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SOLITONS ON SASAKIAN MANIFOLD ADMITTING SEMI-SYMMETRIC NON-METRIC CK-CONNECTION

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Abstract

In this study, different kinds of solitons of Sasakian manifold with SSNMCK (semi-symmetric non-metric CK) connection which can be expanding, shrinking, or steady are examined. We also study the geometric properties of Yamabe soliton, Ricci soliton, etc. Additionally, we examine the SSNMCK-connection's curvature characteristics on Sasakian manifolds. Furthermore, an example is created to illustrate the outcomes.

Keywords: Concircular curvature tensor; Einstein manifold; Invariant submanifold; Sasakian manifold; Semi-symmetric non-metric CK-connection;Yamabe soliton.

AMS Subject Classification: Primary 53C25; Secondary 53C05, 53B20.

1. Introduction

A Sasakian manifold is an odd-dimensional Riemannian manifold that serves as the odd-dimensional counterpart of a Kähler manifold. It refers to a contact metric manifold that fulfills the normality criterion, or, alternatively, whose metric cone is Kähler. Sasakian structures were initially examined by Japanese mathematician S. Sasaki in 1960[15] and subsequently elaborated upon extensively by Blair in 1976 and 2010, as well as by Boyer and Galicki in 2008 [3, 7]. Differential geometry relies significantly on these manifolds due to their connections with contact and CR-geometry, in addition to applications in physics and particularly for string theory and Einstein geometry.

Ricci solitons arise naturally in the study of the Ricci flow as self-similar solutions and serve as natural generalizations of Einstein metrics. Since their introduction by Hamilton (1988)[13], they have played a fundamental role in understanding the formation of singularities in geometric evolution equations. An almost Ricci soliton on a Riemannian manifold (M, g) for a smooth vector field F_m is defined as

$$L_{F_m} g + 2S + 2\alpha g = 0, \tag{1.1}$$

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where L is the Lie derivative, S denotes the Ricci tensor, and α is a smooth function. Whenever α acts as a constant, a metric g that satisfies (1.1) is called a Ricci soliton. A Ricci soliton is expanding, steady, or shrinking if $\alpha > 0$, $\alpha = 0$, or $\alpha < 0$.
An η-Ricci soliton, introduced by Cho and Kimura [9], extends the Ricci soliton by incorporating the Reeb vector field, making it particularly relevant in contact geometry. An almost η-Ricci soliton on a Riemannian manifold (M, g) for a smooth vector field F_m is defined as

$$L_{F_m} g + 2S + 2\alpha g + 2\beta\eta \otimes \eta = 0, \tag{1.2}$$

for α and β, both of which smooth functions. The metric g is called a η-Ricci soliton whenever both α and β are constant.
The Yamabe soliton, introduced by Hamilton [13], arises as a self-similar solution of the Yamabe flow and generalizes metrics of constant scalar curvature. A Yamabe soliton satisfies

$$\frac{1}{2} L_{F_m} g = (r - \gamma) \tag{1.3}$$

in which r is the scalar curvature of the manifold (M, g) and γ acts as a smooth function. Whenever γ remained constant, the almost Yamabe soliton turns into a Yamabe soliton. A Yamabe soliton is expanding, steady, or shrinking if $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$, respectively.
In Riemannian geometry, the concept of linear connections different from the Levi-Civita connection has been of significant interest. A semi-symmetric connection was first introduced by Yano (1970) [18], characterized by a torsion tensor of the type

$$T_c(F_l, F_m) = \eta(F_m)F_l - \eta(F_l)F_m$$

where η is a 1-form. Later, Agashe and Chafle (1992) [1] generalized this idea and defined a semi-symmetric non-metric connection regarding a Riemannian manifold as a linear connection $\widetilde{\nabla}$ that fulfills $\widetilde{\nabla}_{F_1}\, g \neq 0$, and is thus incompatible with the metric [8]. A new kind of such connection is recently defined by S.K. Chanyal and Shankar Kumar, and it is currently referred to as the semi-symmetric non-metric CK-connection [6]. If (M^n, g) is a Riemannian manifold that has Levi-Civita connection ∇, subsequently a linear connection ∇^{CK} is called a semi-symmetric non-metric CK-connection whenever it fulfills:

$$\nabla^{CK}_{F_l} F_m = \nabla_{F_l} F_m + \frac{1}{2}[\eta(F_m)F_l - \eta(F_l)F_m + g(F_l, F_m)\xi] \tag{1.4}$$

in which ξ represents a vector field and η represents a 1-form associated with η. In recent years, the study of Ricci and related solitons on manifolds with non-Levi-Civita connections has gained momentum. Notable works include Ricci, η-Ricci solitons , η-Ricci-Bourguignon solitons, gradient solitons and Ricci–Yamabe type solitons under such connections[11],[17],[4],[10].

In this paper, we analyze several types of Yamabe and Ricci solitons over a Sasakian manifold of dimension n with SSNMCK-connection. Section 2 presents a brief overview of the Sasakian manifold, including curvature tensors and SSNMCK-connection. Section 3 presents the SSNMCK-connection on the Sasakian manifold and develops the formulas for curvature tensor, Ricci tensor, Ricci operator, and scalar curvature. The Ricci solitons for a Sasakian manifold with SSNMCK-connection are examined in section 4. Next section is dedicated to Ricci solitons on submanifolds of Sasakian manifold with connections. The final section discusses Yamabe solitons on an n-dimensional Sasakian manifold with SSNMCK-connection.

2. Preliminaries

Suppose that $(M^n, \phi, \xi, \eta,\ g)$ represents an almost contact metric manifold of dimension $n(n = 2m + 1)$, where g is a Riemannian metric, ϕ represents a (1,1) tensor field, ξ represents the structure vector field, and η represents a 1-form. The structure (ϕ, ξ, η, g) is known for complying with requirements[4]

$$\phi\xi = 0,\ \eta(\phi F_l) = 0,\ \eta(\xi) = 1, \tag{2.1}$$

$$\phi^2 F_l = -F_l + \eta(F_l)\xi,\ g(F_l, \xi) = \eta(F_l), \tag{2.2}$$

$$g(\phi F_l, \phi F_m) = g(F_l, F_m) - \eta(F_l)\eta(F_m), \tag{2.3}$$

$$g(F_l, \phi F_m) = -g(\phi F_l, F_m), \tag{2.4}$$

for any pair of vector fields F_l, F_m on M^n.
Furthermore, whenever

$$\nabla_{F_l}\xi = -\phi F_l, \tag{2.5}$$

$$(\nabla_{F_l}\phi)F_m = g(F_l, F_m)\xi - \eta(F_m)F_l, \tag{2.6}$$

$$g(F_l, \phi F_m) = \nabla\eta(F_l, F_m) \tag{2.7}$$

$$R(F_l, F_m)\xi = \eta(F_m)F_l - \eta(F_l)F_m, \tag{2.8}$$

where ∇ denotes the Levi-civita connection, then $(M^n, \phi, \xi, \eta, g)$ referred to as a Sasakian manifold[15].
The following characteristics are true in a Sasakian manifold [16], [19]:

$$Q\xi = (n - 1)\xi, \tag{2.9}$$

$$S(F_l, \xi) = (n - 1)\eta(F_l), \tag{2.10}$$

for any vector field F_l on M^n.

For constants $\upsilon_1, \upsilon_2, \upsilon_3$ it is claimed that a contact metric structure(η,g) on manifold M^n is generalized η-Einstein manifold if

$$S(F_m, F_n) = \upsilon_1 g(F_m, F_n) + \upsilon_2\eta(F_m)\eta(F_n) + \upsilon_3\psi(F_m, F_n), \tag{2.11}$$

where $\psi(F_m, F_n) = g(\phi F_m, F_n)$.
If $\upsilon_2 = \upsilon_3 = 0$, then M^n is an Einstien Manifold [2].

For a Riemannian manifold (M^n,g) of dimension n, the concircular curvature tensor W_c is defined as [5]:

$$W_c(F_l, F_m)F_n = R(F_l, F_m)F_n - \frac{r}{n(n-1)}[g(F_m, F_n)F_l - g(F_l, F_n)F_m]. \tag{2.12}$$

The W_2-curvature tensor[20], W_3-curvature tensor[21] and W_4-curvature tensor[21] are, respectively given by:

$$W_2(F_l, F_m)F_n = R(F_l, F_m)F_n - \frac{1}{(n-1)}[g(F_m, F_n)QF_l - g(F_l, F_n)QF_m], \tag{2.13}$$

$$W_3(F_l, F_m)F_n = R(F_l, F_m)F_n - \frac{1}{(n-1)}[S(F_l, F_n)F_m - g(F_m, F_n)QF_l], \tag{2.14}$$

$$W_4(F_l, F_m)F_n = R(F_l, F_m)F_n + \frac{1}{(n-1)}[g(F_l, F_n)QF_m - g(F_l, F_m)QF_n]. \tag{2.15}$$

An almost contact metric structure (ϕ, ξ, η, g) including a (1,1) tensor field ϕ, a vector field ξ, a 1-form η, and a Riemannian metric g is linked to an n-dimensional almost contact metric manifold M^n, we have

$$(\nabla^{CK}_{F_l} g)(F_m, F_n) = -\eta(F_m)g(F_l, F_n) - \eta(F_n)g(F_l, F_m) + \eta(F_l)g(F_m, F_n). \tag{2.16}$$

3. Some properties of Sasakian manifold having semi-symmetric non-metric CK-connection

Let R^{CK} be the Riemannian curvature tensor of Sasakian manifold having semi-symmetric non-metric CK-connection and it is defined as

$$R^{CK}(F_l, F_m)F_n = \nabla^{CK}_{F_l}\nabla^{CK}_{F_m}F_n - \nabla^{CK}_{F_m}\nabla^{CK}_{F_l}F_n - \nabla^{CK}_{[F_l,F_m]}F_n. \tag{3.1}$$

Using (2.5), (2.2), (2.9) and (1.4) in (3.1), we get the Riemannian curvature R^{CK} in accordance to semi-symmetric non-metric CK-connection as

$$\begin{aligned} R^{CK}(F_l, F_m)F_n &= R(F_l, F_m)F_n - \frac{1}{4}g(F_l, F_n)F_m + \frac{1}{4}g(F_m, F_n)F_l \\ &- \frac{1}{2}g(F_n, \phi F_l)F_m + \frac{1}{2}g(F_n, \phi F_m)F_l + \frac{1}{2}g(F_m, \phi F_l)F_n \\ &- \frac{1}{2}g(F_l, \phi F_m)F_n - \frac{1}{4}\eta(F_m)g(F_l, F_n)\xi + \frac{1}{4}\eta(F_l)g(F_m, F_n)\xi \\ &- \frac{1}{2}g(F_m, F_n)\phi F_l + \frac{1}{2}g(F_l, F_n)\phi F_m + \frac{1}{4}\eta(F_m)\eta(F_n)F_l \\ &- \frac{1}{4}\eta(F_l)\eta(F_n)F_m. \end{aligned} \tag{3.2}$$

Taking inner product of (3.2) with F_p, we get

$$\begin{aligned} R^{CK}(F_l, F_m, F_n, F_p) &= R(F_l, F_m, F_n, F_p) - \frac{1}{4}g(F_l, F_n)g(F_m, F_p) \\ &+ \frac{1}{4}g(F_m, F_n)g(F_l, F_p) - \frac{1}{2}g(F_n, \phi F_l)g(F_m, F_p) \\ &+ \frac{1}{2}g(F_n, \phi F_m)g(F_l, F_p) + \frac{1}{2}g(F_m, \phi F_l)g(F_n, F_p) \\ &- \frac{1}{2}g(F_l, \phi F_m)g(F_n, F_p) - \frac{1}{4}\eta(F_m)g(F_l, F_n)\eta(F_p) \\ &+ \frac{1}{4}\eta(F_l)g(F_m, F_n)\eta(F_p) - \frac{1}{2}g(F_m, F_n)g(\phi F_l, F_p) \\ &+ \frac{1}{2}g(F_l, F_n)g(\phi F_m, F_p) + \frac{1}{4}\eta(F_m)\eta(F_n)g(F_l, F_p) \\ &- \frac{1}{4}\eta(F_l)\eta(F_n)g(F_m, F_p). \end{aligned} \tag{3.3}$$

At each point on the manifold, let ρ_i represent the orthonormal basis for tangent space. Set $F_l = F_p$ after that contracting equation (3.3) by F_l and F_p, we get

$$\begin{aligned} S^{CK}(F_m, F_n) &= S(F_m, F_n) + \frac{n}{4}g(F_m, F_n) \\ &+ \frac{n-2}{4}\eta(F_m)\eta(F_n) + \frac{2-n}{2}g(F_m, \phi F_n). \end{aligned} \tag{3.4}$$

Also

$$\begin{aligned} S^{CK}(\phi F_m, \phi F_n) &= S(F_m, F_n) + \frac{n}{4}g(F_m, F_n) \\ &- \frac{5n-4}{4}\eta(F_m)\eta(F_n) + \frac{2-n}{2}g(F_m, \phi F_n). \end{aligned} \tag{3.5}$$

For a Sasakian manifold admitting a semi-symmetric non-metric CK-connection, the Ricci operator Q^{CK} is provided by

$$Q^{CK}F_m = QF_m + +\frac{n}{4}F_m + \frac{n-2}{4}\eta(F_m) - \frac{2-n}{2}\phi F_m. \tag{3.6}$$

Put $F_n = F_m = \rho_i$ in (3.4) and summing, we get

$$r^{CK} = r + \frac{n^2+n-2}{4}. \tag{3.7}$$

4. Ricci soliton types on Sasakian manifold exhibiting a semi-symmetric non-metric CK-connection

In this part, we investigate Ricci soliton on n-dimensional Sasakian manifold admitting SSNMCK-connection. Using equation (2.16),(1.4) we have

$$L^{CK}{}_{F_m}g(F_l, F_n) = g(\nabla_{F_l}F_m, F_n) + g(F_l, \nabla_{F_n}F_m) = L_{F_m}g(F_l, F_n) \tag{4.1}$$

On an n-dimensional Sasakian manifold admitting SSNMCK-connection, consider an almost Ricci soliton.
Now from (1.4) and (4.1), we have

$$2S^{CK}(F_l, F_n) = -2\alpha g(F_l, F_n) - g(\nabla_{F_l} F_m, F_n) - g(F_l, \nabla_{F_n} F_m). \tag{4.2}$$

Interchanging F_n and F_m, we have

$$2S^{CK}(F_l, F_m) = -2\alpha g(F_l, F_m) - g(\nabla_{F_l} F_n, F_m) - g(F_l, \nabla_{F_m} F_n). \tag{4.3}$$

Put $F_n = \xi$, we have

$$S^{CK}(F_l, F_m) = -\alpha g(F_l, F_m). \tag{4.4}$$

So M^n with SSNMCK-connection resembles an Einstein manifold.
Conversaly, let us consider an n-dimensional Sasakian manifold admitting SSNMCK-connection be an Einstein manifold, so

$$S^{CK}(F_l, F_m) = Ag(F_l, F_m). \tag{4.5}$$

Then for $F_m = \xi$ in (4.1), we have

$$L^{CK}{}_{\xi} g(F_l, F_n) = 0. \tag{4.6}$$

So putting $F_m = \xi$ in (1.4), we have

$$2S^{CK}(F_l, F_m) + 2\alpha g(F_l, F_m) = 0. \tag{4.7}$$

Comparing (4.5) and (4.7), we have A =-α. Hence we can state the following:

Theorem 4.1. *An n-dimensional Sasakian mnifold M^n is an Einstein manifold w.r.t. SSNMCK-connection iff M^n admits a Ricci soliton and $\alpha + A = 0$.*

From equation (4.4) and (3.4), we have

$$S(F_m, F_n) = \frac{-4\alpha - n}{4} g(F_m, F_n) - \frac{n-2}{4}\eta(F_m)\eta(F_n) - \frac{2-n}{2} g(F_m, \phi F_n) \tag{4.8}$$

For $F_n = \xi$ and using (2.10), we have

$$\alpha = \frac{-3(n-1)}{2}. \tag{4.9}$$

Hence we have the following theorem:

Theorem 4.2. *A triplet (g,ξ, α) on Sasakian manifold M^n endowed with ∇^{CK} is shrinking for all n>1.*

Put $F_l = F_n = e_i$ in (4.4) and summing, we get

$$r^{CK} = -\alpha n. \tag{4.10}$$

Hence, we have the following theorem

Theorem 4.3. *An n-dimensional Sasakian manifold admitting SSNMCK-connection along with an almost Ricci soliton (ξ, α,g) has a scalar curvature of* $r^{CK} = -\alpha n$.

The η-Ricci soliton on an n-dimensional Sasakian manifold with SSNMCK-connection can be obtained by (1.3).

$$(L^{CK}_{F_m} g + 2S^{CK} + 2\alpha g + 2\beta\eta \otimes \eta)(F_l, F_n) = 0, \tag{4.11}$$

From (4.1) and (4.11), we have

$$2S^{CK}(F_l, F_n) = -2\alpha g(F_l, F_n) - g(\nabla_{F_l} F_m, F_n) - g(F_l, \nabla_{F_n} F_m) \tag{4.12}$$

Put $F_m = \xi$, we have

$$S^{CK}(F_l, F_n) = -\alpha g(F_l, F_n) - \beta\eta(F_l)\eta(F_n). \tag{4.13}$$

Thus we have the following theorem:

Theorem 4.4. *If M is an n-dimensional Sasakian manifold with SSNMCK-connection which admits a η-Ricci soliton, then M is a η-Einstein manifold.*

Theorem 4.5. *If a Ricci soliton (g, ξ, λ) w.r.t. SSNMCK-connection is admitted by a concircularly flat Sasakian manifold M, then the Ricci soliton is steady, shrinking or expanding as long as* $r = \frac{n^2-3n+2}{4}$, $r > \frac{n^2-3n+2}{4}$ *or* $r < \frac{n^2-3n+2}{4}$.

Proof. Suppose M be a Sasakian manifold that is concircularly flat w.r.t. SSNMCK-connection i.e. $W_c(F_l, F_m)F_n = 0$, from eqn (2.12) it follows that

$$R^{CK}(F_l, F_m)F_n = \frac{r^{CK}}{n(n-1)}[g(F_m, F_n)F_l - g(F_l, F_n)F_m]. \tag{4.14}$$

The inner product of (4.14) with F_p yields

$$g(R^{CK}(F_l, F_m)F_n, F_p) = \frac{r^{CK}}{n(n-1)}[g(F_m, F_n)g(F_l, F_p) - g(F_l, F_n)g(F_m, F_p)]. \tag{4.15}$$

As an orthonormal frame field is contracted over F_l and F_p, we obtain

$$S^{CK}(F_m, F_n) = \frac{r^{CK}}{n} g(F_m, F_n). \tag{4.16}$$

With $F_n = \xi$, (3.4) and (3.7) in (4.16), we obtain

$$S(F_m, \xi) = \frac{4r - n^2 + 3n - 2}{4n}\eta(F_n). \tag{4.17}$$

By using (4.4) and (4.17), we conclude the result. □

Theorem 4.6. *If a Ricci soliton (g, ξ, λ) is admitted by a W_2 flat Sasakian manifold with SSNMCK-connection M, then the Ricci soliton is steady, shrinking or expanding as long as* $r = \frac{n^2-3n+2}{4}$, $r > \frac{n^2-3n+2}{4}$ *or* $r < \frac{n^2-3n+2}{4}$.

PROOF. Suppose M represent a W_2 flat Sasakian manifold w.r.t. SSNMCK-connection i.e. $W_2(F_l, F_m)F_n = 0$, with eqn (2.13) it follows that

$$R^{CK}(F_l, F_m)F_n = \frac{1}{(n-1)}[g(F_m, F_n)Q^{CK}F_l - g(F_l, F_n)Q^{CK}F_m]. \tag{4.18}$$

Taking the inner product with F_p yields

$$g(R^{CK}(F_l, F_m)F_n, F_p) = \frac{1}{(n-1)}[g(F_m, F_n)S^{CK}(F_l, F_p) - g(F_l, F_n)S^{CK}(F_m, F_p)]. \tag{4.19}$$

By contracting over F_l and F_p with an orthonormal frame field, we obtain

$$nS^{CK}(F_m, F_n) = r^{CK}g(F_m, F_n). \tag{4.20}$$

With $F_n = \xi$ and (3.7), (3.4) in (4.20), we obtain

$$S(F_m, \xi) = \frac{4r - n^2 + 3n - 2}{4n}\eta(F_n). \tag{4.21}$$

By using (4.4) and (4.21), we conclude the result. □

COROLLARY 4.7. *When a W_3 flat Sasakian manifold with SSNMCK-connection M of dimension n = 2m+1 admits a Ricci soliton (g, ξ, λ) , the Ricci soliton is expanding for all n > 1.*

COROLLARY 4.8. *If a Ricci soliton (g, ξ, λ) is admitted by a W_4 flat Sasakian manifold with SSNMCK-connection M of dimension n = 2m+1, then the Ricci soliton expands for all n > 1.*

From equation (4.4) and (3.4), we have

$$S(F_m, F_n) = \frac{-4\alpha - n}{4}g(F_m, F_n) - \frac{n-2}{4}\eta(F_m)\eta(F_n) - \frac{2-n}{2}g(F_m, \phi F_n) \tag{4.22}$$

For $F_n = \xi$ and using (2.10), we have

$$\alpha = \frac{-3(n-1)}{2}. \tag{4.23}$$

Hence we have the following theorem:

THEOREM 4.9. *A triplet (g,ξ, α) on Sasakian manifold M^n endowed with ∇^{CK} is shrinking for all n>1.*

Put $F_l = F_n = e_i$ in (4.4) and summing, we get

$$r^{CK} = -\alpha n. \tag{4.24}$$

Hence, we have the following theorem

THEOREM 4.10. *An n-dimensional Sasakian manifold admitting SSNMCK-connection along with an almost Ricci soliton (ξ, α,g) has a scalar curvature of $r^{CK} = -\alpha n$.*

The η-Ricci soliton on an n-dimensional Sasakian manifold with SSNMCK-connection can be obtained by (1.3).

$$(L^{CK}_{F_m}g + 2S^{CK} + 2\alpha g + 2\beta\eta \otimes \eta)(F_l, F_n) = 0, \tag{4.25}$$

From (4.1) and (4.11), we have

$$2S^{CK}(F_l, F_n) = -2\alpha g(F_l, F_n) - g(\nabla_{F_l}F_m, F_n) - g(F_l, \nabla_{F_n}F_m) \tag{4.26}$$

Put $F_m = \xi$, we have

$$S^{CK}(F_l, F_n) = -\alpha g(F_l, F_n) - \beta\eta(F_l)\eta(F_n). \tag{4.27}$$

Thus we have the following theorem:

Theorem 4.11. *If M is an n-dimensional Sasakian manifold with SSNMCK-connection which admits a η-Ricci soliton, then M is a η-Einstein manifold.*

Theorem 4.12. *If a Ricci soliton (g, ξ, λ) w.r.t. SSNMCK-connection is admitted by a concircularly flat Sasakian manifold M, then the Ricci soliton is steady, shrinking or expanding as long as $r = \frac{n^2-3n+2}{4}$, $r > \frac{n^2-3n+2}{4}$ or $r < \frac{n^2-3n+2}{4}$.*

Proof. Suppose M be a Sasakian manifold that is concircularly flat w.r.t. SSNMCK-connection i.e. $W_c(F_l, F_m)F_n = 0$, from eqn (2.12) it follows that

$$R^{CK}(F_l, F_m)F_n = \frac{r^{CK}}{n(n-1)}[g(F_m, F_n)F_l - g(F_l, F_n)F_m]. \tag{4.28}$$

The inner product with F_p yields

$$g(R^{CK}(F_l, F_m)F_n, F_p) = \frac{r^{CK}}{n(n-1)}[g(F_m, F_n)g(F_l, F_p) - g(F_l, F_n)g(F_m, F_p)]. \tag{4.29}$$

As an orthonormal frame field is contracted over F_l and F_p, we obtain

$$S^{CK}(F_m, F_n) = \frac{r^{CK}}{n}g(F_m, F_n). \tag{4.30}$$

With $F_n = \xi$, (3.4) and (3.7) in (4.30), we obtain

$$S(F_m, \xi) = \frac{4r - n^2 + 3n - 2}{4n}\eta(F_n). \tag{4.31}$$

By using (4.4) and (4.31), we conclude the result. □

Theorem 4.13. *If a Ricci soliton (g, ξ, λ) is admitted by a W_2 flat Sasakian manifold with SSNMCK-connection M, then the Ricci soliton is steady, shrinking or expanding as long as $r = \frac{n^2-3n+2}{4}$, $r > \frac{n^2-3n+2}{4}$ or $r < \frac{n^2-3n+2}{4}$.*

PROOF. Suppose M represent a W_2 flat Sasakian manifold w.r.t. SSNMCK-connection i.e. $W_2(F_l, F_m)F_n = 0$, with eqn (2.13) it follows that

$$R^{CK}(F_l, F_m)F_n = \frac{1}{(n-1)}[g(F_m, F_n)Q^{CK}F_l - g(F_l, F_n)Q^{CK}F_m]. \tag{4.32}$$

Taking the inner product with F_p yields

$$g(R^{CK}(F_l, F_m)F_n, F_p) = \frac{1}{(n-1)}[g(F_m, F_n)S^{CK}(F_l, F_p) - g(F_l, F_n)S^{CK}(F_m, F_p)]. \tag{4.33}$$

By contracting over F_l and F_p with an orthonormal frame field, we obtain

$$nS^{CK}(F_m, F_n) = r^{CK}g(F_m, F_n). \tag{4.34}$$

With $F_n = \xi$ and (3.7), (3.4) in (4.34), we obtain

$$S(F_m, \xi) = \frac{4r - n^2 + 3n - 2}{4n}\eta(F_n). \tag{4.35}$$

By using (4.4) and (4.35), we conclude the result. □

COROLLARY 4.14. *When a W_3 flat Sasakian manifold with SSNMCK-connection M of dimension n=2m+1 admits a Ricci soliton (g, ξ, λ) , the Ricci soliton is expanding for all n > 1.*

COROLLARY 4.15. *If a Ricci soliton (g, ξ, λ) is admitted by a W_4 flat Sasakian manifold with SSNMCK-connection M of dimension n=2m+1, then the Ricci soliton expands for all n > 1.*

5. Ricci Solitons on Submanifolds of Sasakian Manifold

Suppose $\widetilde{M}$ represent a p-dimensional submanifold of n-dimensional manifold M, wherein $p < n$, with the induced metric g. Consider $\widetilde{\nabla}$ and $\widetilde{\nabla}^{\perp}$ represent the Levi-Civita connection on the tangent bundle $T\widetilde{M}$ along with the normal bundle $T^{\perp}\widetilde{M}$ of $\widetilde{M}$, respectively. Now the Gauss and Weingarten formulas are expressed as[19]

$$\nabla_{F_l}F_m = \widetilde{\nabla}_{F_l}F_m + \widetilde{\sigma}(F_l, F_m) \tag{5.1}$$

$$\nabla_{F_l}N = -A_N F_l + \widetilde{\nabla}^{\perp}_{F_l}N \tag{5.2}$$

For every $F_l, F_m \in T\widetilde{M}$ and $N \in T^{\perp}\widetilde{M}$, in which $\widetilde{\sigma}$ and A_N represent the second fundamental form and the shape operator (which corresponds to the normal vector field N), accordingly for the immersion of $\widetilde{M}$ into M. The shape operator A_N and the second fundamental form $\widetilde{\sigma}$ have the following relationship:

$$g(\widetilde{\sigma}(F_l, F_m), N) = g(A_N F_l, F_m) \tag{5.3}$$

for all $F_l, F_m \in T\widetilde{M}$ and $N \in T^{\perp}\widetilde{M}$.
When the structure vector field ξ is tangent to $\widetilde{M}$ at each point of $\widetilde{M}$ and ϕF_l is tangent

to $\widetilde{M}$ for each vector field F_l tangent to $\widetilde{M}$ at at each point of $\widetilde{M}$, then a submanifold $\widetilde{M}$ of an almost contact metric manifold M is said to be invariant.
Let us take Ricci soliton (g, ξ, α) on $\widetilde{M}$ as

$$\widetilde{L}_\xi g(F_m, F_n) + 2\widetilde{S}(F_m, F_n) + 2\alpha g(F_m, F_n) = 0. \tag{5.4}$$

Put $F_m = \xi$ and using (2.9), we have

$$-\phi F_l = \widetilde{\nabla}_{F_l}\xi + \widetilde{\sigma}(F_l, \xi). \tag{5.5}$$

Since $\widetilde{M}$ is invariant therefore $\phi F_l,\ \xi \in \mathrm{T}\widetilde{M}$. Next, by comparing the tangential and normal components, we obtain

$$-\phi F_l = \widetilde{\nabla}_{F_l}\xi. \tag{5.6}$$

$$\widetilde{\sigma}(F_l, \xi) = 0. \tag{5.7}$$

From (5.6), we have

$$(\widetilde{L}_\xi g) = 0. \tag{5.8}$$

Using (5.4) and (5.8), we get

$$\widetilde{S}(F_m, F_n) = -\alpha g(F_m, F_n). \tag{5.9}$$

Again using (5.6), and the curvature formula becomes

$$\widetilde{R}(F_l, F_m)\xi = \eta(F_m)F_l - \eta(F_l)F_m. \tag{5.10}$$

Now, we have

$$\widetilde{S}(F_l, \xi) = (p-1)\eta(F_l). \tag{5.11}$$

Put $F_n = \xi$ into (5.9) and use (5.11), then we arrive at

$$\alpha = -(p-1) < 0. \tag{5.12}$$

From (5.9) and (5.12), we yield the following results:

Theorem 5.1. *If a Ricci soliton (g, ξ, α) is admitted by the invariant submanifold $\widetilde{M}$ of the Sasakian manifold M, then $\widetilde{M}$ is*

1. *Einstein manifold.*
2. *Always shrinking.*

Considering that Levi-Civita connection $\widetilde{\nabla}$ and SSNMCK-connection $\widetilde{\bar{\nabla}}$ over submanifold $\widetilde{M}$ of Sasakian manifold M exhibiting Levi-Civita connection ∇ and SSNMCK-connection $\bar{\nabla}$. In this case, $\widetilde{\bar{\sigma}}$ represents the second fundamental form with regard to the SSNMCK-connection. Then Gauss formula with respect to SSNMCK-connection can be represented as

$$\bar{\nabla}_{F_l}F_m = \widetilde{\bar{\nabla}}_{F_l}F_m + \widetilde{\bar{\sigma}}(F_l, F_m). \tag{5.13}$$

Now, use (1.4) in (5.13), we get

$$\bar{\bar{\widetilde{\nabla}}}_{F_l}F_m + \bar{\bar{\sigma}}(F_l, F_m) = \nabla_{F_l}F_m + \frac{1}{2}[\eta(F_m)F_l - \eta(F_l)F_m + g(F_l, F_m)\xi]. \tag{5.14}$$

Using (5.1) and (5.14), we obtain

$$\bar{\bar{\widetilde{\nabla}}}_{F_l}F_m + \bar{\bar{\sigma}}(F_l, F_m) = \widetilde{\nabla}_{F_l}F_m + \widetilde{\sigma}(F_l, F_m) + \frac{1}{2}[\eta(F_m)F_l - \eta(F_l)F_m + g(F_l, F_m)\xi]. \tag{5.15}$$

Since $\widetilde{M}$ is a invariant submanifold, therefore tangential and normal parts are given by

$$\bar{\bar{\widetilde{\nabla}}}_{F_l}F_m = \widetilde{\nabla}_{F_l}F_m + \frac{1}{2}[\eta(F_m)F_l - \eta(F_l)F_m + g(F_l, F_m)\xi]. \tag{5.16}$$

$$\bar{\bar{\sigma}}(F_l, F_m) = \widetilde{\sigma}(F_l, F_m). \tag{5.17}$$

Thus, we conclude the followings:

Theorem 5.2. *Let $\widetilde{M}$ be considered an invariant submanifold of Sasakian manifold M equipped with two connections namely Levi-civita ∇ and SSNMCK-connections $\bar{\nabla}$, along with $\widetilde{\nabla}$, $\bar{\bar{\widetilde{\nabla}}}$ be the induced Levi-civita and SSNMCK-connection on $\widetilde{M}$, respectively. Therefore we have*

1. *$\widetilde{M}$ admits SSNMCK-connection.*
2. *The second fundamental form are same with respect to ∇ and $\bar{\nabla}$.*

Using the preceding theorem, we conclude the following:

Theorem 5.3. *If $\widetilde{M}$ be a submanifold of M with SSNMCK-connection $\widetilde{\nabla}$ then*

1. *The curvature tensor $\bar{\bar{\widetilde{R}}}$ is expressed as*

$$\begin{aligned}
\bar{\bar{\widetilde{R}}}(F_l, F_m)F_n &= \widetilde{R}(F_l, F_m)F_n - \frac{1}{4}g(F_l, F_n)F_m + \frac{1}{4}g(F_m, F_n)F_l \\
&- \frac{1}{2}g(F_n, \phi F_l)F_m + \frac{1}{2}g(F_n, \phi F_m)F_l + \frac{1}{2}g(F_m, \phi F_l)F_n \\
&- \frac{1}{2}g(F_l, \phi F_m)F_n - \frac{1}{4}\eta(F_m)g(F_l, F_n)\xi + \frac{1}{4}\eta(F_l)g(F_m, F_n)\xi \\
&- \frac{1}{2}g(F_m, F_n)\phi F_l + \frac{1}{2}g(F_l, F_n)\phi F_m + \frac{1}{4}\eta(F_m)\eta(F_n)F_l \\
&- \frac{1}{4}\eta(F_l)\eta(F_n)F_m.
\end{aligned}$$

2. *The Ricci tensor $\bar{\bar{\widetilde{R}}}$ is expressed as*

$$\bar{\bar{\widetilde{S}}}(F_m, F_n) = \widetilde{S}(F_m, F_n) + \frac{n}{4}g(F_m, F_n) + \frac{n-2}{4}\eta(F_m)\eta(F_n) + \frac{2-n}{2}g(F_m, \phi F_n).$$

3. *The scalar curvature is given by*

$$\bar{\bar{\widetilde{r}}} = \widetilde{r} + \frac{n^2 + n - 2}{4}.$$

6. Yamabe Soliton on Sasakian Manifold With SSNMCK-Connection

This part investigates the almost Yamabe soliton on an n-dimensional Sasakian manifold exhibiting SSNMCK-connection.
We now investigate an n-dimensional Sasakian manifold that enables the SSNMCK-connection to deal with an almost Yamabe soliton, as indicated in (1.3).
Therefore, we've got

$$\frac{1}{2}(L_{F_m}{}^{CK} g)(F_l, F_n) = (r^{CK} - \gamma)g(F_l, F_n). \tag{6.1}$$

According to (6.1) and (4.1), we have

$$\frac{1}{2}(g(\nabla_{F_l} F_m, F_n) + g(F_l, \nabla_{F_n} F_m))(F_l, F_n) = (r^{CK} - \gamma)g(F_l, F_n). \tag{6.2}$$

Putting $F_m = \xi$ in the equation above, we get

$$\frac{1}{2}(g(\nabla_{F_l}\xi, F_n) + g(F_l, \nabla_{F_n}\xi))(F_l, F_n) = (r^{CK} - \gamma)g(F_l, F_n). \tag{6.3}$$

In view of (2.4) and (2.5), from (6.3), we obtain

$$r^{CK} = \gamma. \tag{6.4}$$

Thus, we may infer the following conclusion:

Theorem 6.1. *If an almost Yamabe soliton is admitted by an n-dimensional Sasakian manifold with SSNMCK-connection, then the scalar curvature r^{CK} of the manifold is equivalent to γ if F_m and ξ be pairwise collinear in TM.*

Using the preceding theorem, we conclude the following corollaries:

Corollary 6.2. *A Sasakian manifold w.r.t. SSNMCK-connection in n dimensions that admits Yamabe soliton is of constant scalar curvature.*

Corollary 6.3. *The Riemannian metric g is the Yamabe soliton for an n-dimensional Sasakian manifold w.r.t. SSNMCK-connection if it admits a Yamabe soliton and F_m is pairwise collinear with ξ.*

7. Example

This section reconstructs a 5-dimensional Sasakian manifold example that admits a semi-symmetric non-metric CK-connection ∇^{CK}.
Example 1. Let us take a 5-dimensional manifold $M^5 = (f_1, f_2, f_3, f_4, f_5) \in R^5$, where $(f_1, f_2, f_3, f_4, f_5)$ are standard coordinates in R^5. Let $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4, \Upsilon_5$ be the vector fields on M^n given by[14]

$$\Upsilon_1 = e^{f_3}\frac{\partial}{\partial f_1}; \Upsilon_2 = e^{f_3}\frac{\partial}{\partial f_2}; \Upsilon_3 = -\frac{\partial}{\partial f_3} = \xi; \Upsilon_4 = e^{f_3}\frac{\partial}{\partial f_4}; \Upsilon_5 = e^{f_3}\frac{\partial}{\partial f_5}.$$

The metric g is defined by

$$\begin{aligned} g(\Upsilon_i, \Upsilon_j) &= 0, i \neq j; i, j = 1,2,3,4,5 \\ &= 1, i = j. \end{aligned}$$

Consider the 1-form η, which is defined as,

$$\eta(F_l) = g(F_l, \Upsilon_3)$$

for any $F_1 \in \chi$(M). By applying the linearity of g and ϕ, we have

$$\phi^2 F_l = -F_l + \eta(F_l)\xi, \ \eta(\Upsilon_3) = 1.$$

Also,

$$g(\phi F_l, \phi F_m) = g(F_l, F_m) - \eta(F_l)\eta(F_m).$$

For any $F_l, F_m \in \chi$(M). Let ϕ be the (1,1)-tensor field on M^n defined by $\phi\Upsilon_1 = \Upsilon_1, \ \phi\Upsilon_2 = \Upsilon_2, \ \phi\Upsilon_3 = 0, \ \phi\Upsilon_4 = \Upsilon_4 \ and \ \phi\Upsilon_5 = \Upsilon_5$. The structure (ϕ, ξ, η, g) defines an almost contact manifold. For Levi-Civita connection ∇, we have

$$[\Upsilon_1, \Upsilon_3] = -\Upsilon_1; \ [\Upsilon_2, \Upsilon_3] = -\Upsilon_2; \ [\Upsilon_4, \Upsilon_3] = -\Upsilon_4; \ [\Upsilon_5, \Upsilon_3] = -\Upsilon_5 \ and \ [\Omega_i, \Omega_j] = 0,$$

for all others i and j.
Also

$$\begin{array}{ll} \nabla_{\Upsilon_1}\Upsilon_1 = \Upsilon_3 & \nabla_{\Upsilon_1}\Upsilon_3 = -\Upsilon_1 \\ \nabla_{\Upsilon_2}\Upsilon_2 = \Upsilon_3 & \nabla_{\Upsilon_2}\Upsilon_3 = -\Upsilon_2 \\ \nabla_{\Upsilon_4}\Upsilon_3 = -\Upsilon_4 & \nabla_{\Upsilon_4}\Upsilon_4 = \Upsilon_3 \\ \nabla_{\Upsilon_5}\Upsilon_3 = -\Upsilon_5 & \nabla_{\Upsilon_5}\Upsilon_5 = \Upsilon_3. \end{array}$$

Obviously, the manifold be a 5-dimensional Sasakian manifold. The non vanishing components of the curvature tensor are as follows;

$$\begin{array}{ll} R(\Upsilon_1, \Upsilon_2)\Upsilon_1 = \Upsilon_2 & R(\Upsilon_1, \Upsilon_2)\Upsilon_2 = -\Upsilon_1 \\ R(\Upsilon_1, \Upsilon_3)\Upsilon_1 = \Upsilon_3 & R(\Upsilon_1, \Upsilon_3)\Upsilon_3 = -\Upsilon_1 \\ R(\Upsilon_1, \Upsilon_4)\Upsilon_1 = \Upsilon_4 & R(\Upsilon_1, \Upsilon_4)\Upsilon_4 = -\Upsilon_1 \\ R(\Upsilon_1, \Upsilon_5)\Upsilon_1 = \Upsilon_5 & R(\Upsilon_1, \Upsilon_5)\Upsilon_5 = -\Upsilon_1 \\ R(\Upsilon_2, \Upsilon_3)\Upsilon_2 = \Upsilon_3 & R(\Upsilon_2, \Upsilon_3)\Upsilon_3 = -\Upsilon_2 \\ R(\Upsilon_2, \Upsilon_4)\Upsilon_2 = \Upsilon_4 & R(\Upsilon_2, \Upsilon_4)\Upsilon_4 = -\Upsilon_2 \\ R(\Upsilon_2, \Upsilon_5)\Upsilon_2 = \Upsilon_5 & R(\Upsilon_2, \Upsilon_5)\Upsilon_5 = -\Upsilon_2 \\ R(\Upsilon_3, \Upsilon_4)\Upsilon_3 = \Upsilon_4 & R(\Upsilon_3, \Upsilon_4)\Upsilon_4 = -\Upsilon_3 \\ R(\Upsilon_4, \Upsilon_5)\Upsilon_4 = \Upsilon_5 & R(\Upsilon_4, \Upsilon_5)\Upsilon_5 = -\Upsilon_4 \\ R(\Upsilon_3, \Upsilon_5)\Upsilon_3 = \Upsilon_5 & R(\Upsilon_3, \Upsilon_5)\Upsilon_5 = -\Upsilon_3. \end{array}$$

And

$$S(\Upsilon_1, \Upsilon_1) = S(\Upsilon_2, \Upsilon_2) = S(\Upsilon_3, \Upsilon_3) = S(\Upsilon_4, \Upsilon_4) = S(\Upsilon_5, \Upsilon_5) = -4.$$

Using above results, for semi-symmetric non-metric CK-connection ∇^{CK} we can obtain

$$\begin{array}{ll}
\nabla^{CK}_{\Upsilon_1}\Upsilon_1 = \frac{3}{2}\Upsilon_3 & \nabla^{CK}_{\Upsilon_1}\Upsilon_3 = -\frac{1}{2}\Upsilon_1 \\
\nabla^{CK}_{\Upsilon_2}\Upsilon_2 = \frac{3}{2}\Upsilon_3 & \nabla^{CK}_{\Upsilon_2}\Upsilon_3 = -\frac{1}{2}\Upsilon_2 \\
\nabla^{CK}_{\Upsilon_4}\Upsilon_3 = -\frac{1}{2}\Upsilon_4 & \nabla^{CK}_{\Upsilon_4}\Upsilon_4 = \frac{3}{2}\Upsilon_3 \\
\nabla^{CK}_{\Upsilon_5}\Upsilon_3 = -\frac{1}{2}\Upsilon_5 & \nabla^{CK}_{\Upsilon_5}\Upsilon_5 = \frac{3}{2}\Upsilon_3 \\
\nabla^{CK}_{\Upsilon_3}\Upsilon_1 = -\frac{1}{2}\Upsilon_1 & \nabla^{CK}_{\Upsilon_3}\Upsilon_2 = \frac{1}{2}\Upsilon_2 \\
\nabla^{CK}_{\Upsilon_3}\Upsilon_3 = \frac{1}{2}\Upsilon_3 & \nabla^{CK}_{\Upsilon_3}\Upsilon_4 = -\frac{1}{2}\Upsilon_4 \\
\nabla^{CK}_{\Upsilon_3}\Upsilon_5 = -\frac{1}{2}\Upsilon_5. &
\end{array}$$

Considering the results above, we calculate the non-vanishing components of the Riemannian curvature tensor concerning ∇^{CK} as

$$\begin{array}{ll}
R^{CK}(\Upsilon_1,\Upsilon_2)\Upsilon_1 = \frac{3}{4}\Upsilon_2 & R^{CK}(\Upsilon_1,\Upsilon_2)\Upsilon_2 = -\frac{3}{4}\Upsilon_1 \\
R^{CK}(\Upsilon_1,\Upsilon_3)\Upsilon_3 = -\Upsilon_1 & R^{CK}(\Upsilon_2,\Upsilon_3)\Upsilon_3 = -\Upsilon_2 \\
R^{CK}(\Upsilon_1,\Upsilon_4)\Upsilon_1 = \frac{3}{4}\Upsilon_4 & R^{CK}(\Upsilon_1,\Upsilon_4)\Upsilon_4 = -\frac{3}{4}\Upsilon_1 \\
R^{CK}(\Upsilon_1,\Upsilon_5)\Upsilon_1 = \frac{3}{4}\Upsilon_5 & R^{CK}(\Upsilon_1,\Upsilon_5)\Upsilon_5 = -\frac{3}{4}\Upsilon_1 \\
R^{CK}(\Upsilon_2,\Upsilon_4)\Upsilon_2 = \frac{3}{4}\Upsilon_4 & R^{CK}(\Upsilon_2,\Upsilon_4)\Upsilon_4 = -\frac{3}{4}\Upsilon_2 \\
R^{CK}(\Upsilon_2,\Upsilon_5)\Upsilon_2 = \frac{3}{4}\Upsilon_5 & R^{CK}(\Upsilon_2,\Upsilon_5)\Upsilon_5 = -\frac{3}{4}\Upsilon_2 \\
R^{CK}(\Upsilon_3,\Upsilon_4)\Upsilon_3 = \Upsilon_4 & R^{CK}(\Upsilon_3,\Upsilon_5)\Upsilon_3 = \Upsilon_5 \\
R^{CK}(\Upsilon_4,\Upsilon_5)\Upsilon_4 = \frac{3}{4}\Upsilon_5 & R^{CK}(\Upsilon_4,\Upsilon_5)\Upsilon_5 = -\frac{3}{4}\Upsilon_4.
\end{array}$$

In this way Ricci curvature and scalar curvature are given as

$$S^{CK}(\Upsilon_1,\Upsilon_1) = S^{CK}(\Upsilon_2,\Upsilon_2) = S^{CK}(\Upsilon_4,\Upsilon_4) = S^{CK}(\Upsilon_5,\Upsilon_5) = -\frac{9}{4},$$
$$S^{CK}(\Upsilon_3,\Upsilon_3) = -4.$$

For manifold of dimension 5 i.e. $n = 5$, the scalar curvature is $r^{CK} = -13$. So, from this example the equation (3.2), (3.4), (3.7) are verified for $n = 5$.
Now, putting $F_m = F_n = \Upsilon_3$ in (4.1) and (1.2) and using (2.5), we have

$$L_\xi{}^{CK}(\Upsilon_3,\Upsilon_3) = 0.$$

8. Conclusions

This work derives new geometric and curvature results for Sasakian manifolds exhibiting a semi-symmetric non-metric CK-connection through the study of various soliton structures. These results are applicable to geometric flow theory, generalized connections in differential geometry, and related problems in mathematical physics.

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