ricci solitons,*, Balkan J. Geom. Appl., **19** (2014), 13-31.

[12] J. T. Cho and M. Kimura, *Ricci solitons and real hypersurfacesin a complex space form*, Tohoku Math.J. 61 **2** (2009), 205-212.

[13] U. C. De and N. Guha, *On a type of P-Sasakian manifold* , Istanbul Univ. Fen Fak. Mat. Der., (1992), 35-39.

[14] U. C. De and A. Hasseb, *η-Ricci solitons in ϵ-Kenmotsu manifolds*, J. Geom. 110(34) **2**(2019), 12 pages.

[15] U. C. De and P. Majhi, *ϕ-semisymmetric generalized Sasakian spaces forms*, Arab J. Math. Sci 21, **2** (2015), 170-178.

[16] U. C. De and A. Haseeb, *$\star$-conformal η-Ricci solitons on ϵ-para Sasakian manifolds*, Novi Sad J. Math. vol 52, no.2, (2022), 81-94.

[17] A. E. Fischer, *An introduction to conformal Ricci flow*, Class Quantum Grav 21 (3) (2004), 171-218.

[18] T. Hamada, *Real hypersurfaces of complex space form in term of Ricci $\star$-tensor*, Tokyo J. Math.25, **2**(2002), 473-483.

[19] R. S. Hamilton *The Ricci flow on surfaces, mathematics and general relativity*, Contemp. Math. **71** (1988), 237-262 https://doi.org/10.1090/conm/071/954419.

[20] A. Haseeb and R. Prasad $\star$*-conformal* η*-Ricci solitons in* ϵ*-Kenmotsu manifolds*, Publications De L Institute Mathematique. 108 122(2020), 91-102.

[21] C. He and M. Zhu, *Ricci solitons on Sasakian manifolds*, (2011), arxiv:1109.4407V2, [Math DG].

[22] G. Kaimakamis and K. Panagiotidou, $\star$*-Ricci solitons real hypersurfaces is non-flat complex space forms*, J. Geom. Phys. **86** (2014), 408-413.

[23] H. G. Nagaraja and C. R. Premalatha, *Ricci solitons in Kenmotsu manifolds,* Journal of Mathematical Analysis, **3** (2012), 18-24.

[24] E. M. Patterson, *Some theorems on Ricci-recurrent spaces*, J. London Math. Soc. 27, **3** (1952) (2001), 287-295.

[25] D. G. Prakasha and B. S. Hadimani, η*-Ricci solitons on para-Sasakian manifolds*, J. Geo. 108, **2** (2017), 383-392.

[26] R. Prasad, U. K. Gautam and A. Haseeb *On* ϕ*-semisymmetric LP-Kenmotsu manifolds with quarter symmetric non-metric connection*, Kragujevac J. Math. 45, **5** (2021), 815-827.

[27] I. Sato and K. Matsumoto, *On P-Sasakian manifolds satisfying certain conditions*, Tensor N.S., **33** (1979), 173-178.

[28] R. Sharma, *Certain on K-contact and* (k,μ)*-contact manifolds, Journal of Geometry*, **89**(1-2) (2008), 138-147.

[29] M. D. Siddiqi, *Conformal* η*-Ricci solitons in* δ*-Lorentzian trans-Sasakian manifolds*, Inter. J. Maps in Math 1, **1** (2018), 15-34.

[30] R. N. Singh, S. K. Pandey, G. Pandey and K. Tiwari *On a semi-symmetric metric connection is an* ϵ*-Kenmotsu manifold*, Commun. Korean Math. Soc. **29**(2014), 331-343.

[31] T. Takahashi, *Sasakian manifold with pseudo-Riemannian metric*, Tohoku Math. J. **21**(1969), 644-653.

[32] S. Tachibana, *On almost-analytic vectors in almost-kahlerian manifolds*, Tohoku Math. J.11, **2** (1959), 247-265.

[33] M. M. Tripathi, E. Kilic, S. Y. Perktas and S. Keles *Indefinite almost paracontact metric manifolds*, Int. J. Math. Sci.., Art ID 846195 (2010), 19 pages.

[34] M. Turan, C. Yetim and S. K. Chaubey *On quasi Sasakian 3-manifolds admitting* η*-Ricci solitons*, Filomat 33, **15** (2019), 4923-4930.

[35] A. Yidiz, U. C. De and E. Ata, *On a type of Lorentzian para-Sasakian manifolds*, Math. Reports 16(66), **1**(2014), 61-67.

[36] A. M. Yblaga, η*-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl. 20, **1** (2015), 1-13.

Abhishek Singh, Department of Mathematics & Statistics, Dr. Rammanohar Lohia Avadh University, Ayodhya, India
e-mail: abhi.rmlau@gmail.com

Sangeeta Gautam, Department of Mathematics & Statistics, Dr. Rammanohar Lohia Avadh University, Ayodhya, India
e-mail: Sangeeta.ssdc@gmail.com