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HOMOTOPY ANALYSIS METHOD FOR SOLVING THE NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract

The one-dimensional Burger's equation and coupled Burger's equation are considered in the present study. These equations have wide applications in engineering, particularly in modeling of particle transport in fluid suspensions and colloids under the effect of gravity. Various analytical and numerical techniques are available for solving these types of Burger's equation. In the present work, the semi-analytic Homotopy analysis method (HAM) is employed to obtain approximate analytical solutions for the corresponding homogeneous equation for validation. HAM is a powerful tool and effective technique for the solutions of nonlinear partial differential equations due to its convergence control capacity. Finally, the obtained solutions of these equations are presented graphically, and the contour plots are provided to illustrate the solution behavior and demonstrate the accuracy of the proposed method.

Keywords and phrases: Homotopy analysis method (HAM); Burger's equation; coupled Burger's equations.

1. Introduction

Any of the physical systems followed by mathematical modeling are usually nonlinear-type differential equations. These equations can be solved using analytical methods such as perturbation [1, 2], and some are solved using numerical methods. To investigate such a system, authors have used a perturbation theory that contains small perturbation parameters [3, 4]. Applying the perturbation technique to the system where expansion is obtained within the domain and the power series expression is presented to express the approximants. The approximate solution of these equations can be obtained numerically. Henceforth, the Runge-Kutta method in association with discretization techniques is used. To obtain a complete curve of results, it is also costly and time-consuming. Finally, the stability and convergence of the method applied are verified to avoid divergence or inappropriate results. In addition to that, the difficulties that appear in numerical computation have multiple solutions in case singularities arise. Perturbation techniques are rooted in the appearance of very little / very huge parameters, called "perturbation quantity." Unfortunately, numerous nonlinear issues do not hold such kinds of perturbation quantities at all.

Some non-perturbation methods have been ascertained to discover physical issues that do not require a little physical constraint to be used as a perturbation constraint.

The non-restriction development technique [5], the optimized perturbation hypothesis [6], the variation perturbation hypothesis [7, 8], and the linear expansion method [9–13] are typical non-perturbative methods. The most popular nonperturbative technique is the homotopy analysis method (HAM), first proposed by Shi-Jun Liao [14–16], a powerful analytic method for solving linear and nonlinear differential and integral equations. The HAM was successfully applied to solve many nonlinear problems, for instance, nonlinear Riccati DEs employing fractional order [17], the nonlinear Vakhnenko [18] equation, the Glauert-Jet [19] issue, the one-dimensional Burger's equation in longitudinal dispersion phenomena [20], and the comprehensive Kohli et al. [21] unsteady nanofluid flow equations. However, neither of the techniques themselves can give us an easy technique to regulate and manage the convergence area and tempo of a specified approximate sequence. Studies [22–26] demonstrate the effectiveness of HAM in solving highly nonlinear fluid flow, heat transfer, mass transfer, entropy generation, and fractional integro-differential problems, yielding accurate, convergent, and validated semi-analytical solutions.

Novelty of current study

- The present study provides a unified HAM-based analytical solution of both the one-dimensional Burgers' equation and the coupled Burgers' equations within a single framework.
- The study highlights the advantage of HAM in obtaining analytical solution series without the requirement of small or large perturbation parameters.
- A detailed graphical investigation, including solution profiles and contour plots, is presented to illustrate the behavior of the solutions and demonstrate the effectiveness of HAM for solving nonlinear Burgers-type equations.

Moreover, the paper has been arranged as follows. Section 2 deals with the homotopy analysis method of the problems. Section 3 contains the applications of the problem. The description of results and conclusion are presented in Section 4 and Section 5.

2. Homotopy Analysis Method

Any nonlinear differential equation of the form

$$\mathcal{N}[f(x,t)] = 0, \tag{2.1}$$

where $\mathcal{N}$, the operator for nonlinear terms, $f(x,t)$, the function used as per the application of the problem and the space and time dependent variables are displayed as x and t respectively. The general approach for the HAM is to get the zeroth-order deformation equation, which can be expressed as

$$(1-q)\mathcal{L}[F(x,t:q) - f_0(x,t)] = qh\,\mathcal{N}[F(x,t:q)], \quad 0 \le q \le 1 \tag{2.2}$$

where the transformed mapping of $f(x,t)$ is $F(x,t:q)$, the initial guess for the solution is represented as $f_0(x,t)$, $h \neq 0$, an auxiliary parameter, $q \in [0,1]$, the embedding parameter and $\mathcal{L}$ is a linear operator satisfying the property $\mathcal{L}[0] = 0$.

Hence equation (2.1) becomes

$$F(x,t:q) = f_0(x,t), \tag{2.3}$$

$$\mathcal{N}[F(x,t:1)] = 0. \tag{2.4}$$

Therefore, from the initial guess result of $f_0(x,t)$ to the final exact solution $f(x,t)$ the continuous mapping is denoted as $F(x,t:q)$. Expanding $F(x,t:q)$ in Taylor series with respect to the embedded parameter q as

$$F(x,t:q) = f_0(x,t) + \sum_{m=1}^{\infty} f_m(x,t)\, q^m \tag{2.5}$$

were

$$f_m(x,t) = \frac{1}{m!} \left.\frac{\partial^m(x,t:q)}{\partial q^m}\right|_{q=0} \tag{2.6}$$

However, the higher order (mth-order) deformation is obtained by the m times differentiation of equation (2.2) with respect to q about $q = 0$ and then the entire expression is divided by $m!$. Therefore, the expression is

$$\mathcal{L}[f_m(x,t) - \chi_m f_{m-1}(x,t)] = h\mathcal{R}_m(f_{m-1}), \tag{2.7}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{2.8}$$

$$\mathcal{R}_m(x,t) = \frac{1}{(m-1)!} \left.\frac{\partial^{m-1}\mathcal{N}[F(x,t:q)]}{\partial q^{m-1}}\right|_{q=0} \tag{2.9}$$

Finally, at $q = 1$, If the expression (2.5) converges then the exact solution can be expressed as

$$f(x,t) = f_0(x,t) + \sum_{m=1}^{\infty} f_m(x,t) \tag{2.10}$$

The aforesaid expression (2.10) is one of the solutions to the governing equation (Liao [5]). It is one of the convenient ways to choose the parameter h which has to control and adjust the criterion for the rate of convergence [6]. It is clear to state that, $f_m(x,t)$, $m \geq 1$ from the linear equation (2.7) with adequate boundary conditions that arise from the governing problem, it can be solved and the computation is made by the symbolic software MATLAB.

3. Applications

In the present case, we have employed the procedure of HAM to the well-posed $K(2,2)$ Burgers and coupled Burgers of the homogeneous advection equations respectively. The assumptions for both types of equations are expressed by the base function of the form

$$f(x,t) = f_1(x)\sum_{n=0}^{\infty} a_n t^n \tag{3.1}$$

where the coefficients of the respective order in the series are of the form a_n and $f_1(x) \neq 0$, it is determined according to the respective equation.

Moreover, the expression for the linear operator can be imposed as

$$\mathcal{L}[F(x,t:q)] = \frac{\partial F(x,t:q)}{\partial t} \tag{3.2}$$

Satisfied by the property $\mathcal{L}[c_1] = 0$, with constant c_1. $f_1(x)$ is considered as the initial guess defined as $f_0(x,t) = f_1(x)$.

Example-1 Consider the homogeneous advection equation

$$f_t(x,t) = f(x,t)f_x(x,t), \quad f(x,0) = x \tag{3.3}$$

The nonlinear operator is defined as described earlier in the previous article and the initial condition is considered as per the prescribed conditions which is of the form

$$f_0(x,t) = x \tag{3.4}$$

The nonlinear operator is

$$\mathcal{N}_u = f_t(x,t) - f(x,t)f_x(x,t). \tag{3.5}$$

and thus,

$$\mathcal{R}_m(f_{m-1}) = \frac{\partial f_{m-1}(x,t)}{\partial t} - \sum_{i=0}^{m-1} f_i(x,t)\frac{\partial f_{m-1-i}(x,t)}{\partial x} \tag{3.6}$$

Using equation (2.7), equation (3.6) and the defined linear operator in equation (3.2) along with initial conditions $f_m(x,0) = 0$, for $m = 1, 2, 3, \ldots$, we have

$$\begin{aligned}
f_1(x,t) &= x(-ht),\\
f_2(x,t) &= x\left((-ht)^2 - ht - h^2 t\right),\\
f_3(x,t) &= x\left((-ht)^3 + 2h^3t^2 + 2h^2t^2 - ht - 2h^3t - h^3t\right),
\end{aligned}$$

Similarly,

$$f_n(x,t) = x((-ht)^n + Z_1(t,h)).$$

In particular $h = -1$, the polynomial $Z_1(t,h)$ ceases to 0 (zero).

Hence from Eq. (2.10), we get

$$f(x,t) = x\sum_{n=0}^{\infty} t^n. \tag{3.7}$$

The above Eq. (3.7) is the Taylor's series expansion that converge to

$$f(x,t) = \frac{x}{1-t} \tag{3.8}$$

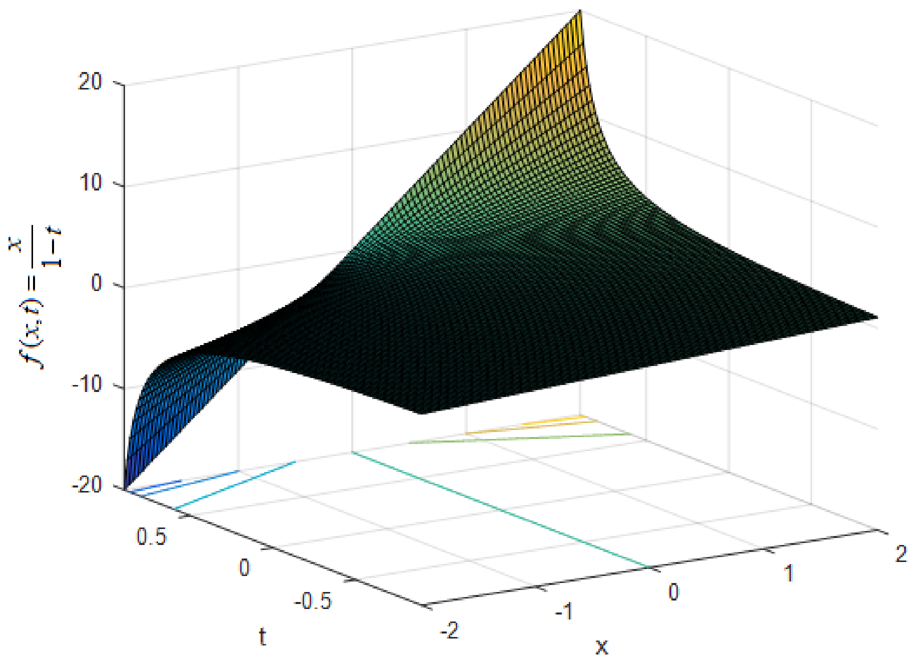

FIGURE 1. Surface plot for the solution of $f(x,t) = \dfrac{x}{1-t}$

Example-2 Consider the following coupled Burger's equations [6];

$$f_t - f_{xx} - 2ff_x + (fg)_x = 0, \tag{3.9}$$

$$g_t - g_{xx} - 2gg_x + (fg)_x = 0, \tag{3.10}$$

The corresponding initial conditions are

$$f(x,0) = \cos x, \qquad g(x,0) = \cos x. \tag{3.11}$$

The nonlinear operator

$$\mathcal{N}_u = \frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} - 2f\frac{\partial f}{\partial x} + \frac{\partial (fg)}{\partial x}, \tag{3.12}$$

and

$$\mathcal{N}_v = \frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} - 2g\frac{\partial g}{\partial x} + \frac{\partial (fg)}{\partial x}. \tag{3.13}$$

Using Eq. (2.9) on Eq. (3.12) and Eq. (3.13), we obtain

$$\begin{aligned}\mathcal{R}_m(f_{m-1}) = {} & \frac{\partial f_{m-1}(x,t)}{\partial t} - \frac{\partial^2 f_{m-1}(x,t)}{\partial x^2} \\ & - \sum_{i=0}^{m-1} 2f_i \frac{\partial f_{m-1-i}}{\partial x} + \frac{\partial (f_i g_{m-1-i})}{\partial x}\end{aligned} \tag{3.14}$$

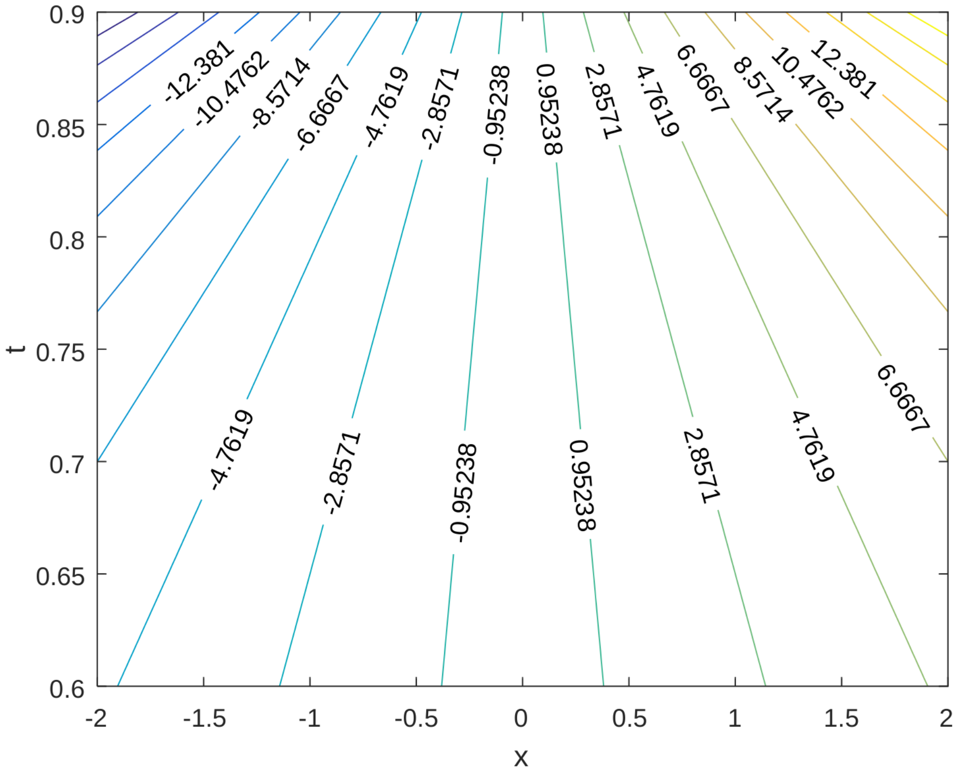

FIGURE 2. Contour plot

$$\mathcal{R}_m(g_{m-1}) = \frac{\partial g_{m-1}(x,t)}{\partial t} - \frac{\partial^2 g_{m-1}(x,t)}{\partial x^2} - \sum_{i=0}^{m-1} 2g_i \frac{\partial g_{m-1-i}}{\partial x} + \frac{\partial (f_i g_{m-1-i})}{\partial x} \tag{3.15}$$

Applying equation (2.7) on equations (3.14) and (3.15), and with the use of equation (3.2), along with initial conditions $f_m(x, 0) = 0$ and $g_m(x, 0) = 0$ where $m = 1, 2, 3, \ldots$, we get

$$\begin{aligned}
f_0(x,t) &= \cos x, \\
g_0(x,t) &= \cos x, \\
f_1(x,t) &= \cos x\,(ht), \\
g_1(x,t) &= \cos x\,(ht), \\
f_2(x,t) &= \cos x\left(\frac{h^2t^2}{2!} + (h+h^2)t\right),
\end{aligned}$$

$$g_2(x,t) = \cos x\left(\frac{h^2t^2}{2!} + (h+h^2)t\right),$$
$$f_3(x,t) = \cos x\left(\frac{h^3t^3}{3!} + (h+2h^2+h^3)t + (h^3+h^2)t\right),$$
$$g_3(x,t) = \cos x\left(\frac{h^3t^3}{3!} + (h+2h^2+h^3)t + (h^3+h^2)t\right),$$

and in similar

$$f_m(x,t) = \cos x\left(\frac{(ht)^m}{m!} + Z_2(h,t)\right),$$
$$g_m(x,t) = \cos x\left(\frac{(ht)^m}{m!} + Z_3(h,t)\right).$$

when $h = -1$, the polynomials $Z_2(h,t) = Z_3(h,t) = 0$.

From Eq. (2.10) the solution for both $f(x,t)$ and $g(x,t)$ are respectively,

$$f(x,t) = \cos x\left(\sum_{m=0}^{\infty}(-1)^m\frac{t^m}{m!}\right),$$
$$g(x,t) = \cos x\left(\sum_{m=0}^{\infty}(-1)^m\frac{t^m}{m!}\right).$$

The closed-form solutions of the above expressions are respectively,

$$f(x,t) = e^{-t}\cos x,$$
$$g(x,t) = e^{-t}\cos x.$$

4. Description of the results

It is worth noting that obtaining analytical solutions for nonlinear ordinary and partial differential equations is generally more challenging than solving their linear counterparts. Traditional methods like the perturbation method and asymptotic methods were widely used for the solution of these types of weakly nonlinear issues. Such techniques strongly depend upon the small/huge parameters. There are many examples in the field of fluid dynamics problem where the viscous flow problems are solved using the perturbation method with Reynolds number is taken to be small enough as a perturbation parameter. Therefore, this is essential to build up an analytical method that is free of the small/huge parameters. Hence, the homotopy analysis method (HAM) provides a guaranteed convergence solution series for the nonlinear and coupled nonlinear differential equations also. The crux of the method homotopy analysis method is, for the option of an auxiliary linear operator and base function HAM provides a large freedom. The homotopy analysis method gives practical analytic to implement for extremely nonlinear problems through singularities and multiple solutions. In the

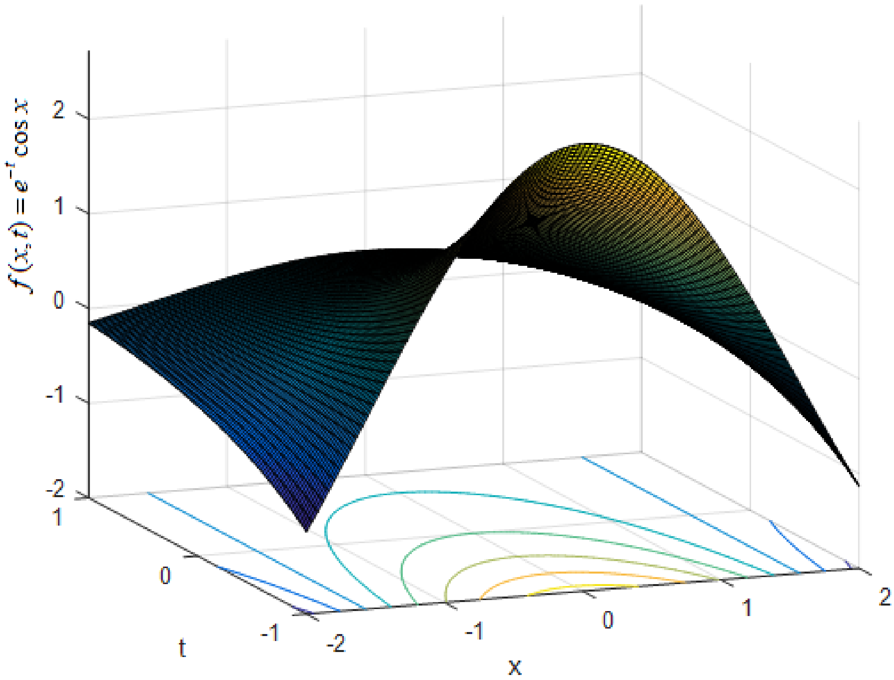

FIGURE 3. Surface plot for the solution of $f(x, t) = e^{-t} \cos x$

present analysis, we have considered Burgers' and coupled Burgers' equations of both linear and nonlinear types of differential equations. Approximate analytical solution of one-dimensional Burgers' and coupled Burgers' equations are obtained and presented through graphs. The homotopy analysis method is used to solve these types of equations and the accurate solutions of the coupled Burgers' equations are obtained. The basic steps of HAM and its application are described elaborately in articles-2 and 3 respectively. Fig.1 exhibits the surface plot for the solution obtained in Example-1, the one-dimensional Burgers' equation also, the contour plot is presented in Fig.2. These plots show that the solution is increasing linearly with respect to the spatial variable x, and the solution gets very large as t gets close to one because of the term $(1 - t)$ in the denominator. Moreover, the mesh grid for both x and t are shown in Fig.1 i.e. $-2 \leq x \leq 2$ and $0 \leq t \leq 1$. In similar, the surface plot and the contour plot for the solution obtained in Example-2 are presented in Figs.3 and 4 respectively. The mesh grid for both x and t are shown in Fig.3 i.e. $-2 \leq x \leq 2$ and $-1 \leq t \leq 1$. The solution varies periodically with respect to the space axis owing to the presence of the cosine function, and its amplitude diminishes as time advances owing to the exponential damping factor e^{-t}. This means that the wave form slowly fades away with the passage of time.

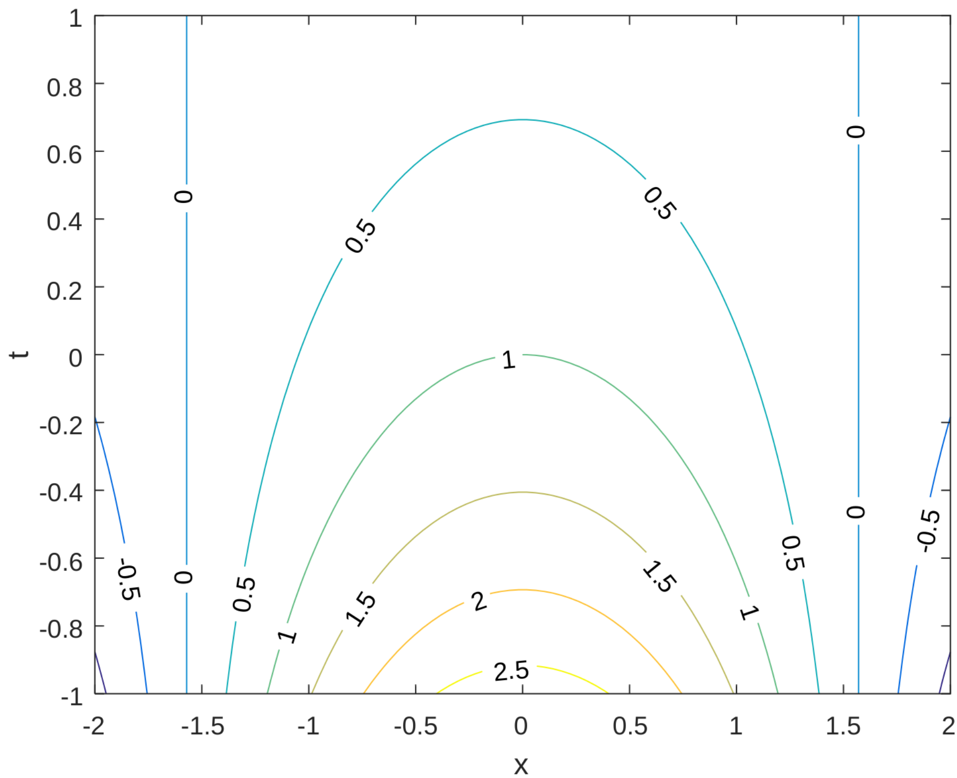

FIGURE 4. Contour plot

5. Conclusion

In the current investigation, the Homotopy Analysis Method (HAM) was used to derive the analytical solution of the one-dimensional Burgers' equation and coupled Burgers' equations. The exact analytical solutions of the associated homogeneous differential equations have been derived analytically. The solutions thus obtained have been plotted as surface and contour plots to show the behavior of solutions. Surface and contour plots show the behavior of the solutions derived analytically. The graphical analysis shows the efficiency of the HAM technique for solving one-dimensional and coupled Burgers' equations.

This study focuses on benchmark Burgers' equations whose analytical solutions are known, and future studies will focus on implementing the suggested method for solving other nonlinear equations which do not have analytical solutions. Further, we conclude that the Burgers' type equations are also solved by different methods like variational iteration method (VIM), differential transformation method (DTM), Adomain Decomposition Method (ADM), etc.

Conflict of Interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

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